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$\mathscr{I}_\lambda$-statistical limit points and $\mathscr{I}_\lambda$-statistical cluster points

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Abstract

In this paper we have extended the notion of λ-statistical limit points of real sequences to $\mathscr{I}_\lambda$-statistical limit points and studied some basic properties of the set of all $\mathscr{I}_\lambda$-statistical limit points and $\mathscr{I}_\lambda$-statistical cluster points of real sequences including their interrelationship. Then we have established $\mathscr{I}_\lambda$-statistical analogue of the monotone sequence theorem. Also introducing additive property of $\mathscr{I}_\lambda$-density zero sets we have established its relationship with $\mathscr{I}_\lambda$-statistical convergence.

Keywords

$\mathscr{I}_\lambda$-statistical convergence, $\mathscr{I}_\lambda$-statistical limit point, $\mathscr{I}_\lambda$-statistical cluster point, $\mathscr{I}_\lambda$-density, $\mathscr{I}_\lambda$-statistical boundedness.

AMS Subject Classification

40G15, 40A35.

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Contents

1. Introduction and background:

As a generalization of the usual notion of convergence of real sequences the notion of statistical convergence was introduced independently by Fast [10] and Schoenberg [31] using the concept of natural density of subsets of $\mathbb{N}$.

A set $\mathscr{M} \subset \mathbb{N}$ is said to have natural density $d(\mathscr{M})$, if

$$d(\mathscr{M}) = \lim_{n\to\infty} \frac{|\mathscr{M}(n)|}{n},$$

where $\mathscr{M}(n) = \{m \leq n : m \in \mathscr{M}\}$ and $|\mathscr{M}(n)|$ represents the number of elements in $\mathscr{M}(n)$.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to ξ if for every $\varepsilon > 0$, $d(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0$.

Study in this line became one of the most active research area in summability theory after the works of Šalát [26] and Fridy [12]. Using the concept of statistical convergence, the notions of statistical limit point and statistical cluster point of real sequences were introduced and studied by Fridy [13].

If $\{x_{k_j}\}_{j\in\mathbb{N}}$ is a subsequence of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ and $\mathscr{Q} = \{k_j : j \in \mathbb{N}\}$, then we use the notation $\{x\}_\mathscr{Q}$ to denote the subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$. In case $d(\mathscr{Q}) = 0$, $\{x\}_\mathscr{Q}$ is called a thin subsequence of x. On the other hand $\{x\}_\mathscr{Q}$ is called a non-thin subsequence of x if $d(\mathscr{Q}) \neq 0$, where $d(\mathscr{Q}) \neq 0$ means that either $d(\mathscr{Q})$ is a positive number or $\mathscr{Q}$ fails to have natural density.

A real number p is called a statistical limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists a non-thin subsequence of x that converges to p.

A real number q is called a statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - q| < \varepsilon\}$ does not have natural density zero.

For more works on this convergence notion one can see [2, 3, 5, 14, 25, 32].

The notion of λ-statistical convergence of real sequences was introduced by Mursaleen [22] using the concept of λ-density of subsets of $\mathbb{N}$.

If $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of positive real numbers tending to ∞ such that $\lambda_1 = 1, \quad \lambda_{n+1} \leq \lambda_n + 1, \quad n \in \mathbb{N}$, then any set $\mathscr{M} \subset \mathbb{N}$ is said to have λ-density

$d_\lambda(\mathscr{M})$, if

$$d_\lambda(\mathscr{M}) = \lim_{n\to\infty} \frac{|\{k \in I_n : k \in \mathscr{M}\}|}{\lambda_n},$$

where $I_n = [n-\lambda_n+1, n]$. The collection of all such sequences λ is denoted by Δ_∞. Throughout this paper λ- stands for such a sequence.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be λ-statistically convergent to ξ if for every $\varepsilon > 0$, $d_\lambda(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0$.

Clearly, if $\lambda_n = n, \forall n \in \mathbb{N}$, then the concepts of λ-density and λ-statistical convergence coincide with natural density and statistical convergence respectively.

Actually the concepts of λ-density and λ-statistical convergence are special cases of A-density and A-statistical convergence (see [1, 4, 11, 16]), where A is an $\mathbb{N}\times\mathbb{N}$ non negative regular summability matrix. An $\mathbb{N}\times\mathbb{N}$ matrix $A = (a_{nk})$ is called a regular summability matrix if for any convergent sequence $x = \{x_k\}_{k\in\mathbb{N}}$ with limit ξ, $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k = \xi$, and A is called nonnegative if $a_{nk} \geq 0, \forall n, k$.

For a non negative regular summability matrix $A = (a_{nk})$, a set $\mathscr{M} \subset \mathbb{N}$ is said to have A-density $\delta_A(\mathscr{M})$, if

$$\delta_A(\mathscr{M}) = \lim_{n\to\infty} \sum_{k\in\mathscr{M}} a_{nk}.$$

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be A-statistically convergent to ξ if for every $\varepsilon > 0$, $\delta_A(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0$.

If $A = A_s = (a_{nk})$, where $a_{nk} = \begin{cases} \frac{1}{\lambda_n} & if\ k \in I_n, \\ 0 & if\ k \notin I_n, \end{cases}$ then A-density and A-statistical convergence coincide with λ-density and λ-statistical convergence respectively. Again, if $\lambda_n = n, \forall n \in \mathbb{N}$, then the matrix $A = A_s$ becomes the Cesaro matrix C_1 and so A-density and A-statistical convergence coincide with natural density and statistical convergence respectively.

The concept of statistical convergence was further generalized to the notion of $\mathscr{I}$-convergence by Kostyrko et al.[17] using the notion of an ideal of subsets of $\mathbb{N}$.

A non-empty family $\mathscr{I}$ of subsets of a non empty set S is called an ideal in S if $\mathscr{I}$ is hereditary (i.e. $\mathscr{A} \in \mathscr{I}, B \subset \mathscr{A} \Rightarrow \mathscr{B} \in \mathscr{I}$) and additive (i.e. $\mathscr{A}, \mathscr{B} \in \mathscr{I} \Rightarrow \mathscr{A} \cup \mathscr{B} \in \mathscr{I}$).

An ideal $\mathscr{I}$ in a non-empty set S is called non-trivial if $S \notin \mathscr{I}$ and $\mathscr{I} \neq \{\emptyset\}$.

A non-trivial ideal $\mathscr{I}$ in $S(\neq \emptyset)$ is called admissible if $\{z\} \in \mathscr{I}$ for each $z \in S$.

Throughout the paper we take $\mathscr{I}$ as a non-trivial admissible ideal in $\mathbb{N}$ unless otherwise mentioned.

A real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}$-convergent to ξ, if for any $\varepsilon > 0$, $\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\} \in \mathscr{I}$. In this case we write $\mathscr{I}\text{-}\lim_{k\to\infty} x_k = \xi$.

Using this notion of an ideal of subsets of $\mathbb{N}$, in [17] the concepts of statistical limit point and statistical cluster point were extended to the notions of $\mathscr{I}$-limit point and $\mathscr{I}$-cluster point respectively.

A real number l is said to be an $\mathscr{I}$-limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists a set $P = \{p_1 < p_2 < \ldots\} \subset \mathbb{N}$ such that $P \notin \mathscr{I}$ and $\lim_{k\to\infty} x_{p_k} = l$.

A real number y is said to be an $\mathscr{I}$-cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |x_k - y| < \varepsilon\} \notin \mathscr{I}$.

For more works on $\mathscr{I}$-convergence one can see [9, 18–20] where other references can be found.

If we take $\mathscr{I} = \mathscr{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $\mathscr{I}_d$-convergence, $\mathscr{I}_d$-limit point and $\mathscr{I}_d$-cluster point coincide with statistical convergence, statistical limit point and statistical cluster point respectively. Again for a non negative regular matrix $A = (a_{nk})$, if one consider $\mathscr{I} = \mathscr{I}_A = \{B \subset \mathbb{N} : \delta_A(B) = 0\}$, then $\mathscr{I}_A$-convergence, $\mathscr{I}_A$-limit point and $\mathscr{I}_A$-cluster point coincide with A-statistical convergence, A-statistical limit point and A-statistical cluster point respectively and in particular for $A = A_s$, $\mathscr{I}_{A_s}$-convergence, $\mathscr{I}_{A_s}$-limit point and $\mathscr{I}_{A_s}$-cluster point coincide with λ-statistical convergence, λ-statistical limit point and λ-statistical cluster point respectively.

Further using the notion of an ideal $\mathscr{I}$ of subsets of $\mathbb{N}$ in [6] a new concept of $\mathscr{I}$-statistical convergence was introduced by Das et al. as a generalization of statistical convergence.

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathscr{I}$-statistically convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - \xi| \geq \varepsilon\}| \geq \delta\} \in \mathscr{I}$.

Applying this concept of $\mathscr{I}$-statistical convergence, the notions of statistical limit point and statistical cluster point were extended to the notions of $\mathscr{I}$-statistical limit point and $\mathscr{I}$-statistical cluster point respectively (see [7, 8, 21, 23]).

In [27] the concept of $\mathscr{I}_\lambda$-statistical convergence was introduced by Savas et al. as a generalization of λ-statistical convergence. Clearly the concept of $\mathscr{I}_\lambda$-statistical convergence includes the ideas of statistical convergence, λ-statistical convergence and $\mathscr{I}$-statistical convergence as special cases.

A real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}_\lambda$-statistically convergent or $\mathscr{I} - S_\lambda$ convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \leq n : |x_k - \xi| \geq \varepsilon\}| \geq \delta\} \in \mathscr{I}$. In this case we write $\mathscr{I}\text{-}S_\lambda\text{-}\lim_{k\to\infty} x_k = \xi$. More works on this summability method can be found in [29, 30] where other references can be found.

The concept of $\mathscr{I}_\lambda$-statistical convergence is a special case of $A^{\mathscr{I}}$-statistical convergence [28], where A is an $\mathbb{N}\times\mathbb{N}$ non negative regular summability matrix.

If $A = (a_{nk})$ is an $\mathbb{N}\times\mathbb{N}$ non negative regular summability matrix, then a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $A^{\mathscr{I}}$-statistically convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : \sum_{k\in B(\varepsilon)} a_{nk} \geq \delta\} \in \mathscr{I}$, where $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}$.

Also in [15], using an $\mathbb{N}\times\mathbb{N}$ non negative regular summability matrix $A = (a_{nk})$, the notion of $A^{\mathscr{I}}$ statistical cluster point was introduced via the concept of $A^{\mathscr{I}}$-density. A subset

$\mathscr{M}$ of $\mathbb{N}$ is said to have $A^{\mathscr{I}}$-density $\delta_{A^{\mathscr{I}}}(\mathscr{M})$, if

$$\delta_{A^{\mathscr{I}}}(\mathscr{M}) = \mathscr{I} - \lim_{n\to\infty} \sum_{k\in\mathscr{M}} a_{nk}.$$

A real number p is said to be an $A^{\mathscr{I}}$-statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for each $\varepsilon > 0$, $\delta_{A^{\mathscr{I}}}(B(\varepsilon)) \neq 0$, where $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - p| < \varepsilon\}$. Note that $\delta_{A^{\mathscr{I}}}(B(\varepsilon)) \neq 0$ means, either $\delta_{A^{\mathscr{I}}}(B(\varepsilon)) > 0$ or $A^{\mathscr{I}}$-density of $B(\varepsilon)$ does not exist. From this notion of $A^{\mathscr{I}}$-statistical cluster point, one can obtain the concept of $\mathscr{I}_\lambda$-statistical cluster point as a special case. Actually, if one consider $A = A_s$, then the notions of $A^{\mathscr{I}}$ statistical convergence and $A^{\mathscr{I}}$ statistical cluster point become $\mathscr{I}_\lambda$-statistical convergence and $\mathscr{I}_\lambda$-statistical cluster point respectively.

In this paper using the notion of $\mathscr{I}_\lambda$-statistical convergence we first extend the concept of λ-statistical limit point to $\mathscr{I}_\lambda$-statistical limit point of sequences of real numbers and then study some properties of $\mathscr{I}_\lambda$-statistical limit points and $\mathscr{I}_\lambda$-statistical cluster points of sequences of real numbers not done earlier. We also study the sets of $\mathscr{I}_\lambda$-statistical limit points and $\mathscr{I}_\lambda$-statistical cluster points of sequences of real numbers including their interrelationship. In section 3 of this paper we establish $\mathscr{I}_\lambda$-statistical analogue of the sequential version of the least upper bound axiom, namely, monotone sequence theorem. Further in section 4 we introduce the condition AP$\mathscr{I}_\lambda$O and study its relationship with $\mathscr{I}_\lambda$-statistical convergence.

2. $\mathscr{I}_\lambda$-statistical limit points and $\mathscr{I}_\lambda$-statistical cluster points

In this section, we first introduce the notion of $\mathscr{I}_\lambda$-statistical limit point (which subsequently includes the notions of statistical limit point, λ-statistical limit point and $\mathscr{I}$-statistical limit point). Then we study $\mathscr{I}_\lambda$-statistical analogue of some results in [13] and [25].

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers respectively and x denotes a real sequence $\{x_k\}_{k\in\mathbb{N}}$.

Definition 2.1. *[15] A set $\mathscr{M} \subset \mathbb{N}$ is said to have $\mathscr{I}_\lambda$-density $d_\lambda^{\mathscr{I}}(\mathscr{M})$ if*

$$d_\lambda^{\mathscr{I}}(\mathscr{M}) = \mathscr{I}\text{-}\lim_{n\to\infty} \frac{|\{m \in I_n : m \in \mathscr{M}\}|}{\lambda_n}.$$

Note 2.2. *From Definition 2.1, it is clear that, if $d_\lambda(\mathscr{A}) = u, \mathscr{A} \subset \mathbb{N}$, then $d_\lambda^{\mathscr{I}}(\mathscr{A}) = u$ for any admissible ideal $\mathscr{I}$ in $\mathbb{N}$.*

In view of Definition 2.1 one can say that: a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is $\mathscr{I}_\lambda$-statistically convergent to ξ if for any $\varepsilon > 0$, $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \xi| \geq \varepsilon\}) = 0$.

If $\{x\}_{\mathscr{Q}}$ is a subsequence of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ and $d_\lambda^{\mathscr{I}}(\mathscr{Q}) = 0$, then $\{x\}_{\mathscr{Q}}$ is called an $\mathscr{I}_\lambda$-thin subsequence of x. On the other hand $\{x\}_{\mathscr{Q}}$ is called an $\mathscr{I}_\lambda$-nonthin subsequence of x if $d_\lambda^{\mathscr{I}}(\mathscr{Q}) \neq 0$, where $d_\lambda^{\mathscr{I}}(\mathscr{Q}) \neq 0$ means that either $d_\lambda^{\mathscr{I}}(\mathscr{Q})$ is a positive number or $\mathscr{Q}$ fails to have $\mathscr{I}_\lambda$-density.

Definition 2.3. *A real number l is said to be an $\mathscr{I}_\lambda$-statistical limit point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists an $\mathscr{I}_\lambda$-nonthin subsequence of x that converges to l. The set of all $\mathscr{I}_\lambda$-statistical limit points of the sequence x is denoted by $\Lambda_x^S(\mathscr{I}_\lambda)$.*

Definition 2.4. *A real number y is said to be an $\mathscr{I}_\lambda$-statistical cluster point of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - y| < \varepsilon\}$ does not have $\mathscr{I}_\lambda$-density zero. The set of all $\mathscr{I}_\lambda$-statistical cluster points of x is denoted by $\Gamma_x^S(\mathscr{I}_\lambda)$.*

Note 2.5. *(i) If $\lambda_n = n, \forall n \in \mathbb{N}$, then the notions of $\mathscr{I}_\lambda$-statistical limit point and $\mathscr{I}_\lambda$-statistical cluster point coincide with the notions of $\mathscr{I}$-statistical limit point and $\mathscr{I}$-statistical cluster point respectively.*

(ii) If $\mathscr{I} = \mathscr{I}_{fin} = \{\mathscr{K} \subset \mathbb{N} : |\mathscr{K}| < \infty\}$, then the notions of $\mathscr{I}_\lambda$-statistical limit point and $\mathscr{I}_\lambda$-statistical cluster point coincide with the notions of λ-statistical limit point and λ-statistical cluster point respectively.

(iii) If $\mathscr{I} = \mathscr{I}_{fin} = \{\mathscr{K} \subset \mathbb{N} : |\mathscr{K}| < \infty\}$ and also $\lambda_n = n, \forall n \in \mathbb{N}$, then the notions of $\mathscr{I}_\lambda$-statistical limit point and $\mathscr{I}_\lambda$-statistical cluster point coincide with the notions of statistical limit point and statistical cluster point respectively.

We also use the notations $\Lambda_x^S(\lambda)$ and $\Gamma_x^S(\lambda)$ to denote the sets of all λ-statistical limit points and λ-statistical cluster points of a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ respectively.

We first present an $\mathscr{I}_\lambda$-statistical analogous of some results in [13].

Theorem 2.6. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. Then $\Lambda_x^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\lambda)$.*

Proof. Let $\xi \in \Lambda_x^S(\mathscr{I}_\lambda)$. So we get a subsequence $\{x_{k_q}\}_{q\in\mathbb{N}}$ of x with $\lim_{q\to\infty} x_{k_q} = \xi$ and $d_\lambda^{\mathscr{I}}(\mathscr{M}) \neq 0$, where $\mathscr{M} = \{k_q : q \in \mathbb{N}\}$. Let $\varepsilon > 0$ be given. Since $\lim_{q\to\infty} x_{k_q} = \xi$, $\mathscr{H} = \{k_q : |x_{k_q} - \xi| \geq \varepsilon\}$ is a finite set. Hence

$$\{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\} \supset \{k_q : q \in \mathbb{N}\} \setminus \mathscr{H}$$

$$\Rightarrow \mathscr{M} = \{k_q : q \in \mathbb{N}\} \subset \{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\} \cup \mathscr{H}.$$

Now if $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \xi| < \varepsilon\}) = 0$, then we have $d_\lambda^{\mathscr{I}}(\mathscr{M}) = 0$, which is a contradiction. Thus ξ is an $\mathscr{I}_\lambda$-statistical cluster point of x. Since $\xi \in \Lambda_x^S(\mathscr{I}_\lambda)$ is arbitrary, $\Lambda_x^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\mathscr{I}_\lambda)$.

Now let $\eta \in \Gamma_x^S(\mathscr{I}_\lambda)$. Then for any $\varepsilon > 0$,

$$d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \eta| < \varepsilon\} \neq 0$$

. Since $\mathscr{I}$ is admissible, $d_\lambda(\{k \in \mathbb{N} : |x_k - \eta| < \varepsilon\} \neq 0$. So, $\eta \in \Gamma_x^S(\lambda)$. Hence $\Lambda_x^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\lambda)$. □

Theorem 2.7. *If $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ are two sequences of real numbers such that $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$, then $\Lambda_x^S(\mathscr{I}_\lambda) = \Lambda_y^S(\mathscr{I}_\lambda)$ and $\Gamma_x^S(\mathscr{I}_\lambda) = \Gamma_y^S(\mathscr{I}_\lambda)$.*

Proof. Let $\zeta \in \Gamma_x^S(\mathscr{I}_\lambda)$ and $\varepsilon > 0$ be given. Then $\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}$ does not have $\mathscr{I}_\lambda$-density zero. Let $\mathscr{H} = \{k \in \mathbb{N} : x_k = y_k\}$. As $d_\lambda^{\mathscr{I}}(\mathscr{H}) = 1$ so $\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\} \cap \mathscr{H}$ does not have $\mathscr{I}_\lambda$-density zero. Thus $\zeta \in \Gamma_y^S(\mathscr{I}_\lambda)$. Since $\zeta \in \Gamma_x^S(\mathscr{I}_\lambda)$ is arbitrary, so $\Gamma_x^S(\mathscr{I}_\lambda) \subset \Gamma_y^S(\mathscr{I}_\lambda)$. By symmetry we have $\Gamma_y^S(\mathscr{I}_\lambda) \subset \Gamma_x^S(\mathscr{I}_\lambda)$. Hence $\Gamma_x^S(\mathscr{I}_\lambda) = \Gamma_y^S(\mathscr{I}_\lambda)$.

Also let $\eta \in \Lambda_x^S(\mathscr{I}_\lambda)$. Then x has an $\mathscr{I}_\lambda$-nonthin subsequence $\{x_{k_q}\}_{q\in\mathbb{N}}$ that converges to η. Let $\mathscr{Q} = \{k_q : q \in \mathbb{N}\}$. Since $d_\lambda^{\mathscr{I}}(\{k_q \in \mathbb{N} : x_{k_q} \neq y_{k_q}\}) = 0$, we have $d_\lambda^{\mathscr{I}}(\{k_q \in \mathbb{N} : x_{k_q} = y_{k_q}\}) \neq 0$. Therefore from the latter set we have an $\mathscr{I}_\lambda$-nonthin subsequence $\{y\}_{\mathscr{Q}'}$ of $\{y\}_{\mathscr{Q}}$ that converges to η. Thus $\eta \in \Lambda_y^S(\mathscr{I}_\lambda)$. As $\eta \in \Lambda_x^S(\mathscr{I}_\lambda)$ is arbitrary, $\Lambda_x^S(\mathscr{I}_\lambda) \subset \Lambda_y^S(\mathscr{I}_\lambda)$. By similar way we get $\Lambda_x^S(\mathscr{I}_\lambda) \supset \Lambda_y^S(\mathscr{I}_\lambda)$. Hence $\Lambda_x^S(\mathscr{I}_\lambda) = \Lambda_y^S(\mathscr{I}_\lambda)$. □

We now investigate some topological properties of the set $\Gamma_x^S(\mathscr{I}_\lambda)$.

Theorem 2.8. *Let $\mathscr{C} \subset \mathbb{R}$ be a compact set and $\mathscr{C} \cap \Gamma_x^S(\mathscr{I}_\lambda) = \emptyset$. Then the set $\{k \in \mathbb{N} : x_k \in \mathscr{C}\}$ has $\mathscr{I}_\lambda$-density zero.*

Proof. Since $\mathscr{C} \cap \Gamma_x^S(\mathscr{I}_\lambda) = \emptyset$, so for every $\alpha \in \mathscr{C}$ there exists a positive real number $\gamma = \gamma(\alpha)$ such that

$$d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \alpha| < \gamma(\alpha)\}) = 0.$$

Let $B_{\gamma(\alpha)}(\alpha) = \{z \in \mathbb{R} : |z - \alpha| < \gamma(\alpha)\}$. Then the family of open sets $\{B_{\gamma(\alpha)}(\alpha) : \alpha \in \mathscr{C}\}$ form an open cover of $\mathscr{C}$. As $\mathscr{C}$ is a compact subset of $\mathbb{R}$ so there exists a finite subcover of the open cover $\{B_{\gamma(\alpha)}(\alpha) : \alpha \in \mathscr{C}\}$ for $\mathscr{C}$, say $\{\mathscr{C}_j = B_{\gamma(\alpha_j)}(\alpha_j) : j = 1,2,...,r\}$. Then $\mathscr{C} \subset \bigcup_{j=1}^{r} \mathscr{C}_i$ and also

$$d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \alpha_j| < \gamma(\alpha_j)\}) = 0 \text{ for } j = 1,2,...,r.$$

Now for every $n \in \mathbb{N}$,

$$|\{k \in I_n : x_k \in \mathscr{C}\}| \leq \sum_{j=1}^{r} \left|\{k \in I_n ; |x_k - \alpha_j| < \gamma(\alpha_j)\}\right|,$$

and by the property of $\mathscr{I}$-convergence,

$$\begin{aligned} &\mathscr{I}\text{-}\lim_{n\to\infty} \frac{|\{k \in I_n : x_k \in \mathscr{C}\}|}{\lambda_n} \\ \leq \quad &\sum_{j=1}^{r} \mathscr{I}\text{-}\lim_{n\to\infty} \frac{\left|\{k \in I_n : |x_k - \alpha_j| < \gamma(\alpha_j)\}\right|}{\lambda_n} = 0. \end{aligned}$$

This gives $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \in \mathscr{C}\}) = 0$. □

Theorem 2.9. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If x has a bounded $\mathscr{I}_\lambda$-nonthin subsequence, then the set $\Gamma_x^S(\mathscr{I}_\lambda)$ is a nonempty closed set.*

Proof. Let $x = \{x_{k_m}\}_{m\in\mathbb{N}}$ is a bounded $\mathscr{I}_\lambda$-nonthin subsequence of x and $\mathscr{C}$ be a compact set such that $x_{k_m} \in \mathscr{C}$ for each $m \in \mathbb{N}$. Let $\mathfrak{Q} = \{k_m : m \in \mathbb{N}\}$. Clearly $d_\lambda^{\mathscr{I}}(\mathfrak{Q}) \neq 0$. Now if $\Gamma_x^S(\mathscr{I}_\lambda) = \emptyset$, then $\mathscr{C} \cap \Gamma_x^S(\mathscr{I}_\lambda) = \emptyset$ and then by Theorem 2.8 we get

$$d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \in \mathscr{C}\}) = 0.$$

Since for every $n \in \mathbb{N}$, $|\{k \in I_n : k \in \mathfrak{Q}\}| \leq |\{k \in I_n : x_k \in \mathscr{C}\}|$, we have $d_\lambda^{\mathscr{I}}(\mathscr{Q}) = 0$, which is a contradiction. Therefore $\Gamma_x^S(\mathscr{I}_\lambda) \neq \emptyset$.

Now to prove $\Gamma_x^S(\mathscr{I}_\lambda)$ is a closed set in $\mathbb{R}$, let ζ be a limit point of $\Gamma_x^S(\mathscr{I}_\lambda)$. Then for any $\varepsilon > 0$, $B_\varepsilon(\zeta) \cap (\Gamma_x^S(\mathscr{I}_\lambda) \setminus \{\zeta\}) \neq \emptyset$. Let $\eta \in B_\varepsilon(\zeta) \cap (\Gamma_x^S(\mathscr{I}_\lambda) \setminus \{\zeta\})$. Now we can choose $\varepsilon' > 0$ so that $B_{\varepsilon'}(\eta) \subset B_\varepsilon(\zeta)$. Since $\eta \in \Gamma_x^S(\mathscr{I}_\lambda)$ so

$$\begin{aligned} &d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \eta| < \varepsilon'\}) \neq 0 \\ \Rightarrow\ &d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}) \neq 0. \end{aligned}$$

Therefore $\zeta \in \Gamma_x^S(\mathscr{I}_\lambda)$. □

Definition 2.10. *A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathscr{I}_\lambda$-statistically bounded if, there exists $M > 0$ such that for all $\delta > 0$, the set*

$$\mathscr{B} = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : |x_k| > M\}| \geq \delta\} \in \mathscr{I}$$

i.e., $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k| > M\}) = 0$.
Equivalently, $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathscr{I}_\lambda$-statistically bounded if, there exists a compact set F in $\mathbb{R}$ such that for all $\delta > 0$, the set $\mathscr{B} = \{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : x_k \notin F\}| \geq \delta\} \in \mathscr{I}$ i.e., $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \notin F\}) = 0$.

Note 2.11. *If $\mathscr{I} = \mathscr{I}_{fin} = \{\mathscr{M} \subset \mathbb{N} : |\mathscr{M}| < \infty\}$, then the notion of $\mathscr{I}_\lambda$-statistical boundedness coincide with the notion of λ-statistical boundedness.*

Corollary 2.12. *If $x = \{x_k\}_{k\in\mathbb{N}}$ is $\mathscr{I}_\lambda$-statistically bounded. Then the set $\Gamma_x^S(\mathscr{I}_\lambda)$ is nonempty and compact.*

Proof. Let $\mathscr{C}$ be a compact set in $\mathbb{R}$ such that $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \notin \mathscr{C}\}) = 0$. Then $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \in \mathscr{C}\}) = 1$ and this implies that $\mathscr{C}$ contains an $\mathscr{I}_\lambda$-nonthin subsequence of x. So by Theorem 2.9, $\Gamma_x^S(\mathscr{I}_\lambda)$ is a nonempty and closed set.

Now to show that $\Gamma_x^S(\mathscr{I}_\lambda)$ is compact it is sufficient to prove that $\Gamma_x^S(\mathscr{I}_\lambda) \subset \mathscr{C}$. If possible let us assume that $\zeta \in \Gamma_x^S(\mathscr{I}_\lambda)$ but $\zeta \notin \mathscr{C}$. Since $\mathscr{C}$ is compact, so there exists $\varepsilon > 0$ such that $B_\varepsilon(\zeta) \cap \mathscr{C} = \emptyset$. So we have

$$\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\} \subset \{k \in \mathbb{N} : x_k \notin \mathscr{C}\}.$$

Therefore $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - \zeta| < \varepsilon\}) = 0$, which is a contradiction to the fact that $\zeta \in \Gamma_x^S(\mathscr{I}_\lambda)$. Therefore $\Gamma_x^S(\mathscr{I}_\lambda) \subset \mathscr{C}$. □

Theorem 2.13. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be an $\mathscr{I}_\lambda$-statistically bounded sequence. Then for any $\varepsilon > 0$ the set*

$$\left\{k \in \mathbb{N} : d(\Gamma_x^S(\mathscr{I}_\lambda), x_k) \geq \varepsilon\right\}$$

has $\mathscr{I}_\lambda$-density zero, where $d(\Gamma_x^S(\mathscr{I}_\lambda), x_k) = \inf_{z\in\Gamma_x^S(\mathscr{I}_\lambda)} |z - x_k|$ the distance from x_k to the set $\Gamma_x^S(\mathscr{I}_\lambda)$.

Proof. Let $\mathscr{C}$ be a compact set such that $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \notin \mathscr{C}\}) = 0$. Then by Corollary 2.12 we get $\Gamma_x^S(\mathscr{I}_\lambda)$ is nonempty and $\Gamma_x^S(\mathscr{I}_\lambda) \subset \mathscr{C}$.

If possible, let $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : d(\Gamma_x^S(\mathscr{I}_\lambda), x_k) \geq \varepsilon'\}) \neq 0$ for some $\varepsilon' > 0$. Now we set $B_{\varepsilon'}(\Gamma_x^S(\mathscr{I}_\lambda)) = \{z \in \mathbb{R} : d(\Gamma_x^S(\mathscr{I}_\lambda), z) < \varepsilon'\}$ and $\mathscr{H} = \mathscr{C} \setminus B_{\varepsilon'}(\Gamma_x^S(\mathscr{I}_\lambda))$. Then $\mathscr{H}$ is a compact set which contains an $\mathscr{I}_\lambda$- nonthin subsequence of x. Then by Theorem 2.8 $\mathscr{H} \cap \Gamma_x^S(\mathscr{I}_\lambda) \neq \emptyset$, which is absurd, since $\Gamma_x^S(\mathscr{I}_\lambda) \subset B_{\varepsilon'}(\Gamma_x^S(\mathscr{I}_\lambda))$. So

$$d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : d(\Gamma_x^S(\mathscr{I}_\lambda), x_k) \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. □

3. $\mathscr{I}_\lambda$-statistical analogous of Completeness Theorem

In this section following the line of Fridy [13], we formulate and prove an $\mathscr{I}_\lambda$-statistical analogue of the theorem concerning sequences that are equivalent to the completeness of the real line.

We consider the sequential version of the least upper bound axiom (in $\mathbb{R}$), namely, Monotone sequence Theorem: every monotone increasing sequence of real numbers which is bounded above, is convergent. The following result is an $\mathscr{I}_\lambda$-statistical analogue of that Theorem.

Theorem 3.1. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers and $\mathscr{Q} = \{k \in \mathbb{N} : x_k \leq x_{k+1}\}$. If $d_\lambda^{\mathscr{I}}(\mathscr{Q}) = 1$ and x is bounded above on $\mathscr{Q}$, then x is $\mathscr{I}_\lambda$-statistically convergent.*

Proof. Since x is bounded above on $\mathscr{Q}$, so let p be the least upper bound of the range of $\{x_k\}_{k\in\mathscr{Q}}$. Then we have
(i) $x_k \leq p$, $\forall k \in \mathscr{Q}$
(ii) for a pre-assigned $\varepsilon > 0$, there exists a natural number $k_0 \in \mathscr{Q}$ such that $x_{k_0} > p - \varepsilon$.
Now let $k \in \mathscr{Q}$ and $k > k_0$. Then $p - \varepsilon < x_{k_0} \leq x_k < p + \varepsilon$. Thus $\mathscr{Q} \cap \{k \in \mathbb{N} : k > k_0\} \subset \{k \in \mathbb{N} : p - \varepsilon < x_k < p + \varepsilon\}$. Since the set on the left hand side of the inclusion is of $\mathscr{I}_\lambda$-density 1, we have $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : p - \varepsilon < x_k < p + \varepsilon\}) = 1$ i.e., $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : |x_k - p| \geq \varepsilon\}) = 0$. Hence x is $\mathscr{I}_\lambda$-statistically convergent to p. □

Theorem 3.2. *Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers and $\mathscr{Q} = \{k \in \mathbb{N} : x_k \geq x_{k+1}\}$. If $d_\lambda^{\mathscr{I}}(\mathscr{Q}) = 1$ and x is bounded below on $\mathscr{Q}$, then x is $\mathscr{I}_\lambda$-statistically convergent.*

Proof. The proof is similar to that of Theorem 3.1 and so is omitted. □

4. Condition AP$\mathscr{I}_\lambda$O

In this section we introduce the condition (AP$\mathscr{I}_\lambda$O) which is similar to the (APO) condition of [3].

Definition 4.1. ***(Additive property for $\mathscr{I}_\lambda$-density zero sets).*** *The $\mathscr{I}_\lambda$-density $d_\lambda^{\mathscr{I}}$ is said to satisfy AP$\mathscr{I}_\lambda$O if, given any countable collection of mutually disjoint sets $\{\mathscr{A}_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ with $d_\lambda^{\mathscr{I}}(\mathscr{A}_m) = 0$, for all $m \in \mathbb{N}$, there exists a collection of sets $\{\mathscr{B}_m\}_{m\in\mathbb{N}}$ in $\mathbb{N}$ with the properties $|\mathscr{A}_m \Delta \mathscr{B}_m| < \infty$ for each $m \in \mathbb{N}$ and $d_\lambda^{\mathscr{I}}(\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m) = 0$.*

Theorem 4.2. *A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real number is $\mathscr{I}_\lambda$-statistically convergent to p implies there exists a subset $\mathscr{H}$ of $\mathbb{N}$ with $d_\lambda^{\mathscr{I}}(\mathscr{H}) = 1$ and $\lim_{\substack{k\in\mathscr{H}\\k\to\infty}} x_k = p$ if and only if $d_\lambda^{\mathscr{I}}$ has the property AP$\mathscr{I}_\lambda$O.*

Proof. Suppose x is $\mathscr{I}_\lambda$-statistically convergent to p implies there exists a subset $\mathscr{H}$ of $\mathbb{N}$ with $d_\lambda^{\mathscr{I}}(H) = 1$ and $\lim_{\substack{k\in\mathscr{H}\\k\to\infty}} x_k = p$. We have to show $d_\lambda^{\mathscr{I}}$ has the property AP$\mathscr{I}_\lambda$O.

Let $\{\mathscr{A}_m\}_{m\in\mathbb{N}}$ be a countable collection of mutually disjoint sets in $\mathbb{N}$ with $d_\lambda^{\mathscr{I}}(\mathscr{A}_m) = 0$, for every $m \in \mathbb{N}$. Let us construct a sequence $\{x_k\}_{k\in\mathbb{N}}$ as follows

$$x_k = \begin{cases} \frac{1}{m} & \text{if } k \in \mathscr{A}_m, \\ 0 & \text{if } k \notin \bigcup_{m=1}^{\infty} \mathscr{A}_m. \end{cases}$$

Let $\varepsilon > 0$ be given. Then there exists $j \in \mathbb{N}$ such that $\frac{1}{j+1} < \varepsilon$. Then we have

$$\{k \in \mathbb{N} : x_k \geq \varepsilon\} \subset \mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \cup \mathscr{A}_j.$$

Since $d_\lambda^{\mathscr{I}}(\mathscr{A}_m) = 0$, $\forall m = 1, 2, ..., j$, we get $d_\lambda^{\mathscr{I}}(\{k \in \mathbb{N} : x_k \geq \varepsilon\}) = 0$. So $\{x_k\}_{k\in\mathbb{N}}$ is $\mathscr{I}_\lambda$-statistically convergent to 0. Then by the assumption there exists a set $\mathscr{B} \subset \mathbb{N}$, $d_\lambda^{\mathscr{I}}(\mathscr{B}) = 0$ such that $\lim_{\substack{k\in\mathbb{N}\setminus\mathscr{B}\\k\to\infty}} x_k = 0$. Therefore for each $m = 1, 2, \ldots$ we have $n_m \in \mathbb{N}$ such that $n_{m+1} > n_m$ and $x_k < \frac{1}{m}$ for all $k \geq n_m$, $k \in \mathbb{N} \setminus \mathscr{B}$. Thus if $x_k \geq \frac{1}{m}$ and $k \geq n_m$ then $k \in \mathscr{B}$.

We set $\mathscr{B}_m = \{k \in \mathbb{N} : k \in \mathscr{A}_m,\ k \geq n_{m+1}\} \cup \{k \in \mathbb{N} : k \in \mathscr{B},\ n_m \leq k < n_{m+1}\}$, $m \in \mathbb{N}$. Clearly for all $m \in \mathbb{N}$ we have $|A_m \Delta B_m| < \infty$. We now show that $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$. Fix $m \in \mathbb{N}$ and let $k \in \mathscr{B}_m$. If $k \in \{j \in \mathbb{N} : j \in \mathscr{B}, n_m \leq j < n_{m+1}\}$, then we are done. If $k \geq n_{m+1}$ and $k \in \mathscr{A}_m$ we have $x_k = \frac{1}{m}$ and so $k \in \mathscr{B}$. Therefore $\mathscr{B}_m \subset \mathscr{B}$ for all $m \in \mathbb{N}$.

Again let $k \in \mathscr{B}$. Then there exists $u \in \mathbb{N}$ such that $n_u \leq k < n_{u+1}$, which implies $k \in \mathscr{B}_u$. Therefore $\mathscr{B} \subset \bigcup_{m=1}^{\infty} \mathscr{B}_m$. Thus $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$ and $d_\lambda^{\mathscr{I}}(\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m) = 0$. This proves that $d_\lambda^{\mathscr{I}}$ has the property AP$\mathscr{I}_\lambda$O.

Conversely suppose that $d_\lambda^{\mathscr{I}}$ has the property AP$\mathscr{I}_\lambda$O. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence such that x is $\mathscr{I}_\lambda$-statistically convergent to p. Then for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - p| \geq \varepsilon\}$ has $\mathscr{I}_\lambda$-density zero. Let $\mathscr{A}_1 = \{k \in \mathbb{N} : |x_k - p| \geq 1\}$, $\mathscr{A}_m = \{k \in \mathbb{N} : \frac{1}{m-1} > |x_k - p| \geq \frac{1}{m}\}$ for $m \geq 2$, $m \in \mathbb{N}$. Then $\{\mathscr{A}_m\}_{m\in\mathbb{N}}$ is a sequence of mutually disjoint sets with $d_\lambda^{\mathscr{I}}(\mathscr{A}_m) = 0$ for every $m \in \mathbb{N}$. Then by the assumption there exists a sequence of sets $\{\mathscr{B}_m\}_{m\in\mathbb{N}}$ with $|\mathscr{A}_m \Delta \mathscr{B}_m| < \infty$ and $d_\lambda^{\mathscr{I}}(\mathscr{B} =$

$\bigcup\limits_{m=1}^{\infty}\mathscr{B}_m) = 0$. We claim that $\lim\limits_{\substack{k\in\mathbb{N}\setminus\mathscr{B}\\ k\to\infty}} x_k = p$. To establish our claim, let $\delta > 0$ be given. Then there exists a positive integer j such that $\frac{1}{j+1} < \delta$. Then $\{k \in \mathbb{N} : |x_k - p| \geq \delta\} \subset \bigcup\limits_{m=1}^{j+1}\mathscr{A}_m$. Now since $|\mathscr{A}_m \Delta \mathscr{B}_m| < \infty$, for each $m = 1, 2, ..., j+1$, there exists $n' \in \mathbb{N}$ such that $\bigcup\limits_{m=1}^{j+1}\mathscr{A}_m \cap (n', \infty) = \bigcup\limits_{m=1}^{j+1}\mathscr{B}_m \cap (n', \infty)$. Now if $k \notin \mathscr{B}$, $k > n'$, then $k \notin \bigcup\limits_{m=1}^{j+1}\mathscr{B}_m$ and consequently $k \notin \bigcup\limits_{m=1}^{j+1}\mathscr{A}_m$, which implies $|x_k - p| < \delta$. This completes the proof. □

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