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A bit on the zeros of $D_\alpha f(z)$ of a polynomial $f(z)$

K. Praveen Kumar[1]* and B. Krishna Reddy[2]

Abstract

In this paper, we prove a variant of enestrom and kakeya theorem. Indeed, for a given a polynomial $f(z)$ with real coefficients, we are providing a bounded region such that any zero of $D_\alpha f(z)$ lie in this region must be a simple zero if coefficients of $D'_\alpha f(z)$ are monotonic, and any zero of $D_\alpha f(z)$ which does not lie in this region must be a simple zero if coefficients of $D'_\alpha f(z)$ are alternative.

Keywords

Polynomial, polar derivative, simple zeros, Eneström-Kakeya theorem.

AMS Subject Classification

30C10, 30C15.

[1] *Department of Mathematics, Government Polytechnic, Vikarabad, DTE, Telangana-501102, India.*

[2] *Department of Mathematics, UCS, OU, HYD, Telangana, India 500007.*

*__Corresponding author__: [1] k.praveen1729@gmail.com; [2]bkrbkr_07@yahoo.com

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Contents

1. Introduction

Given a polynomial $f(z)$ of degree n with complex coefficients, we define the polar derivative of $f(z)$ with respect to a number α. We denote it by $D_\alpha f(z)$ and defined as follows:

$$D_\alpha f(z) = nf(z) + (\alpha - z)f'(z).$$

One notes that $D_\alpha f(z)$ is polynomial of degree at most $n-1$. The polar derivative $D_\alpha f(z)$ generalizes ordinary derivative in the limiting case as α tends to infinity. Indeed, one has

$$\lim_{\alpha \longrightarrow \infty} \frac{D_\alpha f(z)}{\alpha} = f'(z).$$

In [5, 11], Enestrom and kakeya studied the distribution of zeros of $f(z)$. Indeed they obtain the following result:

Theorem 1.1. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients such that for some $0<k_0 \leq k_1 \leq \ldots \leq k_{n-2} \leq k_{n-1} \leq k_n$ then all zeros of $f(z)$ lies in $|z| \leq 1$.*

In [1], Aziz and Mohammad studied the multiplicity of zeros of $f(z)$, and they proved the following result:

Theorem 1.2. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the n^{th} degree polynomial with real coefficients such that for some $0<k_0 \leq k_1 \leq \ldots \leq k_n$ then all zeros of $f(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.*

Gulzar, Zargar,Akhter in [9] are extended the above results to the polar derivatives, in [2–4, 6–8, 10] there exist some generalizations and extentions of Enestrom Kakeya theorems, in this article also $f(z)$ is a polynomial of degree n with real($\mathbb{R}$) coefficients and b_t denotes the coefficient of differentiation of polar derivative $(t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2, 3, 4, .., n$.

2. Main Results

Theorem 2.1. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$b_n \geq b_{n-1} \geq \ldots \geq b_4 \geq b_3 \geq b_2.$$

Then all zeros of $D_\alpha f(z)$, which lie in

$$|z| \leq \frac{|b_2|}{b_n + |b_n| - b_2}$$

are simple.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2, 3, \ldots, n$

Corollary 2.1. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$b_n \geq b_{n-1} \geq \ldots \geq b_4 \geq b_3 \geq b_2 > 0.$$

Then all zeros of $D_\alpha f(z)$, which lie in

$$|z| \leq \frac{|b_2|}{2b_n - b_2}$$

are simple.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Theorem 2.2. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$b_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2.$$

Then all zeros of $D_\alpha f(z)$, which lie in

$$|z| \leq \frac{|b_2|}{b_2 - b_n + |b_n|}$$

are simple.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Corollary 2.2. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$0 < b_n \leq b_{n-1} \leq ... \leq b_4 \leq b_3 \leq b_2$$

then all zeros of $D_\alpha f(z)$, which lie in $|z| \leq 1$ are simple. Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Remark 2.1.
1. *Corollary 2.1 follows thereom 2.1 if $b_t > 0$.*
2. *Corollary 2.2 follows thereom 2.2 if $b_t > 0$.*

Theorem 2.3. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the polynomial of degree n and $\alpha \in \mathbb{R}$, $s \geq 1$ and $\eta > 0$ such that for some*

$$sb_n \geq b_{n-1} \leq b_{n-2} \geq ... \geq b_5 \leq b_4 \geq b_3 \leq b_2 + \eta \;\; if\ n\ is\ even$$

OR

$$sb_n \geq b_{n-1} \leq b_{n-2} \geq ... \leq b_5 \geq b_4 \leq b_3 \geq b_2 - \eta \;\; if\ n\ is\ odd$$

then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2} + b_{n-4} + ... + b_6 + b_4) - 2(b_{n-1} + b_{n-3} + ... + b_5 + b_3)\}$$

are simple if n is even

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2} + b_{n-4} + ... + b_5 + b_3) - 2(b_{n-1} + b_{n-3} + ... + b_6 + b_4)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Corollary 2.3. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$b_n \geq b_{n-1} \leq b_{n-2} \geq ... \geq b_5 \leq b_4 \geq b_3 \leq b_2 \;\; if\ n\ is\ even$$

OR

$$b_n \geq b_{n-1} \leq b_{n-2} \geq ... \leq b_5 \geq b_4 \leq b_3 \geq b_2 \;\; if\ n\ is\ odd$$

then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{b_n + b_2 + |b_2| + 2(b_{n-2} + b_{n-4} + ... + b_6 + b_4) - 2(b_{n-1} + b_{n-3} + ... + b_5 + b_3)\}$$

are simple if n is even

OR

$$|z| \leq \frac{1}{|b_n|}\{b_n - b_2 + |b_2| + 2(b_{n-2} + b_{n-4} + ... + b_5 + b_3) - 2(b_{n-1} + b_{n-3} + ... + b_6 + b_4)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Corollary 2.4. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$0 < b_n \geq b_{n-1} \leq b_{n-2} \geq ... \geq b_5 \leq b_4 \geq b_3 \leq b_2 > 0 \;\; if\ n\ is\ even$$

OR

$$0 < b_n \geq b_{n-1} \leq b_{n-2} \geq ... \leq b_5 \geq b_4 \leq b_3 \geq b_2 > 0 \;\; if\ n\ is\ odd$$

then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{b_n + 2(b_{n-2} + b_{n-4} + ... + b_6 + b_4 + b_2) - 2(b_{n-1} + b_{n-3} + ... + b_5 + b_3)\}$$

are simple if n is even

OR

$$|z| \leq \frac{1}{|b_n|}\{b_n + 2(b_{n-2} + b_{n-4} + ... + b_5 + b_3) - 2(b_{n-1} + b_{n-3} + ... + b_6 + b_4)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$

Theorem 2.4. *Let $f(z) = \sum_{j=0}^{n} k_j z^j$ be the polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$rb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq ... \leq b_5 \geq b_4 \leq b_3 \geq b_2 - \eta \;\; if\ n\ is\ even$$

OR

$$rb_n \leq b_{n-1} \geq b_{n-2} \leq b_{n-3} \geq ... \geq b_5 \leq b_4 \geq b_3 \leq b_2 + \eta \;\; if\ n\ is\ odd$$

then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \le \frac{1}{|b_n|}\{2\eta + |b_n| + |b_2| - b_2 - r(b_n + |b_n|) + 2(b_{n-1} + b_{n-3} + \ldots + b_5 + b_3) - 2(b_{n-2} + b_{n-4} + \ldots + b_6 + b_4)\}$$

are simple if n is even

OR

$$|z| \le \frac{1}{|b_n|}\{2\eta + |b_n| + |b_2| + b_2 - r(b_n + |b_n|) + 2(b_{n-1} + b_{n-3} + \ldots + b_6 + b_4) - 2(b_{n-2} + b_{n-4} + \ldots + b_5 + b_3)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,\ldots,n$

Corollary 2.5. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$b_n \le b_{n-1} \ge b_{n-2} \le b_{n-3} \ge \ldots \le b_5 \ge b_4 \le b_3 \ge b_2 \text{ if } n \text{ is even}$$

OR

$$b_n \le b_{n-1} \ge b_{n-2} \le b_{n-3} \ge \ldots \ge b_5 \le b_4 \ge b_3 \le b_2 \text{ if } n \text{ is odd}$$

then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \le \frac{1}{|b_n|}\{|b_2| - b_n - b_2 + 2(b_{n-1} + b_{n-3} + \ldots + b_5 + b_3) - 2(b_{n-2} + b_{n-4} + \ldots + b_6 + b_4)\}$$

are simple if n is even

OR

$$|z| \le \frac{1}{|b_n|}\{b_2 - b_n + |b_2| + 2(b_{n-1} + b_{n-3} + \ldots + b_6 + b_4) - 2(b_{n-2} + b_{n-4} + \ldots + b_5 + b_3)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,\ldots,n$

Corollary 2.6. *If $f(z) = \sum_{j=0}^{n} k_j z^j$ is a polynomial of degree n and $\alpha \in \mathbb{R}$, such that for some*

$$0 < b_n \le b_{n-1} \ge b_{n-2} \le b_{n-3} \ge \ldots \le b_5 \ge b_4 \le b_3 \ge b_2 > 0$$

if n is even

OR

$$0 < b_n \le b_{n-1} \ge b_{n-2} \le b_{n-3} \ge \ldots \ge b_5 \le b_4 \ge b_3 \le b_2 > 0$$

if n is odd.
Then all zeros of $D_\alpha f(z)$, which does not lie in

$$|z| \le \frac{1}{|b_n|}\{-b_n + 2(b_{n-1} + b_{n-3} + \ldots + b_5 + b_3) - 2(b_{n-2} + b_{n-4} + \ldots + b_6 + b_4)\}$$

are simple if n is even

OR

$$|z| \le \frac{1}{|b_n|}\{-b_n + 2(b_{n-1} + b_{n-3} + \ldots + b_6 + b_4 + b_2) - 2(b_{n-2} + b_{n-4} + \ldots + b_5 + b_3)\}$$

are simple if n is odd.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,\ldots,n$

Remark 2.2.

1. *Corollary 2.3 follows Theorem 2.3 if $s = 1$, $\eta = 0$.*
2. *Corollary 2.4 follows Theorem 2.3 if $s = 1$, $\eta = 0$ and $b_j \ge 0$.*
3. *Corollary 2.5 follows Theorem 2.4 if $r = 1$, $\eta = 0$.*
4. *Corollary 2.6 follows Theorem 2.4 if $b_j \ge 0$ and $r = 1$, $\eta = 0$.*

3. Proof of the theorems

Proof of theorem 2.1
Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + \ldots + k_1 z + k_0$, be the polynomial of deree n with real coefficients. Then by the definition of $D_\alpha f(z)$, we have

$$\begin{aligned} D_\alpha f(z) =& nf(z) + \alpha f'(z) - zf'(z) \\ =& n(k_n z^n + k_{n-1}z^{n-1} + \ldots + k_1 z + k_0) \\ &+ \alpha(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + \ldots + k_1) \\ &- z(nk_n z^{n-1} + (n-1)k_{n-1}z^{n-2} + \ldots + k_1) \\ =& [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} \\ &+ [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + \ldots \\ &+ [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0], \end{aligned}$$

Then,

$$D'_\alpha f(z) = b_n z^{n-2} + b_{n-1}z^{n-3} + b_{n-2}z^{n-4} + \ldots + b_4 z^2 + b_3 z + b_2,$$

where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$.
Now consider $g(z) = z^{n-2}D'_\alpha f(\frac{1}{z})$ and $h(z) = (1-z)g(z)$. So that,

$$\begin{aligned} h(z) =& (1-z)[b_2 z^{n-2} + b_3 z^{n-3} + b_4 z^{n-4} + \ldots + b_{n-2}z^2 \\ &+ b_{n-1}z + b_n] \\ h(z) =& -b_2 z^{n-1} + (b_2 - b_3)z^{n-2} + (b_3 - b_4)z^{n-3} + \ldots \\ &+ (b_{n-2} - b_{n-1})z^2 + (b_{n-1} - b_n)z + b_n \\ |h(z)| \ge& |b_2||z|^{n-2}\Bigg[|z| - \frac{1}{|b_2|}\Big\{|b_2 - b_3| + \frac{|b_3 - b_4|}{|z|} \\ &+ \frac{|b_4 - b_5|}{|z|^2} + \ldots + \frac{|b_{n-1} - b_n|}{|z|^{n-3}} + \frac{|b_n|}{|z|^{n-2}}\Big\}\Bigg] \end{aligned}$$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$, therefore

$$\begin{aligned}|h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{|b_2-b_3|+|b_3-b_4| \\ & +\dots+|b_{n-1}-b_n|+|b_n|\}\Big] \\ \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{b_3-b_2+b_4-b_3 \\ & +\dots+b_n-b_{n-1}+|b_n|\}\Big] \\ |h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{-b_2+b_n+|b_n|\}\Big]\end{aligned}$$

Hence $|h(z)|>0$, provided that $|z|>\frac{-b_2+b_n+|b_n|}{|b_2|}$.
Hence all the zeros of $h(z)$ with $|z|>1$ lie in $|z| \leq \frac{-b_2+b_n+|b_n|}{|b_2|}$.
But those zeros of $h(z)$ whose modulus is less than or equal to 1 already satisfy

$$|z| \leq \frac{-b_2+b_n+|b_n|}{|b_2|}.$$

Since all zeros of $h(z)$ are also the zeros of $g(z)$ lie in

$$|z| \leq \frac{-b_2+b_n+|b_n|}{|b_2|},$$

therefore all the zeros of $g(z)$ lies in

$$|z| \leq \frac{-b_2+b_n+|b_n|}{|b_2|}.$$

But $D'_\alpha f(z) = z^{n-2}g(\frac{1}{z})$ it follows, by choosing z with $\frac{1}{z}$, therefore all zeros of $D'_\alpha f(z)$ lies in

$$|z| \geq \frac{|b_2|}{b_n+|b_n|-b_2}.$$

Therefore $D'_\alpha f(z)$ does not vanish in

$$|z| \leq \frac{|b_2|}{b_n+|b_n|-b_2}.$$

In other words all zeros of $D_\alpha P(z)$ which lie in

$$|z| \leq \frac{|b_2|}{b_n+|b_n|-b_2}$$

are simple.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,\dots,n$
Proof of theorem 2.2
Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + \dots + k_1 z + k_0$, be the polynomial of deree n with real coefficients. Then by the definition of $D_\alpha f(z)$, we have

$$\begin{aligned}D_\alpha f(z) =& nf(z)+\alpha f'(z) - zf'(z) \\ =& n(k_n z^n + k_{n-1}z^{n-1}+\dots+k_1 z+k_0) \\ & +\alpha(nk_n z^{n-1}+(n-1)k_{n-1}z^{n-2}+\dots+k_1) \\ & -z(nk_n z^{n-1}+(n-1)k_{n-1}z^{n-2}+\dots+k_1) \\ =& [n\alpha k_n+(n-(n-1))k_{n-1}]z^{n-1} \\ & +[(n-1)\alpha k_{n-1}+(n-(n-2))k_{n-2}]z^{n-2}+\dots \\ & +[2\alpha k_2+(n-1))k_1]z+[\alpha k_1+nk_0],\end{aligned}$$

Then

$$\begin{aligned}D'_\alpha f(z) =& b_n z^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+\dots \\ & +b_4 z^2+b_3 z+b_2\end{aligned}$$

where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\dots,n$
Now consider $g(z) = z^{n-2}D'_\alpha f(\frac{1}{z})$ and $h(z) = (1-z)g(z)$. So that,

$$\begin{aligned}h(z) =& (1-z)[b_2 z^{n-2}+b_3 z^{n-3}+b_4 z^{n-4}+\dots \\ & +b_{n-2}z^2+b_{n-1}z+b_n] \\ h(z) =& -b_2 z^{n-1}+(b_2-b_3)z^{n-2}+(b_3-b_4)z^{n-3}+\dots \\ & +(b_{n-2}-b_{n-1})z^2+(b_{n-1}-b_n)z+b_n\end{aligned}$$

$$\begin{aligned}|h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{|b_2-b_3|+\frac{|b_3-b_4|}{|z|} \\ & +\frac{|b_4-b_5|}{|z|^2}+\dots+\frac{|b_{n-1}-b_n|}{|z|^{n-3}}+\frac{|b_n|}{|z|^{n-2}}\}\Big]\end{aligned}$$

Also, if $|z|>1$ then $\frac{1}{|z|}<1$, therefore

$$\begin{aligned}|h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{|b_2-b_3|+|b_3-b_4|+\dots \\ & +|b_{n-1}-b_n|+|b_n|\}\Big] \\ |h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{b_2-b_3+b_3-b_4+\dots \\ & +b_{n-1}-b_n+|b_n|\}\Big] \\ |h(z)| \geq & |b_2||z|^{n-2}\Big[|z| - \frac{1}{|b_2|}\{b_2-b_n+|b_n|\}\Big]\end{aligned}$$

Hence $|h(z)|>0$, provided that $|z|>\frac{b_2-b_n+|b_n|}{|b_2|}$.
Hence all the zeros of $h(z)$ with $|z|>1$ lie in $|z| \leq \frac{b_2-b_n+|b_n|}{|b_2|}$.
But those zeros of $h(z)$ whose modulus is less than or equal to 1 already satisfy

$$|z| \leq \frac{b_2-b_n+|b_n|}{|b_2|}.$$

Since all zeros of $h(z)$ are also the zeros of $g(z)$ lie in

$$|z| \leq \frac{b_2 - b_n + |b_n|}{|b_2|},$$

therefore all zeros of $g(z)$ lies in

$$|z| \leq \frac{b_2 - b_n + |b_n|}{|b_2|}.$$

But $D'_\alpha f(z) = z^{n-2} g(\frac{1}{z})$ it follows, by choosing z with $\frac{1}{z}$, therefore all zeros of $D'_\alpha f(z)$ lies in

$$|z| \geq \frac{|b_2|}{b_2 - b_n + |b_n|}.$$

Therefore $D'_\alpha f(z)$ does not vanish in

$$|z| \leq \frac{|b_2|}{b_2 - b_n + |b_n|}.$$

In other words all zeros of $D_\alpha P(z)$ which lie in

$$|z| \leq \frac{|b_2|}{b_2 - b_n + |b_n|}$$

are simple.
Where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,...,n$
Proof of theorem 2.3
Let $f(z) = k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0$, be the polynomial

of deree n with real coefficients. Then by the definition of $D_\alpha f(z)$, we have

$$\begin{aligned} D_\alpha f(z) =& nf(z) + \alpha f'(z) - zf'(z) \\ =& n(k_n z^n + k_{n-1} z^{n-1} + ... + k_1 z + k_0) \\ &+ \alpha(nk_n z^{n-1} + (n-1)k_{n-1} z^{n-2} + ... + k_1) \\ &- z(nk_n z^{n-1} + (n-1)k_{n-1} z^{n-2} + ... + k_1) \\ =& [n\alpha k_n + (n-(n-1))k_{n-1}]z^{n-1} \\ &+ [(n-1)\alpha k_{n-1} + (n-(n-2))k_{n-2}]z^{n-2} + ... \\ &+ [2\alpha k_2 + (n-1))k_1]z + [\alpha k_1 + nk_0], \end{aligned}$$

Then

$$\begin{aligned} D'_\alpha f(z) =& b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + ... \\ &+ b_4 z^2 + b_3 z + b_2 \end{aligned}$$

where $b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,...,n$
Now Consider $g(z) = (1-z)D'_\alpha f(z)$, so that

$$\begin{aligned} g(z) =& (1-z)[b_n z^{n-2} + b_{n-1} z^{n-3} + b_{n-2} z^{n-4} + ... \\ &+ b_4 z^2 + b_3 z + b_2] \\ =& -b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} \\ &+ (b_{n-2} - b_{n-3})z^{n-4} + ... + (b_{m+1} - b_m)z^{m-1} \\ &+ (b_m - b_{m-1})z^{m-2} ... + (b_4 - b_3)z^2 + (b_3 - b_2)z + b_2 \end{aligned}$$

Also, now apply both sides mode, we get the following

$$\begin{aligned} |g(z)| =& |-b_n z^{n-1} + (b_n - b_{n-1})z^{n-2} + (b_{n-1} - b_{n-2})z^{n-3} \\ &+ (b_{n-2} - b_{n-3})z^{n-4} + ... + (b_{m+1} - b_m)z^{m-1} \\ &+ (b_m - b_{m-1})z^{m-2} + ... + (b_4 - b_3)z^2 \\ &+ (b_3 - b_2)z + b_2| \\ |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} \\ &+ \frac{|b_{n-2} - b_{n-3}|}{|z|^2} + ... + \frac{|b_3 - b_2|}{|z|^{n-3}} + \frac{|b_2|}{|z|^{n-2}}\Big\}\Big] \end{aligned}$$

If $|z|>1$ then $\frac{1}{|z|}<1$, we have the following

$$\begin{aligned} |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{|b_n - b_{n-1}| + |b_{n-1} - b_{n-2}| \\ &+ ... + |b_4 - b_3| + |b_3 - b_2| + |b_2|\Big\}\Big] \\ \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{|b_n - sb_n + sb_n - b_{n-1}| \\ &+ ... + |b_3 - (b_2 + \eta) - \eta| + |b_2|\Big\}\Big] \\ \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{|b_n - sb_n| + |sb_n - b_{n-1}| \\ &+ ... + |b_3 - (b_2 + \eta)| + |\eta| + |b_2|\Big\}\Big] \end{aligned}$$

$$\begin{aligned} |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{(1-s)|b_n| + |sb_n - b_{n-1}| \\ &+ ... + |b_3 - (b_2 + \eta)| + |\eta| + |b_2|\Big\}\Big] \\ \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{(1-s)|b_n| + sb_n - b_{n-1} + b_{n-2} \\ &- b_{n-1} + ... + b_2 + \eta - b_3 + \eta + |b_2|\Big\}\Big] \end{aligned}$$

$$\begin{aligned} |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{(s-1)|b_n| + sb_n + b_2 + |b_2| \\ &+ 2\eta + 2(b_{n-2} + b_{n-4} + ... + b_6 + b_4) \\ &- 2(b_{n-1} + b_{n-3} + ... + b_5 + b_3)\Big\}\Big] \quad if\ n\ is\ even \end{aligned}$$

OR

$$\begin{aligned} |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{(s-1)|b_n| + sb_n - b_2 + |b_2| \\ &+ 2\eta + 2(b_{n-2} + b_{n-4} + ... + b_5 + b_3) \\ &- 2(b_{n-1} + b_{n-3} + ... + b_6 + b_4)\Big\}\Big] \quad if\ n\ is\ odd, \end{aligned}$$

$$\begin{aligned} |g(z)| \geq& |b_n||z|^{n-2}\Big[|z| - \frac{1}{|b_n|}\Big\{(s-1)|b_n| + sb_n + b_2 + |b_2| \\ &+ 2\eta + 2(b_{n-2} + b_{n-4} + ... + b_6 + b_4) \\ &- 2(b_{n-1} + b_{n-3} + ... + b_5 + b_3)\Big\}\Big] \quad if\ n\ is\ even \end{aligned}$$

OR

$$|g(z)| \geq |b_n||z|^{n-2}\Bigg[|z| - \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\}\Bigg] \quad if\ n\ is\ odd$$

Hence $|g(z)| > 0$ provided

$$|z| > \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\}.$$

OR

$$|z| > \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\}.$$

This shows that all the zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\},$$

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\},$$

it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\},$$

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\},$$

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\},$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\},$$

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\},$$

In other words all the zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n + b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_6+b_4) - 2(b_{n-1}+b_{n-3}+\ldots+b_5+b_3)\},$$

are simple if n is even.

OR

$$|z| \leq \frac{1}{|b_n|}\{(s-1)|b_n| + sb_n - b_2 + |b_2| + 2\eta + 2(b_{n-2}+b_{n-4}+\ldots+b_5+b_3) - 2(b_{n-1}+b_{n-3}+\ldots+b_6+b_4)\},$$

are simple if n is odd, where
$b_t = (t-1)[t\alpha k_t + (n-(t-1))k_{t-1}]$ for $t = 2,3,4,\ldots,n$

Proof of Theorem 2.4

Let $f(z) = k_n z^n + k_{n-1}z^{n-1} + \ldots + k_1 z + k_0$, be the polynomial of deree n with real coefficients. Then by the definition of $D_\alpha f(z)$, we have

$$\begin{aligned}D_\alpha f(z) =& nf(z)+\alpha f^{'}(z)-zf^{'}(z)\\ =& n(k_nz^n+k_{n-1}z^{n-1}+...+k_1z+k_0)\\ &+\alpha(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)\\ &-z(nk_nz^{n-1}+(n-1)k_{n-1}z^{n-2}+...+k_1)\\ =& [n\alpha k_n+(n-(n-1))k_{n-1}]z^{n-1}\\ &+[(n-1)\alpha k_{n-1}+(n-(n-2))k_{n-2}]z^{n-2}+...\\ &+[2\alpha k_2+(n-1))k_1]z+[\alpha k_1+nk_0],\ Then,\\ D^{'}_\alpha f(z) =& b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...\\ &+b_4z^2+b_3z+b_2\end{aligned}$$

where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$
Now Consider $g(z)=(1-z)D^{'}_\alpha f(z)$, so that

$$\begin{aligned}g(z) =& (1-z)[b_nz^{n-2}+b_{n-1}z^{n-3}+b_{n-2}z^{n-4}+...\\ &+b_4z^2+b_3z+b_2]\\ =& -b_nz^{n-1}+(b_n-b_{n-1})z^{n-2}+(b_{n-1}-b_{n-2})z^{n-3}\\ &+(b_{n-2}-b_{n-3})z^{n-4}+...+(b_{m+1}-b_m)z^{m-1}\\ &+(b_m-b_{m-1})z^{m-2}...+(b_4-b_3)z^2\\ &+(b_3-b_2)z+b_2\end{aligned}$$

Also, now apply both sides mode, we get the following

$$\begin{aligned}|g(z)| =& |-b_nz^{n-1}+(b_n-b_{n-1})z^{n-2}+(b_{n-1}-b_{n-2})z^{n-3}\\ &+(b_{n-2}-b_{n-3})z^{n-4}+...+(b_{m+1}-b_m)z^{m-1}\\ &+(b_m-b_{m-1})z^{m-2}+...+(b_4-b_3)z^2\\ &+(b_3-b_2)z+b_2|\\ |g(z)| \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{|b_n-b_{n-1}|+\frac{|b_{n-1}-b_{n-2}|}{|z|}\\ &+\frac{|b_{n-2}-b_{n-3}|}{|z|^2}+...+\frac{|b_3-b_2|}{|z|^{n-3}}+\frac{|b_2|}{|z|^{n-2}}\}\Bigg]\end{aligned}$$

If $|z|>1$ then $\frac{1}{|z|}<1$, we have the following

$$\begin{aligned}|g(z)| \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{|b_n-b_{n-1}|+|b_{n-1}-b_{n-2}|\\ &+...+|b_4-b_3|+|b_3-b_2|+|b_2|\}\Bigg]\\ \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{|b_n-b_{n-1}|+|b_{n-1}-b_{n-2}|\\ &+...+|b_3-b_2|+|b_2|\}\Bigg]\\ \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{|b_n-rb_n+rb_n-b_{n-1}|\\ &+|b_{n-1}-b_{n-2}|+...+|b_4-b_3|\\ &+|b_3-(b_2-\eta)-\eta|+|b_2|\}\Bigg]\\ |g(z)| \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{|b_n-rb_n|+|rb_n-b_{n-1}|\\ &+|b_{n-1}-b_{n-2}|+...+|b_4-b_3|\\ &+|b_3-(b_2-\eta)|+|\eta|+|b_2|\}\Bigg]\\ \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{(1-r)|b_n|+|rb_n-b_{n-1}|\\ &+|b_{n-1}-b_{n-2}|+...+|b_4-b_3|\\ &+|b_3-(b_2-\eta)|+|\eta|+|b_2|\}\Bigg]\end{aligned}$$

$$\begin{aligned}&|g(z)|\\ &\geq |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{(1-r)|b_n|+b_{n-1}-rb_n+b_{n-1}\\ &-b_{n-2}+b_{n-3}-b_{n-2}+...+b_5-b_4\\ &+b_3-b_4+b_3-(b_2-\eta)+\eta+|b_2|\}\Bigg]\quad if\ n\ is\ even\end{aligned}$$

OR

$$\begin{aligned}&|g(z)|\\ &\geq |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{(1-r)|b_n|+b_{n-1}-rb_n+b_{n-1}\\ &-b_{n-2}+b_{n-3}-b_{n-2}+...+b_4-b_5\\ &+b_4-b_3+b_2+\eta-b_3\eta+|b_2|\}\Bigg]\quad if\ n\ is\ odd,\end{aligned}$$

$$\begin{aligned}|g(z)| \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2\\ &-r(b_n+|b_n|)+2(b_{n-1}+b_{n-3}+...+b_5+b_3)\\ &-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}\Bigg]\quad if\ n\ is\ even\end{aligned}$$

OR

$$\begin{aligned}|g(z)| \geq& |b_n||z|^{n-2}\Bigg[|z|-\frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2\\ &-r(b_n+|b_n|)+2(b_{n-1}+b_{n-3}+...+b_6+b_4)\\ &-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}\Bigg]\quad if\ n\ is\ odd,\end{aligned}$$

Hence $|g(z)|>0$ provided

$$\begin{aligned}|z|>& \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)\\ &+2(b_{n-1}+b_{n-3}+...+b_5+b_3)\\ &-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.\end{aligned}$$

OR

$$|z| > \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

This shows that all the zeros of $g(z)$ whose modulus is greater than 1 are lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

Since the zeros of $g(z)$ whose modulus is less than or equal to 1 are already lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

it follows that all the zeros of $g(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

Since all the zeros of $g(z)$ are also the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

Thus all the zeros of $D'_\alpha f(z)$ lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\}.$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}.$$

In other words all the zeros of $D_\alpha f(z)$ which does not lie in

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|-b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_5+b_3)$$
$$-2(b_{n-2}+b_{n-4}+...+b_6+b_4)\} \quad ,$$
$$are\ simple\ if\ n\ is\ even,$$

OR

$$|z| \leq \frac{1}{|b_n|}\{2\eta+|b_n|+|b_2|+b_2-r(b_n+|b_n|)$$
$$+2(b_{n-1}+b_{n-3}+...+b_6+b_4)$$
$$-2(b_{n-2}+b_{n-4}+...+b_5+b_3)\}$$
$$are\ simple,\ if\ n\ is\ odd,$$

where $b_t=(t-1)[t\alpha k_t+(n-(t-1))k_{t-1}]$ for $t=2,3,4,...,n$

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