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Type II Topp-Leone Dagum distribution for modeling failure times data

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Abstract

In this paper, we introduce four parameter continuous probability distribution and named it as Type II Topp-Leone Dagum distribution which is generated using Type II Topp-Leone generated family of distributions. We observe different desirable properties of Type II Topp-Leone Dagum distribution. We present expressions for important statistical measures such as moments, moment generating function, cumulant generating function, inverted moments, probability weighted moments, reliability function, hazard rate function, reversed hazard function, cumulative hazard function, second failure rate function, mean waiting time, mean residual life, Bonferroni index, Lorenz curve and generalized entropy. For the proposed distribution, the parameters of the distribution are estimated by using maximum likelihood method. Finally, we used failure time of air conditioners data to study performance of the proposed distribution.

Keywords

Type II Topp-Leone generated family, Dagum distribution, Generalized entropy, Probability weighted moment, Maximum likelihood estimation.

AMS Subject Classification

60E05, 62F10, 62E10.

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Contents

1. Introduction

Statistical modeling is the best and ultimate way of studying uncertainty of any phenomena and statistical analysis of lifetime data are random in applied sciences such as biological sciences, medical sciences, environment sciences, engineering, finance, and actuarial science, among others. Particularly, life time data plays very crucial role in fields like insurance and finance due to dynamic of nature and unique nature of data; hence there is a clear necessity for extension and modificate form of existing standard statistical distributions. In fact, numerous attempts have been made to introduce new classes of lifetime distribution to generalize many family of distributions to provide greater flexibility to the new model. Past few decades, new class of life time distributions were contributed by many researchers and it is given in the statistical literature. Here we listed prominent generated family of distribution like Gupta et al.,[10] introduced Exponentiated-G (E-G), Shaw and Buckley [18] introduced the Quadratic Rank Transmutation Map (QRTM), Cordeiro et al., [3] introduced Kumaraswamy family (Kw-G), Hasssan and Elgarhy [11] introduced Exponentiated Weibull-G (EW-G).

Camilo Dagum proposed a continuous probability distribution known as Dagum distribution in the year 1977 for modeling income and wealth data. Dagum distribution is alternative to heavy tailed distributions such as lognormal, Pareto, and generalized beta distribution.

A continuous random variable X is said to have a three parameter Dagum distribution if its probability density function and cumulative distribution function are given by

$$g(x;\sigma,\theta,\beta) = \sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-\beta-1}. \tag{1.1}$$

where, $x>0, \sigma>0, \theta>0$ and $\beta>0$.
and

$$G(x;\sigma,\theta,\beta) = \left(1+\sigma x^{-\theta}\right)^{-\beta}, x>0, \sigma>0, \theta>0, \text{ and } \beta>0. \tag{1.2}$$

where σ is scale parameter, while θ and β are shape parameters. Its very important note that, if $\sigma=1$, the pdf of Dagum distribution become a Burr III distribution and if $\theta=1$, the pdf of Dagum distribution become a Log-Logistic or Fisk distribution.

In this paper, we introduce a new probability distribution called Type II Topp-Leone Dagum distribution using Type II Topp-Leone generated family studied by M. Elgarhy et al.,[9]. The main advantage of this family has one additional shape parameter which given more flexibility. For this proposed new distribution, we obtain important mathematical and statistical properties.

The content of the research paper is organized as follows: Section 1 presents Introduction, about proposed distribution in Section 2. In Section 3 we present various Reliability measures. Section 4 discusses the various mathematical and statistical properties like moment, moment generating function, characteristic function, cumulant generating function, inverted moment, central moment and probability weighted moment. In Section 5 we study order statistics, generalized entropy, Lorenz curve, Bonferroni index, Zenga index. Section 6 provides maximum likelihood estimator. The real data set present in Section 7. Finally, conclusion is given in Section 8.

2. Proposed Distribution

2.1 Type II Topp-Leone Generated Family

Type II Topp-Leone Generated family is proposed by M. Elgarhy et al., [9]. The probability density function and cumulative distribution function are respectively given by

$$f(x) = 2\tau g(x)G(x)[1-G^2(x)]^{\tau-1}, \tau>0. \tag{2.1}$$

and

$$F(x) = 1-[1-G^2(x)]^{\tau}, \tau>0. \tag{2.2}$$

where, τ is a shape parameter. G(x) and g(x) are cumulative distribution function and probability density function of the base distribution.

2.2 Type II Topp-Leone Dagum Distribution

A random variable X is said to have Type II Topp-Leone Dagum distribution with parameters τ, σ, θ and β. If its probability density function is given by

$$f(x;\tau,\sigma,\theta,\beta) = 2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1} \times [1-(1+\sigma x^{-\theta})^{-2\beta}]^{\tau-1} \tag{2.3}$$

where, $x>0, \tau>0, \sigma>0, \theta>0$ and $\beta>0$.
Consider binomial series,

$$(x-y)^r = \sum_{j=0}^{\infty}\binom{r}{j}(-1)^j x^{r-j}y^j$$

Hence,

$$\left[1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right]^{\tau-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j} \times \left(1+\sigma x^{-\theta}\right)^{-2\beta j} \tag{2.4}$$

Using (2.3) in (2.4), we have the probability density function of X is

$$f(x;\tau,\sigma,\theta,\beta) = \sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta\beta x^{-\theta-1} \times (1+\sigma x^{-\theta})^{-2\beta(j+1)-1} \tag{2.5}$$

and the cumulative distribution function (cdf) is

$$F(x;\tau,\sigma,\theta,\beta) = 1-\left[1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right]^{\tau} \tag{2.6}$$

with, $x>0, \tau>0, \sigma>0, \theta>0$ and $\beta>0$.
where, σ is a scale parameter, θ and β are shape parameter, τ is parameter of Type II Topp-Leone Generated family. Type II Topp-Leone Dagum distribution is symbolically denoted by TIITLD $(\tau,\sigma,\theta,\beta)$. Figures 1 to 8, depict the shape of Pdf, Cdf for various values of the parameters of Type II Topp-Leone Dagum distribution.

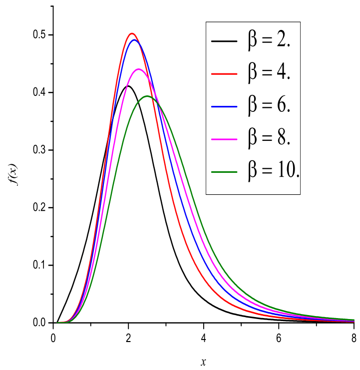

Figure 1. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 1$, $\sigma = 2$, $\theta = 4$ and different values of β.

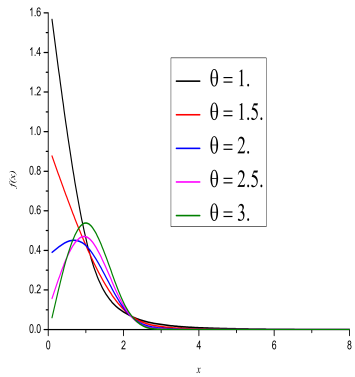

Figure 2. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 4$, $\sigma = 2$, $\beta = 0.5$ and different values of θ.

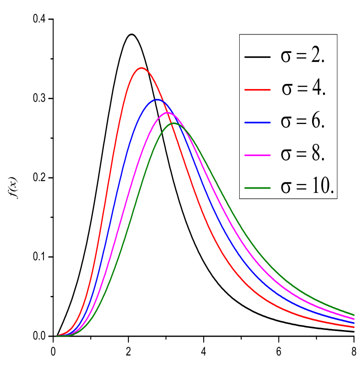

Figure 3. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 1$, $\theta = 3$, $\beta = 2$ and different values of σ.

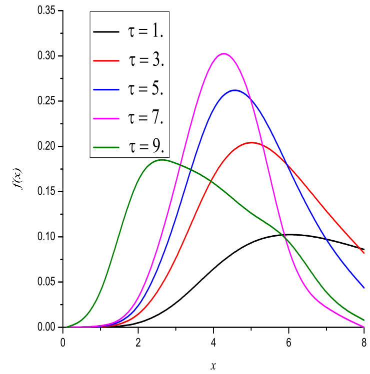

Figure 4. Pdfs of Type II Topp-Leone Dagum distribution for fixed value of $\sigma = 7$, $\theta = 2$, $\beta = 4$ and different values of τ.

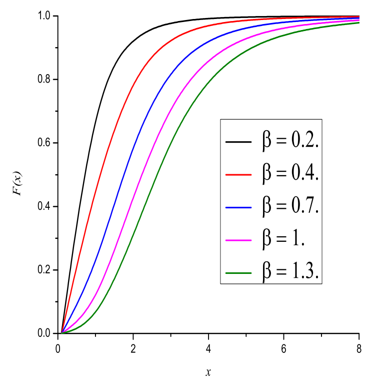

Figure 5. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 2$, $\sigma = 4$, $\theta = 2$ and different values of β.

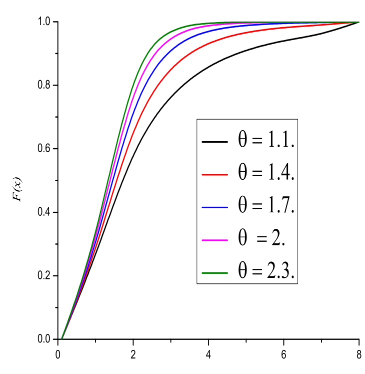

Figure 6. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3$, $\sigma = 2$, $\beta = 1$ and different values of θ.

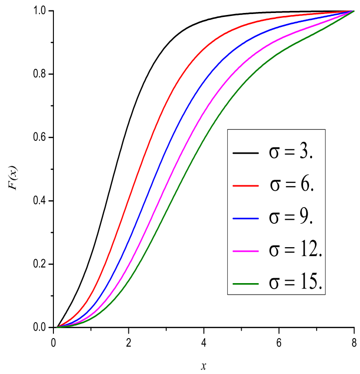

Figure 7. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3$, $\theta = 2$, $\beta = 1$ and different values of σ.

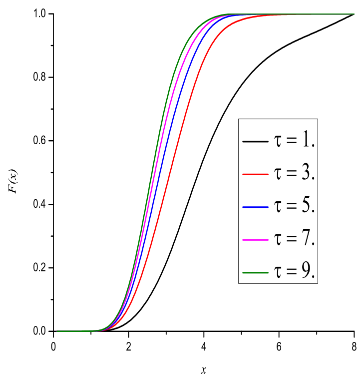

Figure 8. Cdfs of Type II Topp-Leone Dagum distribution for fixed value of $\sigma = 6$, $\theta = 4$, $\beta = 12$ and different values of τ.

3. Reliability Measures

If X $\sim$ TIITLDD $(\tau, \sigma, \theta, \beta)$ then the different reliability measures of random variable are given by

3.1 Reliability function

$$R(x) = 1 - \left[1 - \left(1 - \left(1 + \sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right] \tag{3.1}$$

3.2 Hazard rate function

$$h(x) = \frac{2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1}}{\left[1-\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right]} \times [1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}]^{\tau-1} \tag{3.2}$$

3.3 Reversed hazard function

$$r(x) = \frac{2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1}}{1-\left[1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right]^{\tau}} \times \left[1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right]^{\tau-1} \tag{3.3}$$

3.4 Cumulative hazard function

$$H(x) = -log\left[1-\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right] \tag{3.4}$$

3.5 Second failure rate function

$$h(x) = log\left[\frac{1-\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)}{\left(1-\left(1-\left(1-\left(1+\sigma(x+1)^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right)}\right] \tag{3.5}$$

3.6 Mean waiting time

Mean waiting time is defined by

$$\varphi(x) = x - \left[\frac{1}{F(x)}\int_0^x xf(x)dx\right]$$

$$\varphi(x) = x - \left[\frac{1}{1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}} \times \int_0^x \left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta\beta x^{-\theta-1} \times \left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx\right] \tag{3.6}$$

The mean waiting time of Type II Topp-Leone Dagum distribution is given by

$$\varphi(x) = x - \left[\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}}{1-\left[1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right]^{\tau}}\right] \times B(1-\frac{1}{\theta}, 2\beta(j+1)+\frac{1}{\theta}; y) \tag{3.7}$$

3.7 Mean residual life

Mean residual life is defined by

$$\phi(x) = \frac{1}{s(x)}\int_x^{\infty} xf(x)dx - x$$

$$\phi(x) = \frac{1}{1-\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]} \times \int_0^{\infty} x\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta \times \beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx - x \tag{3.8}$$

The mean residual life function of Type II Topp-Leone Dagum distribution is given by

$$\phi(x) = \frac{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},2\beta(j+1)+1\frac{1}{\theta}\right)}{1-\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]} - x \tag{3.9}$$

Figures (9) to (16) deficts the shape of the reliability function and hazard rate of the Type II Topp-Leone Dagum distribution.

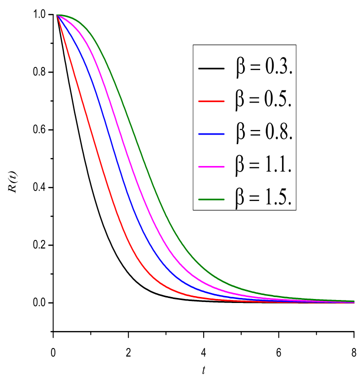

Figure 9. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 2.2$, $\sigma = 3.8$, $\theta = 2.3$ and different values of β.

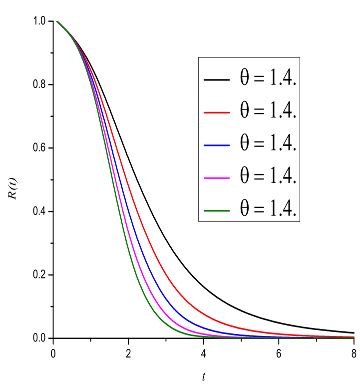

Figure 10. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3.3$, $\sigma = 2.4$, $\beta = 1.4$ and different values of θ.

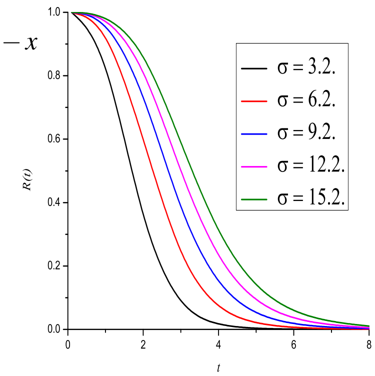

Figure 11. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3.3$, $\theta = 2.3$, $\beta = 1.2$ and different values of σ.

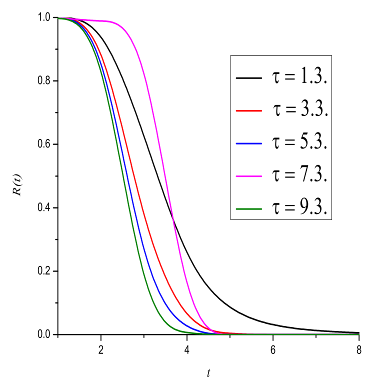

Figure 12. Reliability function of Type II Topp-Leone Dagum distribution for fixed value of $\sigma = 5.8$, $\theta = 4.3$, $\beta = 10$ and different values of τ.

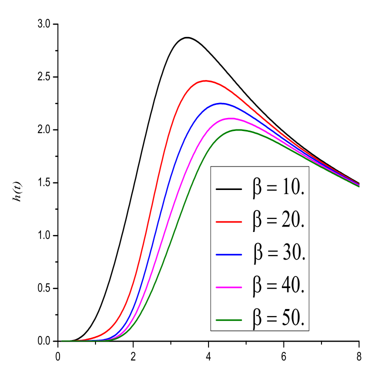

Figure 13. Hazard rate function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 3$, $\sigma = 2$, $\theta = 4$ and different values of β.

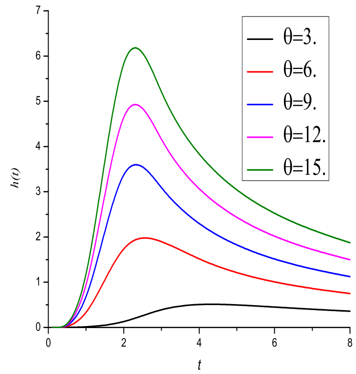

Figure 14. Hazard rate function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 1$, $\sigma = 4$, $\beta = 5$ and different values of θ.

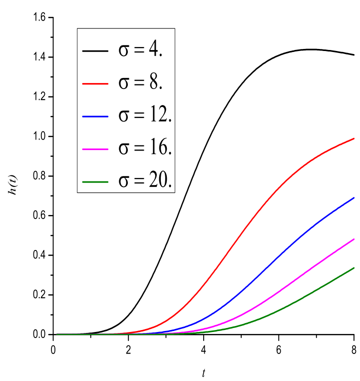

Figure 15. Hazard rate function of Type II Topp-Leone Dagum distribution for fixed value of $\tau = 8$, $\theta = 2$, $\beta = 5$ and different values of σ.

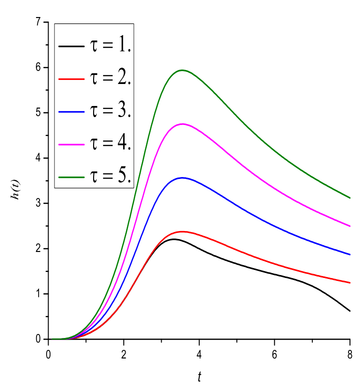

Figure 16. Hazard rate function of Type II Topp-Leone Dagum distribution for fixed value of $\sigma = 20$, $\theta = 5$, $\beta = 3$ and different values of τ.

4. Statistical Properties

4.1 Moments

The moment about the origin is defined by

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx, \text{when X is continuous.}$$

$$\mu_r' = \int_0^{\infty} x^r \Big[\sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\sigma\theta\beta x^{-\theta-1} \left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\Big] dx \tag{4.1}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{r}{\theta}} \int_0^{\infty} \frac{u^{-\frac{r}{\theta}}}{(1+u)^{2\beta(j+1)+1}} du$$

Hence, the r^{th} moment of Type II Topp-Leone Dagum distribution is given by

$$\mu_r' = \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{r}{\theta}} B\Big(1-\frac{r}{\theta}, 2\beta(j+1)+\frac{r}{\theta}\Big) \tag{4.2}$$

In particular,

$$E(X) = \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{1}{\theta}} \times B\Big(1-\frac{1}{\theta}, 2\beta(j+1)+\frac{1}{\theta}\Big)$$

$$E(X^2) = \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{2}{\theta}} \times B\Big(1-\frac{2}{\theta}, 2\beta(j+1)+\frac{2}{\theta}\Big)$$

$$E(X^3) = \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{3}{\theta}} \times B\Big(1-\frac{3}{\theta}, 2\beta(j+1)+\frac{3}{\theta}\Big)$$

$$E(X^4) = \sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{4}{\theta}} \times B\Big(1-\frac{4}{\theta}, 2\beta(j+1)+\frac{4}{\theta}\Big)$$

Variance,

$$V(X) = \Big[\sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{2}{\theta}} \times B\Big(1-\frac{2}{\theta}, 2\beta(j+1)+\frac{2}{\theta}\Big)\Big] - \Big[\sum_{j=0}^{\infty} (-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{1}{\theta}} \times B\Big(1-\frac{1}{\theta}, 2\beta(j+1)+\frac{1}{\theta}\Big)\Big]^2 \tag{4.3}$$

4.2 Moment generating function

The moment generating function of the random variable X is defined by

$$M_X(t)=\int_{-\infty}^{\infty} e^{tx} f(x)dx. \text{ where, } e^{tx}=\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$$

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx$$

The moment generating function of Type II Topp-Leone Dagum distribution is given by

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r}{\theta}} \times B\left(1-\frac{r}{\theta},2\beta(j+1)+\frac{r}{\theta}\right)\right] \tag{4.4}$$

4.3 Characteristic function

The characteristic function of the random variable X is defined by

$$\Phi_X(t)=\int_{-\infty}^{\infty} e^{itx} f(x)dx. \text{ where, } e^{itx}=\sum_{r=0}^{\infty}\frac{(itx)^r}{r!}; i^2=-1$$

$$\Phi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty} x^r\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta \times \beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx \tag{4.5}$$

The characteristic function of Type II Topp-Leone Dagum distribution is given by

$$\Phi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r}{\theta}} \times B\left(1-\frac{r}{\theta},2\beta(j+1)+\frac{r}{\theta}\right)\right] \tag{4.6}$$

4.4 Cumulant generating function

Cumulant generating function is defined by

$$K_X(t)=log\, M_x(t)$$

The cumulant generating function of Type II Topp-Leone Dagum distribution is given by

$$K_X(t)=log\left[\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r}{\theta}} \times B\left(1-\frac{r}{\theta},2\beta(j+1)+\frac{r}{\theta}\right)\right]\right] \tag{4.7}$$

4.5 Inverted moments

The r^{th} inverted moment is defined by

$$\mu_r^*=\int_{-\infty}^{\infty} x^{-r} f(x)dx$$

$$\mu_r^*=\int_0^{\infty} x^{-r}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta \times \beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx \tag{4.8}$$

The r^{th} inverted moment of Type II Topp-Leone Dagum distribution is given by

$$\mu_r^*=\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{-r}{\theta}} \times B\left(1+\frac{r}{\theta},2\beta(j+1)-\frac{r}{\theta}\right)\right] \tag{4.9}$$

The r^{th} inverted moment measure is useful to find the harmonic mean. The harmonic mean of Type II Topp-Leone Dagum distribution is given by

$$\frac{1}{\mu_r^*}=\frac{1}{\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{-r}{\theta}}\right]} \times \frac{1}{\left[B\left(1+\frac{r}{\theta},2\beta(j+1)-\frac{r}{\theta}\right)\right]} \tag{4.10}$$

4.6 Central moments

The r^{th} central moment is defined by

$$\mu_r=\int_{-\infty}^{\infty}(x-\mu_1')^r f(x)dx = \sum_{m=0}^{r}\binom{r}{m}(-1)^m(\mu_1')^m(\mu_r'-m)$$

The r^{th} central moment of Type II Topp-Leone Dagum distribution is given by

$$\mu_r=\sum_{m=0}^{r}\binom{r}{m}(-1)^m\times\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}} \times B\left(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta}\right)\right]^m \times\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r-m}{\theta}} \times B\left(1-\frac{r-m}{\theta},2\beta(j+1)+\frac{r-m}{\theta}\right)\right] \tag{4.11}$$

4.7 Probability weighted moments

The probability weighted moment of the random variable X is defined by

$$\tau_{r,h}=E\left[X^rF(x)^h\right]=\int_{-\infty}^{\infty} x^r f(x)F(x)^h dx$$

$$\tau_{r,h}=\int_0^{\infty} x^r\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta \times \beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right] \times\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]^h dx \tag{4.12}$$

Here we use binomial series expansion

$$(x-y)^r=\sum_{j=0}^{\infty}\binom{r}{j}(-1)^j x^{r-j}y^j$$

It is to be noted that,

$$\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]^{h} = \sum_{l,m=0}^{\infty}\binom{h}{l}(-1)^{l+m}\binom{\tau l}{m}\left(1+\sigma x^{-\theta}\right)^{-2\beta m}$$

Hence,

$$\begin{aligned}\tau_{r,h} =& \sum_{j,l,m=0}^{\infty}(-1)^{j+l+m}\binom{\tau-l}{j}\binom{h}{l}\binom{\tau l}{m}2\tau\sigma\theta\beta \\ &\times\int_0^{\infty}x^{r-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1-2\beta m}dx \\ =& \sum_{j,l,m=0}^{\infty}(-1)^{j+l+m}\binom{\tau-l}{j}\binom{h}{l}\binom{\tau l}{m}2\tau\beta\sigma^{\frac{r}{\theta}} \\ &\times\int_0^{\infty}\frac{u^{-\frac{r}{\theta}}}{(1+u)^{2\beta(j+ml+1)+1}}du \qquad (4.13)\end{aligned}$$

Thus, the probability weighted moment of Type II Topp-Leone Dagum distribution is given by

$$\begin{aligned}\tau_{r,h} =& \sum_{j,l,m=0}^{\infty}(-1)^{j+l+m}\binom{\tau-l}{j}\binom{h}{l}\binom{\tau l}{m}2\tau\beta\sigma^{\frac{r}{\theta}} \\ &\times B\left(1-\frac{r}{\theta},2\beta(j+m+l)+\frac{r}{\theta}\right) \qquad (4.14)\end{aligned}$$

5. Order Statistics

Let Y_i be the i^{th} order statistics of the random sample $X_1, X_2, \ldots, X_n$. The pdf of the j^{th} order statistics for Type II Topp-Leone Dagum distribution is given by

$$\begin{aligned}f_X(x) =& \frac{n!}{(j-1)(n-j)!}\Big[2\tau\sigma\theta\beta x^{-\theta-1} \\ &\times\left(1+\sigma x^{-\theta}\right)^{-2\beta-1}\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau-1}\Big] \\ &\times\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-\beta}\right)^{\tau}\right]^{j-1} \\ &\times\left[1-\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right]^{n-1} \qquad (5.1)\end{aligned}$$

The pdf of the smallest order statistics X_1 is given by

$$\begin{aligned}f_{(1:n)}x =& n\left[1-\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right]^{n-1} \\ &\times\Big[2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1} \\ &\times\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau-1}\Big] \qquad (5.2)\end{aligned}$$

The pdf of the largest order statistics X_n is given by

$$\begin{aligned}f_{(n:n)}(x) =& n\left[\left(1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right]^{n-1} \\ &\times\Big[2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1} \\ &\times\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau-1}\Big] \qquad (5.3)\end{aligned}$$

and the pdf of the median order statistics is given by

$$\begin{aligned}f_{m+1:n}(x) =& \frac{(2m+1)}{m!m!}\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]^{m} \\ &\times\left[\left(1-\left(1-\left(\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right)\right)\right)^{\alpha}\right]^{m} \\ &\times\Big[2\tau\sigma\theta\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta-1} \\ &\times\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau-1}\Big] \qquad (5.4)\end{aligned}$$

The joint distribution of the i^{th} and j^{th} order statistics for $1 \le i < j \le n$ is given by

$$\begin{aligned}f_{1;j;n}(x_i,x_j) =& C\left[F(x_i)\right]^{i-1}\times\left[F(x_j)-F(x_i)\right]^{j-i-1} \\ &\times\left[1-F(x_j)\right]^{n-j}f(x_i)f(x_j)\end{aligned}$$

where,

$$\begin{aligned}C =& \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \\ f_{1;j;n}&(x_i,x_j) \\ =& \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \\ &\times\left[1-\left(1-(q_i^{-2\beta})\right)^{\tau}\right]^{i-1} \\ &\times\left[1-\left(1-(q_j^{-2\beta})\right)^{\tau}-1-\left(1-(q_i^{-2\beta})\right)^{\tau}\right]^{j-i-1} \\ &\times\left[1-\left(1-\left(1-(q_i^{-2\beta})\right)^{\tau}\right)\right]^{n-j} \\ &\times\left[2\tau\sigma\theta\beta x_i^{-\theta-1}q_i^{-2\beta-1}\left(1-(q_i^{-2\beta})\right)^{\tau-1}\right] \\ &\times\left[2\tau\sigma\theta\beta x_i^{-\theta-1}q_j^{-2\beta-1}\left(1-(q_j^{-2\beta})\right)^{\tau-1}\right] \qquad (5.5)\end{aligned}$$

where,

$$q_i = \left(1+\sigma x_i^{-\theta}\right), q_j = \left(1+\sigma x_j^{-\theta}\right).$$

Special case if $i = 1$ and $j = n$. We get the joint distribution of minimum and maximum of order statistics

$$\begin{aligned}f_{1;n;n}&(x_1,x_n) \\ =& n(n-1)\left[F(x_{(n)})-F(x_{(1)})\right]^{n-2}f(x_1)f(x_n) \\ f_{1;n;n}&(x_1,x_n) \\ =& n(n-1) \\ &\times\left[\left(1-\left(1-(q_1^{-2\beta})\right)^{\tau}\right)-\left(1-\left(1-(q_1^{-2\beta})\right)^{\tau}\right)\right]^{n-2} \\ &\times\left[2\tau\sigma\theta\beta x_1^{-\theta-1}q_1^{-2\beta-1}\left(1-(q_1^{-\beta})\right)^{\tau-1}\right] \\ &\times\left[2\tau\sigma\theta\beta x_n^{-\theta-1}q_n^{-2\beta-1}\left(1-(q_n^{-\beta})\right)^{\tau-1}\right] \qquad (5.6)\end{aligned}$$

where,

$$q_1 = \left(1+\sigma x_1^{-\theta}\right), q_n = \left(1+\sigma x_n^{-\theta}\right).$$

5.1 Generalized entropy

The generalized entropy proposed by Cowell [2] and it is defined by

$$GE(w,\delta)=\frac{1}{\delta(\delta-1)\mu^{\delta}}\left[\int_0^{\infty}x^{\delta}f(x)dx\right]-1$$

where, $\delta>0$.

$$GE(w,\delta)=\frac{C}{D}-1 \tag{5.7}$$

where,

$$C=\int_0^{\infty}x^{\delta}\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta\beta x^{-\theta-1}\times\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx$$

$$D=\delta(\delta-1)\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}\times B\big(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta}\big)\right]^{\delta}$$

Note that,

$$\int_0^{\infty}x^{\delta}f(x)dx=\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}\times B\big(1-\frac{\delta}{\theta},2\beta(j+1)+\frac{\delta}{\theta}\big)$$

The Generalized entropy of Type II Topp-Leone Dagum distribution is given by

$$GE(w,\delta)=\frac{A}{B}-1 \tag{5.8}$$

where

$$A=\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{\delta}{\theta}}\times B\left(\left(1-\frac{\delta}{\theta}\right),2\beta(j+1)+\frac{\delta}{\theta}\right)$$

$$B=\delta(\delta-1)\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{\delta}{\theta}}\times B\left(\left(1-\frac{\delta}{\theta}\right),2\beta(j+1)+\frac{\delta}{\theta}\right)\right]$$

5.2 Lorenz curve

Lorenz [15] curve is defined by

$$L(x)=\frac{1}{\mu}\int_0^{x}xf(x)dx$$

$$L(x)=\frac{1}{\mu}\int_0^{x}x\left[\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\sigma\theta\times\beta x^{-\theta-1}\left(1+\sigma x^{-\theta}\right)^{-2\beta(j+1)-1}\right]dx$$

The Lorenz curve of Type II Topp-Leone Dagum distribution is given by

$$L(x)=\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}B\big(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta};y\big)}{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r}{\theta}}B\big(1-\frac{r}{\theta},2\beta(j+1)+\frac{r}{\theta}\big)} \tag{5.9}$$

5.3 Bonferroni index

The Bonferroni index is introduced by Bonferroni [1]. The Bonferroni index is defined by

$$B(x)=\frac{L(x)}{F(x)}$$

The Bonferroni index of Type II Topp-Leone Dagum distribution is given by

$$B(x)=\frac{E}{F} \tag{5.10}$$

where,

$$E=\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}\times B\big(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta};y\big)$$

$$F=\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{r}{\theta}}B\big(1-\frac{r}{\theta},2\beta(j+1)+\frac{r}{\theta}\big)\times\left[1-\left(1-(1+\sigma x^{-\theta})^{-2\beta}\right)^{\tau}\right]$$

5.4 Zenga Index

Zenga [20] index is defined by

$$Z=1-\frac{\bar{\mu}_x}{\mu_x^{+}}$$

where,

$$\bar{\mu}_x=\frac{1}{F(x)}\int_0^{x}xf(x)dx$$

$$\mu_x^{+}=\frac{1}{1-F(x)}\int_0^{\infty}xf(x)dx$$

$$\bar{\mu}_x=\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta};y\right)}{\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]}$$

and

$$\mu_x^{+}=\frac{\sum_{j=0}^{\infty}(-1)^j\binom{\tau-1}{j}2\tau\beta\sigma^{\frac{1}{\theta}}B\left(1-\frac{1}{\theta},2\beta(j+1)+\frac{1}{\theta}\right)}{\left[1-\left(1-\left(1+\sigma x^{-\theta}\right)^{-2\beta}\right)^{\tau}\right]}$$

The Zenga index of Type II Topp-Leone Dagum distribution is given by

$$Z = 1 - \frac{G}{H} \tag{5.11}$$

where

$$G = \sum_{j=0}^{\infty}(-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{1}{\theta}} B\big(1-\frac{1}{\theta}, 2\beta(j+1)+\frac{1}{\theta}; y\big) \times \big[1-\big(1-\big(1-(1+\sigma x^{-\theta})^{-2\beta}\big)^{\tau}\big)\big]$$

$$H = \sum_{j=0}^{\infty}(-1)^j \binom{\tau-1}{j} 2\tau\beta\sigma^{\frac{1}{\theta}} B\big(1-\frac{1}{\theta}, 2\beta(j+1)+\frac{1}{\theta}\big) \times \big[\big(1-\big(1-(1+\sigma x^{-\theta})^{-2\beta}\big)^{\tau}\big)\big]$$

6. Parameter estimation

The principle of maximum likelihood essentially assumes that the sample is representative of the population and chooses as the estimator that value of the parameter which is maximizes the probability density function. If $X_1, X_2, ..., X_n$ are iid random variables with probability density function $f_\theta(x_i)$ and the likelihood function is $L(\theta/x_i)$.

$$L\big(\theta; x_1, x_2, ..., x_n\big) = \prod_{i=1}^{n} f_\theta(x_i)$$

Hence, the likelihood function of Type II Topp-Leone Dagum distribution is given by

$$L(\theta) = \prod_{i=1}^{n}\Big[2\tau\sigma\theta\beta x_i^{-\theta-1}\big(1+\sigma x_i^{-\theta}\big)^{-2\beta-1} \times \big(1-\big(1+\sigma x_i^{-\theta}\big)^{-2\beta}\big)^{\tau-1}\Big] \tag{6.1}$$

and the log likelihood is

$$\begin{aligned} logL(\theta) =& nlog2 + nlog\tau + nlog\sigma + nlog\theta + nlog\beta \\ & m + (-\theta-1)\sum_{i=1}^{n} logx_i + (-2\beta-1) \\ & \times \sum_{i=1}^{n} log\big(1+\sigma x_i^{-\theta}\big) + (\tau-1)\sum_{i=1}^{n} \\ & \times log\big(1-\big(1+\sigma x_i^{-\theta}\big)^{-2\beta}\big) \end{aligned}$$

The partial derivatives first order of log L(θ) with respect to parameter τ, σ, θ and β is given by

$$\frac{\partial logL}{\partial \tau} = 0, \frac{\partial logL}{\partial \sigma} = 0, \frac{\partial logL}{\partial \theta} = 0 \text{ and } \frac{\partial logL}{\partial \beta} = 0.$$

$$\frac{\partial logL}{\partial \tau} = \frac{n}{\tau} + \sum_{i=1}^{n}\frac{(\tau-1)}{\big(1-\big(1+\sigma x_i^{-\theta}\big)^{-2\beta}\big)} = 0 \tag{6.2}$$

$$\begin{aligned} \frac{\partial logL}{\partial \sigma} =& \frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(-2\beta-1)x_i}{\big(1+\sigma x_i^{-\theta}\big)} \\ & + \sum_{i=1}^{n}\frac{(\tau-1)2\beta\big(1+\sigma x_i^{-\theta}\big)^{-2\beta-1}x_i^{-\theta}}{\big(1-\big(1+\sigma x_i^{-\theta}\big)^{-2\beta}\big)} = 0 \end{aligned} \tag{6.3}$$

$$\frac{\partial logL}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} logx_i + \sum_{i=1}^{n}\frac{(-\beta-1)\sigma x_i^{-\theta} log(-\theta)}{\big(1+\sigma x_i^{-\theta}\big)} = 0 \tag{6.4}$$

and

$$\begin{aligned} \frac{\partial logL}{\partial \beta} =& \frac{n}{\beta} - 2\sum_{i=1}^{n} log\big(1+\sigma x_i^{-\theta}\big) \\ & - \sum_{i=1}^{n}\frac{(\tau-1)\big(1+\sigma x_i^{-\theta}\big)^{-2\beta} log(-2\beta)}{\big(1-\big(1+\sigma x_i^{-\theta}\big)^{-2\beta}\big)} = 0 \end{aligned} \tag{6.5}$$

The above mentioned four nonlinear equations can not solve analytically. We can solve these four nonlinear equations through numerical methods like Quasi–Newton method. However, we can also solve the above-mentioned nonlinear equations using the *R* software.

7. Application

We consider the data set given in Proschan [16]. The data set is about the duration of time between successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes. This data set studied by different authors including Haung and Oluyede [12] and Elbatal and Aryal [8]. The descriptive statistics is provided in Table 7.1 for Type II Topp-Leone Dagum distribution. We estimated the parameters for type II Topp-Leone Dagum distribution using method of estimation of maximum likelihood function. The estimated parameters values are given in table 7.2.

Table 7.1

n	**Mean**	**Median**	**Min**	**Max**	Q_1	Q_3
188	92.07	54.00	1.00	603	20.75	603.00

Table 7.2

Model	σ	θ	β	λ	τ	ϕ
TIITLDD	0100.998	1.928	0.246	-	4.592	-
KD	0005.035	4.385	0.376	-	-	21.704
TDD	1574.602	1.659	0.674	0.167	-	-
DD	0094.153	1.263	1.239	-	-	-

Table 7.3

Model	-2LL	AIC	AICC	BIC
T2TLDD	**2066.5**	**2074.5**	**2074.7**	**2087.5**
KD	2066.9	2074.9	2075.2	2087.5
TDD	2074.1	2082.09	2082.31	2095.04
DD	2078.4	2084.40	2084.53	2094.11

The statistical model selection is based on using the statistical measures like, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC) and Bayesian information criterion (BIC) and it is provided in table 7.3 for proposed a new distribution. The data is fitted for four probability distributions Dagum distribution (DD), Transmuted Dagum distribution (TDD), Kumaraswamy distribution (KD), and Type II Topp-Leone Dagum distribution (TIITLDD) we fitted for the data . Note that smaller values of statistical measures like AIC, CAIC, and BIC better fit of data set. Hence, The Type II Topp- Leone Dagum distribution provided the better fit compared to other three models.

8. Conclusion

In this research paper, we introduced the Type II Topp-Leone Dagum distribution. This proposed distribution is generated from Type II Topp-Leone generating family distribution studied by M. Elghargy et al., [9]. For the proposed distribution, we have study graphical behavior of probability density function, cumulative distribution function, hazard rate function and reliability function for the different parameters values. We have obtained some mathematical and statistical properties like moment, moment generating function, characteristic function, cumulant generating function, inverted moment, central moment, probability weighted moment, order statistics, Generalized entropy, Lorenz curve, Bonferroni index, Zenga index. And we obtained expression for reliability function, hazard rate function, reversed hazard rate function, cumulant hazard function, second failure rate function, mean waiting time and mean residual life function. The proposed distribution parameters are estimated by method of maximum likelihood. Finally, a real data set used to fit the proposed distribution, and it provided greater flexibility compared to other suitable probability models.

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