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On isolate domination in hypergraphs

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Abstract. In this paper we introduced the notion of an isolate domination in hypergraphs. A set $D \subseteq V$ is called a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that u and v are adjacent. A dominating set I of a hypergraph $\mathcal{H}$ is called an isolate dominating set of $\mathcal{H}$ if it contains at least one vertex $v \in I$ such that v is not adjacent to any vertex of I. The minimum cardinality of an isolate dominating set of $\mathcal{H}$ is called the isolate domination number γ_0 of $\mathcal{H}$. We determine the isolate domination number for some hypergraphs while the study on this parameter has been initiated. Furthermore, the effects of the removal of a vertex or an edge from the hypergraph upon the isolate domination number are examined.

AMS Subject Classifications: 05C65.

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Contents

1. Introduction and Background

The concept of domination in graphs was initiated by de Jaenisch [14] during 1862 when he attempted to determine the minimum number of queens required to cover or dominate an $n \times n$ chess board. Similar problems posed by Ball [3] were studied by Yaglom brothers [18]. Berge [4] in 1958 and Ore [16] in 1962 introduced the idea of domination in graphs. Berge named domination as external stability and domination number as a coefficient of external stability while Ore used the words domination and domination number for the same idea. A survey of Cockayne and Hedetniemi [7] about domination motivates many researchers to work on it. Since then many researchers have been working on this topic and extending their contributions through research articles and books. An excellent treatment of fundamentals of domination in graph is given in Haynes et. al [11] while several advanced topics for domination can studied in [10]. Several variants of domination have been introduced and well-studied in the present literature such as edge domination, total domination, connected domination, global domination, equitable domination etc. and many others are being studied. For a detailed bibliography of papers on the concept of domination, the readers may to refer Hedetniemi and Laskar [12]. The notion of an isolate domination in graphs was introduced by Hamid and Balamurugan [8]. The theory of domination in graphs is well developed on the other hand, domination in hypergraph is a recent problem to study. However, as in case

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of graphs, the domination in hypergraphs also has many interesting applications. The concept of domination in hypergraphs was initiated by Acharya [1], [2] and thereafter many researchers began to study domination in hypergraph. Reader may refer to the second part of the book [9] by Haynes et. al for the domination in hypergraph. Domination and related subset problems such as independence, irredundance, vertex covering and matching has became an extensively researched branch of graph theory, due to its wide applications and potential to solve many real life problems involving design and analysis of communication network as well as defense surveillance.

In this paper we introduced a new variant of domination in hypergraph and studied two new parameters of this domination. Later several important properties are studied and some results are found.

2. Preliminaries

We begin with recalling some basic definitions and results from [5], [6], [15], [13], [17] required for our purpose.

Definition 2.1. *A hypergraph $\mathcal{H}$ is a pair $\mathcal{H}(V,E)$ where V is a finite nonempty set and E is a collection of subsets of V. The elements of V are called vertices and the elements of E are called edges or hyperedges. And $\cup_{e_i \in E} e_i = V$ and $e_i \neq \phi$ are required for all $e_i \in E$. The number of vertices in $\mathcal{H}$ is called the order of the hypergraph and is denoted by $|V|$. The number of edges in $\mathcal{H}$ is called the size of $\mathcal{H}$ and is denoted by $|E|$. A hypergraph of order n and size m is called a (n,m) hypergraph. The number $|e_i|$ is called the degree (cardinality) of the edges e_i. The rank of a hypergraph $\mathcal{H}$ is $r(\mathcal{H}) = max_{e_i \in E}|e_i|$.*

Definition 2.2. *For any vertex v in a hypergraph $\mathcal{H}(V,E)$, the set*

$$N[v] = \{u \in V : u \text{ is adjacent to } v\} \cup \{v\}$$

is called the closed neighborhood of v in $\mathcal{H}$ and each vertex in the set $N[v] - \{v\}$ is called neighbor of v. The open neighborhood of the vertex v is the set $N[v] \setminus \{v\}$. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Definition 2.3. *A simple hypergraph (or sperner family) is a hypergraph $\mathcal{H}(V,E)$ where $E = \{e_1, e_2, \cdots, e_m\}$ such that $e_i \subset e_j$ implies $i = j$.*

Definition 2.4. *For any hypergraph $\mathcal{H}(V,E)$ two vertices v and u are said to be adjacent if there exists an edge $e \in E$ that contains both v and u and non-adjacent otherwise.*

Definition 2.5. *For any hypergraph $\mathcal{H}(V,E)$ two edges are said to be adjacent if their intersection is nonempty. If a vertex $v_i \in V$ belongs to an edge $e_j \in E$ then we say that they are incident to each other.*

Definition 2.6. *The vertex degree of a vertex v is the number of vertices adjacent to the vertex v in $\mathcal{H}$. It is denoted by $d(v)$. The maximum (minimum) vertex degree of a hypergraph is denoted by $\Delta(\mathcal{H})$ $(\delta(\mathcal{H}))$.*

Definition 2.7. *The edge degree of a vertex v is the number of edges containing the vertex v. It is denoted by $d_E(v)$.*

The maximum (minimum) edge degree of a hypergraph is denoted by $\Delta_E(\mathcal{H})(\delta_E(\mathcal{H}))$. A vertex of a hypergraph which is incident to no edge is called an isolated vertex.

Definition 2.8. *A star hypergraph is an intersecting family of edges having a common element v. It is denoted by $\mathcal{H}(v)$ and the vertex v is called the center of $\mathcal{H}(v)$.*

Definition 2.9. *The hypergraph $\mathcal{H}(V,E)$ is called connected if for any pair of its vertices, there is a path connecting them. If $\mathcal{H}$ is not connected then it consists of two or more connected components, each of which is a connected hypergraph.*

Definition 2.10. *For $0 \leq r \leq n$, we define the complete r-uniform hypergraph to be the simple hypergraph $K_n^r = \mathcal{H}(V, E)$ such that $|V| = n$ and $E(K_n^r)$ coincides with all the r-subsets of V.*

Definition 2.11. *A complete r-partite hypergraph is an r-uniform hypergraph $\mathcal{H}(V, E)$ such that the set V can be partitioned into r non-empty parts, each edge contains precisely one vertex from each part, and all such subsets form E. It is denoted by $K^r_{n_1,n_2,\ldots,n_r}$, where n_i is the number of vertices in part V_i.*

Definition 2.12. *Let S be a set of vertices of a hypergraph $\mathcal{H}$ and let $u \in S$. Then the vertex v is said to be a private neighbor of u (with respect to S) if $N[v] \cap S = \{u\}$. The set of all private neighbors of u with respect to S is called private neighbor set of u with respect to S and is denoted by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.*

Definition 2.13. *For a hypergraph $\mathcal{H}(V, E)$, a set $D \subseteq V$ is called a dominating set of $\mathcal{H}$ if for every $v \in V \setminus D$ there exists $u \in D$ such that u and v are adjacent in $\mathcal{H}$, that is there exists $e \in E$ such that $u, v \in e$.*

Definition 2.14. *A dominating set D of a hypergraph $\mathcal{H}$ is called a minimal dominating set, if no proper subset of D is a dominating set of $\mathcal{H}$. The minimum(maximum) cardinality of a minimal dominating set in a hypergraph $\mathcal{H}$ is called the domination(upper domination) number of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})(\Gamma(\mathcal{H}))$.*

3. Isolate Domination

In this section the notion of an isolate domination is given while the parameters like isolate domination number and upper isolate domination number are defined and verified with examples. Later we determine the values of these parameters for some hypergraphs and some bounds in terms of elements of $\mathcal{H}$ are obtained. Lastly, we investigate the properties of the hypergraphs for which $\gamma_0(\mathcal{H}) = n - \Delta(\mathcal{H})$.

Definition 3.1. *A dominating set I of a hypergraph $\mathcal{H}$ is called an isolate dominating set of $\mathcal{H}$ if it contains at least one vertex $v \in I$ such that v is not adjacent to any vertex of I i.e. $N(v) \cap I = \phi$, for at least one vertex $v \in I$.*

Definition 3.2. *An isolate dominating set I of a hypergraph $\mathcal{H}$ is called a minimal isolate dominating set if no proper subset of I is an isolate dominating set of $\mathcal{H}$.*

Definition 3.3. *The minimum (maximum) cardinality of a minimal isolate dominating set in a hypergraph $\mathcal{H}$ is called the isolate (upper isolate) domination number of $\mathcal{H}$ and is denoted by $\gamma_0(\mathcal{H})(\Gamma_0(\mathcal{H}))$.*
An isolate dominating set of cardinality $\gamma_0(\Gamma_0)$ is called a γ_0-set (Γ_0-set).

Example 3.4. *Consider the hypergraph $\mathcal{H}(V, E)$ where $V = \{v_1, v_2, \ldots, v_{14}\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned}
e_1 &= \{v_1, v_2, v_3, v_4, v_5, v_6\},\\
e_2 &= \{v_5, v_6, v_7, v_8\},\\
e_3 &= \{v_6, v_9\},\\
e_4 &= \{v_2, v_3, v_{10}, v_{11}\},\\
e_5 &= \{v_1, v_2, v_{12}, v_{13}, v_{14}\}.
\end{aligned}$$

Then the sets $I_1 = \{v_2, v_7, v_9\}$, $I_2 = \{v_4, v_6, v_{10}, v_{12}\}$ and $I_3 = \{v_4, v_7, v_9, v_{10}, v_{12}\}$ are the isolate dominating sets of $\mathcal{H}$. But among these only I_1 and I_3 are minimal isolate dominating sets but not I_2. In fact, I_1 is a minimal dominating set of $\mathcal{H}$ with minimum cardinality and I_3 is that of maximum cardinality. Hence $\gamma_0(\mathcal{H}) = 3$ and $\Gamma_0(\mathcal{H}) = 5$.

Theorem 3.5. *Let $\mathcal{H}$ be a disconnected hypergraph having $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots, \mathcal{H}_k$ as its components then*

1. $\gamma_0(\mathcal{H})$= $min_{1\leq i\leq k}\{s_i\}$, *where* $s_i = \gamma_0(\mathcal{H}_i) + \sum_{j=1,j\neq i}^{k} \gamma(\mathcal{H}_j)$.

2. $\Gamma_0(\mathcal{H})$= $max_{1\leq i\leq k}\{r_i\}$, *where* $r_i = \Gamma_0(\mathcal{H}_i) + \sum_{j=1,j\neq i}^{k} \Gamma(\mathcal{H}_j)$.

Proof. **1)** Suppose s_1 = $\min\{s_1, s_2, ..., s_k\}$. Let I be a γ_0-set of $\mathcal{H}_1$ and D_i be a γ-sets of $\mathcal{H}_i$ for all $i \geq 2$. Then the set $I \cup (\cup_{i=2}^{k} D_i)$ is an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq \gamma_0(\mathcal{H}_1) + \sum_{j=2}^{k} \gamma(\mathcal{H}_j) = s_1 = \min_{1\leq i\leq k}\{s_i\}$.

Now let I be any minimal isolate dominating set of $\mathcal{H}$. Then the intersection of I and the vertex of $V(\mathcal{H}_i)$ of each component $\mathcal{H}_i$ is non-empty. In fact, the set $I \cap V(\mathcal{H}_i)$ a minimal dominating set of $\mathcal{H}_i$, for all $i = 1, 2, \ldots, k$. Further, for at least one i, say j we have $I \cap V(\mathcal{H}_j)$ is an isolate dominating set of $\mathcal{H}_j$. Therefore $|I| \geq \gamma_0(\mathcal{H}_j) + \sum_{i=1,i\neq j}^{k} \gamma(\mathcal{H}_i) = s_j \geq \min\{s_i\}$. Hence $\gamma_0(\mathcal{H})$= $\min_{1\leq i\leq k}\{s_i\}$.

2) Every Γ_0-set of $\mathcal{H}_i$ together with the set $\cup_{j=1,j\neq i}^{k} D_j$ forms a minimal isolate dominating set of $\mathcal{H}$, where D_j is a Γ-set of $\mathcal{H}_j$ and $1 \leq i \leq k$. Hence $\Gamma_0(\mathcal{H}) \geq \max_{1\leq i\leq k}\{r_i\}$.

Now let I be any minimal isolate dominating set of $\mathcal{H}$. Then $I \cap V(\mathcal{H}_i)$ is a minimal dominating set of $\mathcal{H}$ for every $i = 1, 2, \ldots, k$. Further for at least one i, say j we have $I \cap V(\mathcal{H}_j)$ is an isolate dominating set of $\mathcal{H}_j$.

Therefore $|I| \leq \Gamma_0(\mathcal{H}_j) + \sum_{i=1,i\neq j}^{k} \Gamma(\mathcal{H}_i) = r_j \leq \max_{1\leq i\leq k}\{r_i\}$.

Hence $\Gamma_0(\mathcal{H})$=$\max_{1\leq i\leq k}\{r_i\}$. ■

Observations 3.6. *If a hypergraph* $\mathcal{H}$ *contains an isolated vertex then* $\gamma_0(\mathcal{H}) = \gamma(\mathcal{H})$ *and* $\Gamma_0(\mathcal{H}) = \Gamma(\mathcal{H})$.

In light of the above observation, we restrict our attention to connected hypergraphs in the rest of this paper unless otherwise stated.

Theorem 3.7. *For complete* r*-uniform hypergraph* $\mathcal{H} = K_n^r$*, for* $r \geq 2$*,* $\gamma_0(\mathcal{H}) = \Gamma_0(\mathcal{H}) = 1$ *and for complete* r*-partite hypergraph* $\mathcal{H} = K^r_{n_1,n_2,\ldots,n_r}$*,* $\gamma_0(\mathcal{H})$=$min\{n_1, n_2, \ldots, n_r\}$*,* $\Gamma_0(\mathcal{H})$=$max\{n_1, n_2, \ldots, n_r\}$.

Proof. Any vertex in complete r-uniform hypergraph is adjacent to all vertices of $\mathcal{H}$. Hence $\gamma_0(K_n^r) = \Gamma_0(K_n^r) = 1$. Further from the definition of complete r-partite hypergraph $\mathcal{H}$, each r parts are the minimal isolate dominating sets of $\mathcal{H}$. Hence maximum and minimum values of the set $\{n_1, n_2, \ldots, n_r\}$ will be the $\gamma_0(\mathcal{H})$ and $\Gamma_0(\mathcal{H})$ respectively. ■

Observations 3.8. *If* I *is a minimal isolate dominating set of* $\mathcal{H}$ *then* $V \setminus I$ *is a dominating set of* $\mathcal{H}$.

In view of the above observation, complement of a minimal isolate dominating set is dominating but need not be an isolate dominating. But following theorem proves that like domination number of $\mathcal{H}$, the isolate domination number $\gamma_0(\mathcal{H})$ does not exceed half of the order of $\mathcal{H}$.

Theorem 3.9. *For a connected hypergraph* $\mathcal{H}$*,* $\gamma_0(\mathcal{H}) \leq \frac{n}{2}$*, where* n *is the number of vertices of* $\mathcal{H}$*. Moreover, if* p *and* q *are positive integers such that* $q \geq 2p$ *then there exists a hypergraph* $\mathcal{H}$ *of order* q *with* $\gamma_0(\mathcal{H}) = p$.

Proof. Let $\mathcal{H}$ be a connected hypergraph. Let D be a minimum dominating set of $\mathcal{H}$. If for any $v \in D$, we have $N(v) \cap D = \phi$ then D itself is a minimal isolate dominating set of $\mathcal{H}$ and the result follows. If $N(v) \cap D \neq \phi$, for every $v \in D$ then every vertex $v \in D$ has at least one private neighbor in $V \setminus D$ with respect to D. Let w be a vertex in D with minimum number of private neighbors, say m with respect to D. Then $\gamma(\mathcal{H}) + \gamma(\mathcal{H})m \leq n$. Further, the set $D - \{w\} \cup I$, where I is γ_0-set of $pn\,[w, D]$ is an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq \gamma(\mathcal{H}) - 1 + m$. Now we prove that $\gamma(\mathcal{H}) - 1 + m \leq \frac{\gamma(\mathcal{H})+\gamma(\mathcal{H})m}{2}$. The inequality is true when $\gamma(\mathcal{H}) = 2$. Now if $2(\gamma(\mathcal{H}) - 1 + m) > \gamma(\mathcal{H}) + \gamma(\mathcal{H})m$ and $\gamma(\mathcal{H}) \neq 2$, then we have $(\gamma(\mathcal{H}) - 2) > m(\gamma(\mathcal{H}) - 2)$, getting a contradiction as $m \geq 1$. Hence $\gamma_0(\mathcal{H}) \leq \gamma(\mathcal{H}) - 1 + m \leq \frac{\gamma(\mathcal{H})+\gamma(\mathcal{H})m}{2} \leq \frac{n}{2}$.

Now let p and q be any two positive integers such that $q > 2p$. Construct a hypergraph $\mathcal{H}$ of order q with $\gamma_0(\mathcal{H}) = p$. Firstly we consider an edge e' of cardinality p. Then the hypergraph $\mathcal{H}$ is obtained from that edge e' by attaching exactly one vertex at each $p - 1$ vertices and then adding one edge e containing the remaining one vertex from e' and $q - 2p + 1$ new vertices. It is clear to see that $\mathcal{H}$ is a hypergraph of order q with $\gamma_0(\mathcal{H}) = p$. ■

Observations 3.10. *For any vertex v in a hypergraph $\mathcal{H}$, the set $V \setminus N(v)$ is always an isolate dominating set of $\mathcal{H}$ and consequently $\gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$.*

Theorem 3.11. *Let $\mathcal{H}$ be a hypergraph of order n with $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ and let w be a vertex of degree $\Delta(\mathcal{H})$. Then $V \setminus N[w]$ is independent and $\Delta(\mathcal{H}) \geq \frac{n}{2}$.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Suppose $V \setminus N[w]$ is not independent then there exists two vertices $p, q \in V \setminus N[w]$ such that p and q are adjacent. Consequently, the set $I = (V \setminus N[w] - \{p\}) \cup \{w\}$ is an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H}) - 1$, a contradiction. Hence $V \setminus N[w]$ is independent. Now we prove that $\Delta(\mathcal{H}) \geq \frac{n}{2}$. Suppose $\Delta(\mathcal{H}) < \frac{n}{2}$. Before proving this, first we claim that each vertex of $N(w)$ is adjacent to at most one vertex in $V \setminus N[w]$. Suppose there exists a vertex $u \in N(w)$ having at least two neighbors say x and y in $V \setminus N[w]$. Since $\Delta(\mathcal{H}) < \frac{n}{2}$, it follows that $V \setminus N[w]$ contains at least $\Delta(\mathcal{H})$ vertices. Hence there exists a vertex z in $V \setminus N[w]$ which is not adjacent to u. Therefore the set $I = (V \setminus N[w] - \{x, y\}) \cup \{u, w\}$ is an isolate dominating set of $\mathcal{H}$ with cardinality less than or equal to $n - \Delta(\mathcal{H}) - 1$, which is a contradiction. Hence each vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$. Further $|V \setminus N[w]| \geq \Delta(\mathcal{H})$, together with the facts that $V \setminus N[w]$ is independent and each vertex of $N(w)$ has at most one neighbor in $V \setminus N[w]$, it follows that the sets $V \setminus N[w]$ and $N(w)$ have equal number of vertices . Hence a vertex in $N(w)$ together with its non-neighbors in $V \setminus N[w]$ form an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H}) - 1$, a contradiction. Hence $\Delta(\mathcal{H}) \geq \frac{n}{2}$. ■

Theorem 3.12. *Let $\mathcal{H}$ be a connected hypergraph and let w be a vertex of degree $\Delta(\mathcal{H})$. If $V \setminus N[w]$ is an independent set and every vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$ then either $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ or $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n - 1$. Further if $N(w)$ contains a vertex of degree 1 then $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Let I be an isolate dominating set of $\mathcal{H}$ with $|I| = \gamma_0(\mathcal{H})$. It is easy to see that the set $V \setminus N(w)$ is an isolate dominating set of $\mathcal{H}$ with cardinality $n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$. Since $V \setminus N[w]$ is independent and every vertex in $N(w)$ has at most one neighbor in $V \setminus N[w]$, it follows that $|I| \geq |V \setminus N[w]| = n - \Delta(\mathcal{H}) - 1$. Therefore $n - \Delta(\mathcal{H}) - 1 \leq \gamma_0(\mathcal{H}) \leq n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$ or $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n - 1$. Further if $N(w)$ contains a vertex of degree 1. Let $u \in N(w)$ such that $d(u) = 1$. Then I must contain either u or w. Also I contains at least $|V \setminus N[w]|$ vertices for dominating all the vertices of $V \setminus N[w]$. Therefore $\gamma_0(\mathcal{H}) = |I| \geq n - \Delta(\mathcal{H})$. Hence $\gamma_0(\mathcal{H}) + \Delta(\mathcal{H}) = n$. This completes the proof. ■

4. Vertex Removal and Edge Removal

This section deals with the effects of vertex removal or edge removal on the isolate domination number and study the characteristics of vertices whose removal decreases or increases the isolate domination number of a hypergraph $\mathcal{H}$.

Definition 4.1. *[5] Let $\mathcal{H}$ be a hypergraph and $v \in V$. Then $\mathcal{H} \setminus \{v\}$ is a sub-hypergraph with vertex set $V \setminus \{v\}$ and edge set $\{e \setminus \{v\} : e \in E, e \setminus \{v\} \neq \phi\}$.*

Definition 4.2. *[5] Let $\mathcal{H}$ be a hypergraph and $e \in E$. Then $\mathcal{H} \setminus \{e\}$ is a sub-hypergraph with edge set $E \setminus \{e\}$, whose vertex set contains all vertices of $\mathcal{H}$ which are not pendant vertices in the deleted edge e.*

Theorem 4.3. *For a hypergraph $\mathcal{H}$ and $v \in V$, $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H}) - 1$.*

Proof. Let v be the vertex in $\mathcal{H}$ such that $\gamma_0(\mathcal{H} \setminus v) < \gamma_0(\mathcal{H})$. Let I be a γ_0-set of $\mathcal{H} \setminus v$. Then $N(v) \cap I = \phi$, otherwise I would be an isolate dominating set of $\mathcal{H}$ with cardinality less than $\gamma_0(\mathcal{H})$, which is a contradiction. Therefore the set $I \cup \{v\}$ forms an isolate dominating set of $\mathcal{H}$ with a vertex v such that $N(v) \cap I = \phi$. Thus $\gamma_0(\mathcal{H}) \leq |I \cup \{v\}| \leq \gamma_0(\mathcal{H} \setminus v) + 1$. Hence $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H}) - 1$. ■

Proposition 4.4. *Let $\mathcal{H}$ be a complete r-partite hypergraph with r-partitions $V_1, V_2, ...V_r$ then*

1. *If $|V_i| = 1$, for exactly one i, then $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H})$, for $v \in V$.*
2. *If $|V_i| = 1$, for more than one i, then isolate domination number remains unchanged on removal of any vertex v from H.*
3. *If each V_i contains at least two vertices then $\gamma_0(\mathcal{H} \setminus v) \leq \gamma_0(\mathcal{H})$, for $v \in V$.*

Proof.
1. Let $V_1 = \{w\}$. Then by definition, w dominates all the vertices of hypergraph $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) = 1$. Now if we remove a vertex w from hypergraph $\mathcal{H}$ then $\mathcal{H} \setminus w$ is a complete $(r-1)$ partite hypergraph with each part having at least two vertices. Thus $\gamma_0(\mathcal{H} \setminus w) \geq 2$, by theorem 3.7. Further, the removal of any vertex $v \neq w$ will not affect the value of $\gamma_0(\mathcal{H})$, as w is still there, to dominate all the vertices of hypergraph $\mathcal{H} \setminus v$. Hence $\gamma_0(\mathcal{H} \setminus v) \geq \gamma_0(\mathcal{H})$.
2. Let $V_1 = \{w_1\}$ and $V_2 = \{w_2\}$. Clearly V_1 and V_2 are the isolate dominating sets of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) = 1$. Also the removal of any vertex v from $\mathcal{H}$ does not affect the value of $\gamma_0(\mathcal{H})$ as either w_1 or w_2 is present in $\mathcal{H} \setminus v$. Hence the result follows.
3. Let min$\{|V_i|\} = p$. Let the part V_k contains p vertices. Then by theorem 3.7, V_k is a γ_0-set of $\mathcal{H}$. Also each vertex of V_k is the only private neighbor of itself. Hence $\gamma_0(\mathcal{H} \setminus v) < \gamma_0(\mathcal{H})$, for $v \in V_k$. Further on removing any vertex $v \in V_i$ and $V_i \neq V_k$, we have $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H})$.

■

Theorem 4.5. *Let $\mathcal{H}$ be a hypergraph with $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H}) - 1$ iff there is a γ_0-set I with at least two vertices $u \in I$ such that $N(u) \cap I = \phi$ and $pn[v, I] = \{v\}$.*

Proof. Let $\gamma_0(\mathcal{H} \setminus v) = \gamma_0(\mathcal{H}) - 1$ and let I be a γ_0-set of $\mathcal{H} \setminus v$. Then $N(v) \cap I = \phi$. Thus the set $I \cup \{v\}$ is a γ_0-set of $\mathcal{H}$ with at least two vertices $u \in I$ such that $N(u) \cap I = \phi$ and also $pn[v, I] = \{v\}$. Conversely, suppose I be a γ_0-set of $\mathcal{H}$ with given conditions. Since $pn[v, I] = \{v\}$ and for at least two vertices of I, we have $N(v) \cap I = \phi$, it follows the set $I - v$ is an isolate dominating set of $\mathcal{H} \setminus v$. Therefore $\gamma_0(\mathcal{H} \setminus v) \leq |I| - 1 = \gamma(\mathcal{H}) - 1$. Hence by theorem 4.3, the result follows. ■

Theorem 4.6. *Let $\mathcal{H}$ be a hypergraph with at most one isolate vertex then $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$ if and only if*

1. *v is in every γ_0-set of $\mathcal{H}$.*
2. *No subset of $I \subseteq V \setminus N[v]$ with cardinality less than or equal to $\gamma_0(\mathcal{H})$ can be an isolate dominating set of $\mathcal{H} \setminus v$.*

Proof. Let $\mathcal{H}$ be a given hypergraph and $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. Suppose v does not belong to γ_0-set I of $\mathcal{H}$. Then I will be an isolate dominating set of $\mathcal{H} \setminus v$. Consequently, $\gamma_0(\mathcal{H} \setminus v) \leq |I|$, which is a contradiction. Hence v is in every γ_0-set of $\mathcal{H}$ and 2 is obvious. Now conversely let 1) and 2) hold. Let I be a γ_0-set of $\mathcal{H} \setminus v$. If $I \subseteq V|N[v]$ then $|I| > \gamma_0(\mathcal{H})$, by condition 2. Hence $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. If $I \cap N(v) \neq \phi$. Then I would be an isolate dominating set of $\mathcal{H}$. Hence $\gamma_0(\mathcal{H}) \leq |I|$. But by condition 1, $|I| > \gamma_0(\mathcal{H})$. Consequently, $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$. ■

Theorem 4.7. *Let $\mathcal{H}$ be a hypergraph with at most one isolated vertex. If u and v be the vertices in $\mathcal{H}$ such that $\gamma_0(\mathcal{H} \setminus u) < \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H} \setminus v) > \gamma_0(\mathcal{H})$ then u and v are not adjacent.*

Proof. Let $\mathcal{H}$ be a given hypergraph. Suppose u and v are adjacent. Let I be a γ_0-set of $\mathcal{H} \setminus u$. Then $I \cap N(u) = \phi$, otherwise I would form an isolate dominating set of $\mathcal{H}$ with cardinality less than $\gamma_0(\mathcal{H})$. Since u and v are adjacent, it follows $v \notin I$. Therefore the set $I \cup \{u\}$ would form a γ_0-set of $\mathcal{H}$, which is contradiction to the condition 1 of theorem 4.6,. Hence u and v are not adjacent. ■

Remark 4.8. *The following example illustrates that the converse is not true.*

Example 4.9. *Let $\mathcal{H}(V,E)$ be a hypergraph, where $V = \{v_1, v_2, \ldots, v_{11}\}$ and $E = \{e_1, e_2, \ldots, e_5\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned}
e_1 &= \{v_1, v_2, v_3, v_4, v_5\},\\
e_2 &= \{v_1, v_2, v_6, v_7\},\\
e_3 &= \{v_1, v_8\},\\
e_4 &= \{v_4, v_5, v_9, v_{10}\},\\
e_5 &= \{v_4, v_{11}\}.
\end{aligned}$$

The vertices v_8, v_{11} are not adjacent in $\mathcal{H}$ with $\gamma_0(\mathcal{H} \setminus v_8) < \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H} \setminus v_{11}) < \gamma_0(\mathcal{H})$. And the vertices v_6, v_9 are not adjacent in $\mathcal{H}$ with $\gamma_0(\mathcal{H} \setminus v_6) = \gamma_0(\mathcal{H})$, $\gamma_0(\mathcal{H} \setminus v_9) = \gamma_0(\mathcal{H})$.

Observations 4.10. *The isolate domination number $\gamma_0(\mathcal{H})$ of a hypergraph $\mathcal{H}$ may increase, decrease or remains unaltered when we remove an edge e from hypergraph $\mathcal{H}$. Moreover the differences $\gamma_0(\mathcal{H} \setminus e) - \gamma_0(\mathcal{H})$ and $\gamma_0(\mathcal{H}) - \gamma_0(\mathcal{H} \setminus e)$ can be made arbitrarily large.*

The following examples give the illustration of the above observation.

Example 4.11. *Consider two star hypergraphs $\mathcal{H}_1(u)$ and $\mathcal{H}_2(v)$ of size p whose centers are connected by an edge $e' = \{u, v\}$. Let $\mathcal{H}$ be that hypergraph. Then $\gamma_0(\mathcal{H}) = 1 + p$. Thus removing an edge e' from hypergraph $\mathcal{H}$ decrease the isolate domination number of $\mathcal{H}$ by $p - 1$.*

Example 4.12. *Consider the hypergraph $\mathcal{H}(V,E)$ where $V = \{v_1, v_2, \ldots, v_{p+2}, u_1, u_2, \ldots, u_{p+2}\}$ where p be any positive integer and $E = \{e_1, e_2, \ldots, e_{p+4}\}$. In which the edges of $\mathcal{H}$ are defined as follows:*

$$\begin{aligned}
e_1 &= \{v_1, u_1\},\\
e_2 &= \{v_2, u_2\},\\
&\ \vdots\\
e_{p+2} &= \{v_{p+2}, u_{p+2}\},\\
e_{p+3} &= \{v_1, v_2, \ldots v_{p+2}\},\\
e_{p+4} &= \{u_1, u_2, \ldots u_{p+2}\}.
\end{aligned}$$

Clearly, $\{v_1, u_2\}$ is an isolate dominating set of $\mathcal{H}$ and $\gamma_0(\mathcal{H}) = 2$. However, $\gamma_0(\mathcal{H} \setminus e_{p+3}) = p + 2$ and $\gamma_0(\mathcal{H} \setminus e_1) = 2$.

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