Fixed point theorems for orthogonal F-Suzuki contraction mappings on O-complete metric space with an applications

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Abstract

In this paper, we introduce the concepts of generalized orthogonal F-contraction and orthogonal F-Suzuki contraction mappings and prove some fixed point theorems for a self mapping in orthogonal metric space. The proved results generalize and extend some of the well known results in the literature. An example to support our result is presented. As applications of the main results, we apply our main results to show the existence of a unique solution of the first-order ordinary differential equation.

Keywords

Orthogonal set, orthogonal metric space, orthogonal continuous, orthogonal preserving, generalized orthogonal F-contraction, orthogonal F-Suzuki contraction, fixed point.

AMS Subject Classification

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1. Introduction

The concept of an orthogonal set has many applications in several branches in mathematics and it has many types of the orthogonality. Gordji, Ramezani, De La Sen and Cho [1] introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with the new orthogonality. Furthermore, they gave the application of this results for claming the existence and uniqueness of solution of the first-ordinary differential equation while the Banach contraction mapping can not be applied in this problem. Eshaghi Gordji and Habibi [2] proved fixed point theory in generalized orthogonal metric space. Sawangsup, Sintunavarat and Cho [3] introduced the new concept an orthogonal F-contraction mappings and proved the fixed point theorems on orthogonal-complete metric space. The orthogonal contractive type mappings has been studied by many authors and important results have been obtained by [4], [5], [6], [7], [8]. Some interesting concepts and application see [11–17]. In this paper, we introduced the new concepts generalized orthogonal F-contraction and orthogonal F-Suzuki contraction mappings and prove the fixed point theorems on orthogonal complete metric space.

2. Preliminaries

Throughout this paper, we denote by W, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ the nonempty set, the set of positive real numbers, the set of positive integers and the set of nonnegative integers, respectively. Firstly, we recall the concept of a control function which is introduced by Wardowski [9]. Let $\mathfrak{I}$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

(F_1) F_1 is strictly increasing;

(F_2) for each sequence $\{\alpha_n\}$ of positive numbers, we have

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Piri and Kumam [10] introduced a large class of functions by replacing the condition (F_3) in the definition of F-contractions with the following

(F_3') F is continuous on $(0,\infty)$.

Let the set $\mathfrak{D} = \{F : \mathbb{R}_+ \to \mathbb{R} \mid F$ satisfies the conditions $(F_1), (F_2)$ and $(F_3')\}$. He also proved the following existence and uniqueness of a fixed point for F-contraction mappings:

Theorem 2.1. *[10] Let (W,d) be a complete metric space and a mapping $Y : W \to W$. Suppose that there exist $F \in \mathfrak{D}$ and $\tau > 0$ such that, for all $u, v \in W$*

$$d(Yu, Yv) > 0 \Rightarrow \tau + F(d(Yu, Yv)) \leq F(d(u,v)).$$

Then Y has a unique fixed point $t \in W$ and for every $u \in W$, the sequence $\{Y^n u\}$ converges to t.

Next, he introduce notion of F-Suzuki contraction and proved existence and uniqueness of a fixed point theorem.

Definition 2.2. *[10] Let (W,d) be a metric space. A mapping $Y : W \to W$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $u, v \in W$ with $Yu \neq Yv$,*

$$\frac{1}{2} d(u, Yu) < d(u,v) \Rightarrow \tau + F(d(Yu, Yv)) \leq F(d(u,v)),$$

where $F \in \mathfrak{D}$.

Theorem 2.3. *[10] Let (W,d) be a complete metric space and $Y : W \to W$ be an F-suzuki contraction. Then Y has a unique fixed point $u^* \in W$ and for every $u_0 \in W$, the sequence $\{Y^n u_0\}_{n=1}^{\infty}$ converges to u^*.*

We give some examples of functions belonging to $\mathfrak{D}$ as folows:

Example 2.4. *Let functions F_1, F_2, F_3, $F_4 : \mathbb{R}^+ \to \mathbb{R}$ be defined by:*

(1) $F_1(\beta) = \frac{-1}{\beta}$ *for $\beta > 0$;*

(2) $F_2(\beta) = \frac{-1}{\beta} + \beta$ *for all $\beta > 0$;*

(3) $F_3(\beta) = \frac{1}{1 - e^\beta}$ *for all $\beta > 0$;*

(4) $F_4(\beta) = \frac{1}{e^\beta - e^{-\beta}}$ *for all $\beta > 0$.*

Then $F_1, F_2, F_3, F_4 \in \mathfrak{D}$.

On the other hand, Gordji, Ramezani, De La Sen and Cho [1] introduced the concept of an orthogonal set (or O-set), some examples and some properties of the orthogonal sets as follows:

Definition 2.5. *[1] Let $W \neq \phi$ and $\perp \subseteq W \times W$ be a binary relation. If $\perp$ satisfies the following condition:*

$$\exists w_0 \in W : (\forall w \in W, w \perp w_0) \quad or \quad (\forall w \in W, w_0 \perp w),$$

then it is called an orthogonal set (briefly O-set). We denote this O-set by $(W, \perp)$.

Example 2.6. *[1] Let W be the set all people in the world. Define the binary relation $\perp$ on W by $u \perp w$ if u can give blood to w. Accoding to the Table 1, if w_0 is a person such that his(her) blood type is O-, then we have $w_0 \perp w$ for all $w \in W$. This means that $(W, \perp)$ is an O-set. In this O-set, w_0 (in Definition 2.5) is not unique. Note that, in this example, w_0 may be a person with blood type AB+. In this case, we have $w \perp w_0$ for all $w \in W$.*
Table 1

Type	You can give blood to	You can receive blood from
A+	A+AB+	A+A-O+O-
O+	O-A+B+AB+	O+O-
B+	B+AB+	B+B-O+O-
AB+	AB+	Everyone
A-	A+A-AB+AB-	A-O-
O-	Everyone	O-
B-	B+B-AB+AB-	B-O-
AB-	AB+AB-	AB-B-O-A-

Example 2.7. *[1] Let $W = \mathbb{Z}$. Define the binary relation $\perp$ on W by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(W, \perp)$ is an O-set.*

Example 2.8. *[1] Let (W,d) be a metric space and $Y : W \to W$ be a Picard operator, that is, Y has a unique fixed point $w^* \in W$ and $\lim_{n\to\infty} Y^n(u) = w^*$ for all $u \in W$. We define the binary relation $\perp$ on W by $u \perp w$ if*

$$\lim_{n\to\infty} d(w, Y^n(u)) = 0.$$

Then, $(W, \perp)$ is an O-set.

Example 2.9. *[1] Let $W = [0, \infty)$ and define $w \perp u$ if $wu \in \{w, u\}$. Then, by setting $w_0 = 0$ or $w_0 = 1$, $(W, \perp)$ is an O-set.*

Now, we give the concepts of an *O*-sequence, a $\perp$-continuous mapping, an *O*-complete orthogonal metric space, a $\perp$-preserving mapping and a weakly $\perp$-preserving mapping.

Definition 2.10. *[1] Let $(W, \perp)$ be an O-set. A sequence $\{w_n\}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall n \in \mathbb{N}, w_n \perp w_{n+1}) \quad or \quad (\forall n \in \mathbb{N}, w_{n+1} \perp w_n).$$

Definition 2.11. *[1] The triplet $(W, \perp, d)$ is called an orthogonal metric space if $(W, \perp)$ is an O-set and (W, d) is a metric space.*

Definition 2.12. *[1] Let $(W, \perp, d)$ be an orthogonal metric space. Then, a mapping $Y : W \to W$ is said to be orthogonally continuous (or $\perp$-continuous) in $w \in W$ if for each O-sequence $\{w_n\}$ in W with $w_n \to w$ as $n \to \infty$, we have $Y(w_n) \to Y(w)$ as $n \to \infty$. Also, Y is said to be $\perp$-continuous on W if Y is $\perp$-continuous in each $w \in W$.*

Remark 2.13. *[1] Every continuous mapping is $\perp$-continuous and the converse is not true.*

Definition 2.14. *[1] Let $(W, \perp, d)$ be an orthogonal metric space. Then, W is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.*

Remark 2.15. *[1] Every complete metric space is O-complete and the converse is not true.*

Definition 2.16. *[1] Let $(W, \perp)$ be an O-set. A mapping $Y : W \to W$ is said to be $\perp$-preserving if $Yw \perp Yu$ whenever $w \perp u$. Also $Y : W \to W$ is said to be weakly $\perp$-preserving if $Y(w) \perp Y(u)$ or $Y(u) \perp Y(w)$ whenever $w \perp u$.*

In this paper, we modify the concepts of *F*-contraction and *F*-Suzuki contraction mappings to orthogonal sets and prove some fixed point theorems for *F*-contraction and *F*-Suzuki contraction mappings in orthogonality complete metric spaces. Also, we give some example to illustrate our results. Furthermore, we apply our results to show the existence and uniqueness of a solution of the first-order ordinary differential equation.

3. Main Results

In this section, inspired by the notions of an *F*-contraction mapping, *F*-Suzuki contraction and an orthogonal set, we introduce a new *F*-contraction mapping, new *F*-Suzuki contraction and prove some fixed point theorems for these contraction mappings in an orthogonal metric space.

Definition 3.1. *Let $(W, \perp, d)$ be a orthogonal metric space. A map $Y : W \to W$ is said to be a generalized orthogonal F-contraction mapping (briefly, generalized $Y_\perp$- contraction) on $(W, \perp, d)$ if there are $F \in \mho$ and $\tau > 0$ such that the following condition holds:*

$$\forall w, u \in W \text{with } w \perp u \big[d(Yw, Yu) > 0$$
$$\Rightarrow \tau + F(d(Yw, Yu)) \leq F(d(w, u)) \big].$$

Definition 3.2. *Let $(W, \perp, d)$ be a orthogonal metric space. A map $Y : W \to W$ is said to be a orthogonal F-Suzuki contraction mapping (briefly, $Y_\perp$-Suzuki contraction) on $(W, \perp, d)$ if there are $F \in \mho$ and $\tau > 0$ such that the following condition holds:*

$$\forall w, u \in W \text{with } w \perp u \big[d(Yw, Yu) > 0,$$
$$\frac{1}{2} d(w, Yw) < d(w, u)$$
$$\Rightarrow \tau + F(d(Yw, Yu)) \leq F(d(w, u)) \big].$$

Now, we give the first fixed point theorem for generalized orthogonal *F*-contraction mapping in an *O*-complete orthogonal metric space $(W, \perp, d)$.

Theorem 3.3. *Let $(W, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal element w_0 and a mapping $Y : W \to W$. Suppose that there exist $F \in \mho$ and $\tau > 0$ such that the following conditions hold:*

(i) Y is $\perp$-preserving;

(ii) Y is generalized $Y_\perp$-contraction mapping;

(iii) Y is $\perp$-continuous.

Then Y has a unique fixed point $t \in W$ and for every $w \in W$, the sequence $\{Y^n w\}$ converges to t.

Proof. Since $(W, \perp)$ is an *O*-set,

$$\exists w_0 \in W : (\forall w \in W, w \perp w_0) \quad or \quad (\forall w \in W, w_0 \perp w).$$

It follows that $w_0 \perp Yw_0$ or $Yw_0 \perp w_0$. Let

$$w_1 := Yw_0, w_2 := Yw_1 = Y^2 w_0...$$
$$..., w_{n+1} := Yw_n = Y^{n+1} w_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $w_n = w_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that w_n is a fixed point of *Y*. Assume that $w_n \neq w_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have $d(Yw_n, Yw_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since *Y* is $\perp$-preserving, we have

$$w_n \perp w_{n+1} \quad or \quad w_{n+1} \perp w_n \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{w_n\}$ is an *O*-sequence. Since *Y* is generalized $Y_\perp$-contraction mapping, we have

$$F(d(w_n, w_{n+1})) = F(d(Yw_{n-1}, Yw_n))$$
$$\leq F(d(w_{n-1}, w_n)) - \tau, \quad \forall n \in \mathbb{N}.$$

Repeating this process, we get

$$F(d(Yw_{n-1}, Yw_n)) \leq F(d(w_{n-1}, w_n)) - \tau$$
$$= F(d(Yw_{n-2}, Yw_{n-1})) - \tau$$
$$\leq F(d(w_{n-2}, w_{n-1})) - 2\tau$$
$$= F(d(Yw_{n-3}, Yw_{n-2})) - 2\tau$$
$$\leq F(d(w_{n-3}, w_{n-2})) - 3\tau$$
$$\vdots$$
$$\leq F(d(w_0, w_1)) - n\tau. \tag{3.2}$$

From (3.2), we obtain $\lim_{n\to\infty} F(d(w_n, Yw_n)) = -\infty$.
By the property (F_2), we have

$$\lim_{n\to\infty} d(w_n, Yw_n) = 0. \tag{3.3}$$

Now, we claim that $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\varepsilon > 0$ and sequence $\{i(n)\}_{n=1}^{\infty}$ and $\{j(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$i(n) > j(n) > n, \quad d(w_{i(n)}, w_{j(n)}) \geq \varepsilon,$$
$$d(w_{i(n)-1}, w_{j(n)}) < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

So, we have

$$\varepsilon \leq d(w_{i(n)}, w_{j(n)}) \leq d(w_{i(n)}, w_{i(n)-1}) + d(w_{i(n)-1}, w_{j(n)})$$
$$< d(w_{i(n)}, w_{i(n)-1}) + \varepsilon$$
$$= d(w_{i(n)-1}, Yw_{i(n)-1}) + \varepsilon.$$

It follows from (3.3) and the above inequality that

$$\lim_{n\to\infty} d(w_{i(n)}, w_{j(n)}) = \varepsilon. \tag{3.5}$$

On the other hand, from (3.3) there exists $N \in \mathbb{N}$, such that

$$d(w_{i(n)}, Yw_{i(n)}) < \frac{\varepsilon}{4} \quad \text{and} \quad d(w_{j(n)}, Yw_{j(n)}) < \frac{\varepsilon}{4}, \tag{3.6}$$

$\forall n \in \mathbb{N}$. Next, we claim that

$$d(Yw_{i(n)}, Yw_{j(n)}) = d(w_{i(n)+1}, w_{j(n)+1}) > 0, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Arguing by contradiction, there exists $m \geq \mathbb{N}$ such that

$$d(w_{i(m)+1}, w_{j(m)+1}) = 0. \tag{3.8}$$

It follows from (3.4), (3.6) and (3.8) that

$$\varepsilon \leq d(w_{i(m)}, w_{j(m)}) \leq d(w_{i(m)}, w_{i(m)+1})$$
$$+ d(w_{i(m)+1}, w_{j(m)})$$
$$\leq d(w_{i(m)}, w_{i(m)+1})$$
$$+ d(w_{i(m)+1}, w_{j(m)+1})$$
$$+ d(w_{j(m)+1}, w_{j(m)})$$
$$= d(w_{i(m)}, Yw_{i(m)})$$
$$+ d(w_{i(m)+1}, w_{j(m)+1})$$
$$+ d(w_{j(m)}, Yw_{j(m)})$$
$$< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This contradiction establishes the relation (3.7). Since Y is $\perp$-preserving, we have

$$w_{i(n)} \perp w_{j(n)} \quad \text{or} \quad w_{j(n)} \perp w_{i(n)}.$$

Since Y is generalized orthogonal F-contraction mapping, we have

$$\tau + F(d(Yw_{i(n)}, Yw_{j(n)})) \leq F(d(w_{i(n)}, w_{j(n)})), \forall n \geq \mathbb{N}. \tag{3.9}$$

From (F_3'), (3.5) and (3.9), we obtain $\tau + F(\varepsilon) \leq F(\varepsilon)$. This contradiction shows that $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since W is complete, there exists $t \in W$ such that $\lim_{n\to\infty} w_n = t$. Since Y is $\perp$-continuous, we have

$$Yt = Y(\lim_{n\to\infty} w_n) = \lim_{n\to\infty} w_{n+1} = t$$

and so t is a fixed point of Y.

Next, we show that t is a unique fixed point of Y. Indeed, if $s, r \in W$ be two distinct fixed points of Y, that is $Ys = s \neq r = Yr$. Therefore,

$$d(Ys, Yr) = d(s, r) > 0. \tag{3.10}$$

Since Y is $\perp$-preserving, we have

$$s \perp r \quad \text{or} \quad r \perp s.$$

Since Y is generalized orthogonal F-contraction mapping, we have

$$F(d(s, r)) = F(d(Ys, Yr)) < \tau + F(d(Ys, Yr))$$
$$\leq F(d(s, r)),$$

which is a contradiction. Hence, Y has a unique fixed point. $\square$

Next, we give the fixed point theorem for an orthogonal F-Suzuki contraction mapping in an O-complete orthogonal metric space $(W, \perp, d)$.

Theorem 3.4. *Let $(W, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal element w_0 and a mapping $Y : W \to W$. Suppose that there exist $F \in \eth$ and $\tau > 0$ such that the following conditions hold:*

(i) Y is $\perp$-preserving;

(ii) Y is an orthogonal F-Suzuki contraction mapping;

(iii) Y is $\perp$-continuous.

Then Y has a unique fixed point $t \in W$ and for every $w \in W$, the sequence $\{Y^n w\}$ converges to t.

Proof. Since $(W, \perp)$ is an O-set,

$$\exists w_0 \in W : (\forall w \in W, w \perp w_0) \quad \text{or} \quad (\forall w \in W, w_0 \perp w).$$

It follows that $w_0 \perp Yw_0$ or $Yw_0 \perp w_0$. Let

$$w_1 := Yw_0, w_2 := Yw_1 = Y^2 w_0 \dots$$
$$\dots, w_{n+1} := Yw_n = Y^{n+1} w_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $w_n = w_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then it is clear that w_n is a fixed point of Y. Assume that $w_n \neq w_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have $d(Yw_n, Yw_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$\frac{1}{2} d(w_n, Yw_n) < d(w_n, Yw_n), \quad \forall n \in \mathbb{N}.$$

Since Y is $\perp$-preserving, we have

$$w_n \perp w_{n+1} \quad \text{or} \quad w_{n+1} \perp w_n \tag{3.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{w_n\}$ is an O-sequence. Since Y is an orthogonal F-Suzuki contraction mapping, we have

$$F(d(Yw_{n-1}, Yw_n)) \leq F(d(w_{n-1}, w_n)) - \tau, \quad \forall n \in \mathbb{N}.$$

Repeating this process, we get

$$\begin{aligned} F(d(Yw_{n-1}, Yw_n)) &\leq F(d(w_{n-1}, w_n)) - \tau \\ &= F(d(Yw_{n-2}, Yw_{n-1})) - \tau \\ &\leq F(d(w_{n-2}, w_{n-1})) - 2\tau \\ &= F(d(Yw_{n-3}, Yw_{n-2})) - 2\tau \\ &\leq F(d(w_{n-3}, w_{n-2})) - 3\tau \\ &\vdots \\ &\leq F(d(w_0, w_1)) - n\tau. \end{aligned} \tag{3.12}$$

From (3.12), we obtain $\lim_{n \to \infty} F(d(w_n, Yw_n)) = -\infty$. By the property (F_2), we have

$$\lim_{n \to \infty} d(w_n, Yw_n) = 0. \tag{3.13}$$

Now, we claim that $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\varepsilon > 0$ and sequence $\{i(n)\}_{n=1}^{\infty}$ and $\{j(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$\begin{aligned} i(n) > j(n) > n, \quad d(w_{i(n)}, w_{j(n)}) \geq \varepsilon, \\ d(w_{i(n)-1}, w_{j(n)}) < \varepsilon, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.14}$$

So, we have

$$\begin{aligned} \varepsilon \leq d(w_{i(n)}, w_{j(n)}) &\leq d(w_{i(n)}, w_{i(n)-1}) + d(w_{i(n)-1}, w_{j(n)}) \\ &< d(w_{i(n)}, w_{i(n)-1}) + \varepsilon \\ &= d(w_{i(n)-1}, Yw_{i(n)-1}) + \varepsilon. \end{aligned}$$

It follows from (3.13) and the above inequality that

$$\lim_{n \to \infty} d(w_{i(n)}, w_{j(n)}) = \varepsilon. \tag{3.15}$$

On the other hand, from (3.13) there exists $N \in \mathbb{N}$, such that

$$d(w_{i(n)}, Yw_{i(n)}) < \frac{\varepsilon}{4} \tag{3.16}$$

$$\text{and} \, d(w_{j(n)}, Yw_{j(n)}) < \frac{\varepsilon}{4}, \quad \forall n \in \mathbb{N}. \tag{3.17}$$

Next, we claim that

$$d(Yw_{i(n)}, Yw_{j(n)}) = d(w_{i(n)+1}, w_{j(n)+1}) > 0, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

Arguing by contradiction, there exists $m \geq N$ such that

$$d(w_{i(m)+1}, w_{j(m)+1}) = 0. \tag{3.19}$$

It follows from (3.14), (3.17) and (3.19) that

$$\begin{aligned} \varepsilon \leq d(w_{i(m)}, w_{j(m)}) &\leq d(w_{i(m)}, w_{i(m)+1}) \\ &\quad + d(w_{i(m)+1}, w_{j(m)}) \\ &\leq d(w_{i(m)}, w_{i(m)+1}) \\ &\quad + d(w_{i(m)+1}, w_{j(m)+1}) \\ &\quad + d(w_{j(m)+1}, w_{j(m)}) \\ &= d(w_{i(m)}, Yw_{i(m)}) \\ &\quad + d(w_{i(m)+1}, w_{j(m)+1}) \\ &\quad + d(w_{j(m)}, Yw_{j(m)}) \\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This contradiction establishes the relation (3.18). Since Y is $\perp$-preserving, we have

$$w_{i(n)} \perp w_{j(n)} \quad \text{or} \quad w_{j(n)} \perp w_{i(n)}.$$

From (3.13) and (3.15), we can choose a positive integer $N \in \mathbb{N}$ such that

$$\frac{1}{2} d(w_{i(n)}, Yw_{i(n)}) < \frac{1}{2}\varepsilon < d(w_{i(n)}, w_{j(n)}), \quad \forall n \in \mathbb{N}.$$

Since Y is F-Suzuki contraction mapping, we have

$$\tau + F(d(Yw_{i(n)}, Yw_{j(n)})) \leq F(d(w_{i(n)}, w_{j(n)})), \forall n \geq N. \tag{3.20}$$

From (F_3'), (3.15) and (3.20), we obtain $\tau + F(\varepsilon) \leq F(\varepsilon)$. This contradiction shows that $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (W, d), $\{w_n\}_{n=1}^{\infty}$ converges to some point $t \in W$. Therefore,

$$\lim_{n \to \infty} d(w_n, t) = 0. \tag{3.21}$$

Now, we claim that

$$\frac{1}{2} d(w_n, Yw_n) < d(w_n, t)$$

$$\text{or} \quad \frac{1}{2} d(Yw_n, Y^2 w_n) < d(Yw_n, t), \quad \forall n \in \mathbb{N}. \tag{3.22}$$

Again, assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2} d(w_m, Yw_m) \geq d(w_m, t)$$

$$\text{and} \quad \frac{1}{2} d(Yw_m, Y^2 w_m) \geq d(Yw_m, t). \tag{3.23}$$

Therefore,

$$2d(w_m,t) \le d(w_m,Yw_m) \le d(w_m,t)+d(t,Yw_m),$$

which implies that

$$d(w_m,t) \le d(t,Yw_m). \tag{3.24}$$

It follows from (3.23) and (3.24) that

$$d(w_m,t) \le d(t,Yw_m) \le \frac{1}{2}d(Yw_m,Y^2w_m). \tag{3.25}$$

Since $\frac{1}{2}d(w_m,Yw_m) < d(w_m,Yw_m)$, by the assumption of the theorem, we get

$$\tau + F(d(Yw_m,Y^2w_m)) \le F(d(w_m,Yw_m)).$$

Since $\tau > 0$, this implies that

$$F(d(Yw_m,Y^2w_m)) < F(d(w_m,Yw_m)).$$

So, from (F_1), we get

$$d(Yw_m,Y^2w_m) < d(w_m,Yw_m). \tag{3.26}$$

It follows that from (3.23), (3.25) and (3.26) that

$$\begin{aligned}
d(Yw_m,Y^2w_m) &< d(w_m,Yw_m) \\
&\le d(w_m,t)+d(t,Yw_m) \\
&\le \frac{1}{2}d(Yw_m,Y^2w_m)+\frac{1}{2}d(Yw_m,Y^2w_m) \\
&= d(Yw_m,Y^2w_m).
\end{aligned}$$

This is a contradiction. Hence, (3.22) holds. So, from (3.22), for every $n \in \mathbb{N}$, either

$$\tau + F(d(Yw_n,Yt)) \le F(d(w_n,t)),$$
$$\text{or}$$
$$\tau + F(d(Y^2w_n,Yt)) \le F(d(Yw_n,t)) = F(d(w_{n+1},t))$$

holds. In the first case, from (3.21), we get

$$\lim_{n\to\infty} F(d(Yw_n,Yt)) = -\infty.$$

By the property (F_2), we have

$$\lim_{n\to\infty} d(Yw_n,Yt) = 0.$$

Therefore,

$$d(t,Yt) = \lim_{n\to\infty} d(w_{n+1},Yt) = \lim_{n\to\infty} d(Yw_n,Yt) = 0.$$

Also, in the second case, from (3.21), we get

$$\lim_{n\to\infty} F(d(Y^2w_n,Yt)) = -\infty.$$

By the property (F_2), we have

$$\lim_{n\to\infty} d(Y^2w_n,Yt) = 0.$$

Therefore,

$$d(t,Yt) = \lim_{n\to\infty} d(w_{n+2},Yt) = \lim_{n\to\infty} d(Y^2w_n,Yt) = 0.$$

Hence, t is a fixed point of Y. Next, we show that t is a unique fixed point of Y. Indeed, if $s,r \in W$ be two distinct fixed points of Y, that is $Ys = s \ne r = Yr$, then $d(Ys,Yr) > 0$. Since Y is $\perp$-preserving, we have

$$s \perp r \quad \text{or} \quad r \perp s.$$

We have $0 = \frac{1}{2}d(s,Ys) < d(s,r)$. Since Y is F-Suzuki contraction mapping, we have

$$\begin{aligned}
F(d(s,r)) = F(d(Ys,Yr)) &< \tau + F(d(Ys,Yr)) \\
&\le F(d(s,r)), \tag{3.27}
\end{aligned}$$

which is a contradiction. Hence, Y has a unique fixed point. $\qquad\square$

4. Example

Example 4.1. *Let $W = [0,\infty)$ and $d : W \times W \to [0,\infty)$ be a mapping defined by $d(u,v) = |u-v|$ for all $u,v \in W$. Consider the sequence $\{S_p\}_{p\in\mathbb{N}}$ defined as*

$$S_p = \frac{p(p+1)}{2}, \quad \forall p \in \mathbb{N} \cup \{0\}.$$

Define a relation $\perp$ on X by

$$u \perp v \iff uv \in \{u,v\} \subseteq \{S_p\}.$$

Thus $(W,\perp,d)$ is an O-complete metric space. Now, we will define a mapping $Y : W \to W$ by

$$Yu = \begin{cases} S_1 & if \quad S_0 \le u \le S_1, \\ S_{p-1} & if \quad S_p \le u \le S_{p+1} \quad \forall p > 1. \end{cases} \tag{4.1}$$

It is easy to see that Y is $\perp$-continuous and Y is $\perp$-preserving. Let $u,v \in W$ with $u \perp v$ and $d(Yu,Yv) > 0$. Without loss of generality, we may assume that $u < v$. This implies that $u \in \{S_0,S_1\}$ and $v = S_i$ for some $i \in \mathbb{N} \setminus \{1\}$.

$$\begin{aligned}
\lim_{p\to\infty} \frac{d(Yu,Yv)}{d(u,v)} &= \lim_{p\to\infty} \frac{S_{p-1}-1}{S_p-1} \\
&= \lim_{p\to\infty} \frac{\frac{p(p-1)}{2}-1}{\frac{p(p+1)}{2}-1} \\
&= \lim_{p\to\infty} \frac{p(p-1)-2}{p(p+1)-2} = 1, \tag{4.2}
\end{aligned}$$

Y is not a Banach contraction and a orthogonal F-Suzuki contraction. On the other hand taking $F(\beta) = \frac{-1}{\beta} + \beta \in \partial$, we obtain the result that Y is an orthogonal F-Suzuki contraction with $\tau = 2$. To see this, let us consider the following calculation. First observe that

$$\begin{aligned}
\frac{1}{2}d(S_p,YS_p) &< d(S_p,S_q) \\
&\iff \big[(1 = p < q) \vee (1 \le q < p) \\
&\qquad \vee (1 < p < q)\big]. \tag{4.3}
\end{aligned}$$

For $1 < p < q$, we have

$$|Y(S_q) - Y(S_1)| = |S_{q-1} - S_1|$$
$$= 2 + 3 + 4 + \cdots + q - 1,$$
$$|S_q - S_1| = 2 + 3 + \cdots + q. \qquad (4.4)$$

Since $q > 1$ and $\frac{-1}{2+3+\cdots+q-1} < \frac{-1}{2+3+\cdots+q}$, we have

$$2 - \frac{-1}{2+3+\cdots+q-1} + [2+3+\cdots+q-1]$$
$$< 2 - \frac{1}{2+3+\cdots+q} + [2+3+\cdots+q-1]$$
$$\leq \frac{-1}{2+3+\cdots+q} + [2+3+\cdots+q-1] + q.$$

So, from (4.4), we get

$$2 - \frac{1}{|Y(S_q)-Y(S_1)|} + |Y(S_q)-Y(S_1)|$$
$$< -\frac{1}{|S_q - S_1|} + |S_q - S_1|.$$

For $1 \leq p < q$, similar to $1 = p < q$, we have

$$2 - \frac{1}{|Y(S_q)-Y(S_1)|} + |Y(S_q)-Y(S_1)|$$
$$< -\frac{1}{|S_q - S_1|} + |S_q - S_1|.$$

For $1 < p < q$, we have

$$|Y(S_q) - Y(S_p)| = p + (p+1) + (p+2) + \cdots + q - 1,$$
$$|S_q - S_p| = p + (p+1) + (p+2) + \cdots + q. \qquad (4.5)$$

Since $q > p > 1$ and $\frac{-1}{p+(p+1)+(p+2)+\cdots+q-1} < \frac{-1}{p+(p+1)+(p+2)+\cdots+q}$. Therefore

$$2 - \frac{1}{p+(p+1)+(p+2)+\cdots+q-1}$$
$$+ [p+(p+1)+(p+2)+\cdots+q-1]$$
$$< 2 - \frac{1}{p+(p+1)+(p+2)+\cdots+q}$$
$$+ [p+(p+1)+(p+2)+\cdots+q-1]$$
$$\leq -\frac{1}{p+(p+1)+(p+2)+\cdots+q}$$
$$+ [p+(p+1)+(p+2)+\cdots+q-1+q]$$

So, from (4.5), we get

$$2 - \frac{1}{|Y(S_q)-Y(S_p)|} + |Y(S_q)-Y(S_p)|$$
$$< -\frac{1}{|S_q - S_p|} + |S_q - S_p|.$$

Therefore $\tau + F(d(Y(S_q), Y(S_p))) \leq F(d(S_q, S_p))$ for all $p, q \in \mathbb{N}$. Hence Y is an orthogonal F-Suzuki contraction and $Y(S_1) = S_1$.

5. Application to ordinary differential equations

Recall that, for any $1 \leq p < \infty$, the space $L^p(W, F, \gamma)$ (or $L^p(W)$) consists of all complex-valued measurable ζ on the underlying space W satisfying

$$\int_W |\zeta(v)|^p d\gamma(v),$$

where F is the σ-algebra of measurable sets and γ is the measure. When $p = 1$, the space $L^1(W)$ consists of all integrable functions ζ on W and we define the L^1-norm of ζ by

$$\|\zeta\|_1 = \int_W |\zeta(v)| d\gamma(v).$$

In this section, using Theorem 3.3, we show the existence of a solution of the following differential equation:

$$\begin{cases} w'(t) = Y(t, w(t)), \text{a.e.} \quad t \in I := [0, T]; \\ w(0) = c, c \geq 1, \end{cases} \qquad (5.1)$$

where $Y : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(i) $Y(s, x) \geq 0$ for all $x \geq 0$ and $s \in I$;

(ii) for each $i, j \in L^1(I)$ with $i(s)j(s) \geq i(s)$ or $i(s)j(s) \geq j(s)$ for all $s \in I$, there exist $\zeta \in L^1(I)$ and $\tau > 0$ such that

$$|Y(s, i(s)) - Y(s, j(s))| \leq \frac{\zeta(s)}{1 + \tau\zeta(s)}|i(s) - j(s)| \qquad (5.2)$$

and

$$|i(s) - j(s)| \leq \zeta(s)e^{D(s)}$$

for all $s \in I$, where $D(s) := \int_0^s |\zeta(r)| dr$.

Theorem 5.1. *Consider the differential Eq.(5.1). If (i) and (ii) are satisfied, then the differential Eq. (5.1) has a unique positive solution.*

Proof. Let $W = \{w \in C(I, \mathbb{R}) : w(t) > 0 \quad \text{for all} \quad t \in I\}$. Define the orthogonality relation $\perp$ on V by

$$i \perp j \iff i(t)j(t) \geq i(t) \quad or \quad i(t)j(t) \geq j(t), \forall t \in I.$$

Since $D(t) = \int_0^t |\zeta(s)| ds$, we have $D'(t) = |\zeta(t)|$ for almost everywhere $t \in I$. Define a mapping $d : W \times W \to [0, \infty)$ by

$$d(i, j) = \|i - j\|_D = \sup_{t \in I} e^{-D(t)}|i(t) - j(t)|$$

for all $i, j \in W$. Thus, (W, d) is a metric space and also a complete metric space (see, [1] for details). Define a mapping $G : W \to W$ by

$$(Gi)(t) = c + \int_0^t Y(s, i(s)) ds.$$

Then, we see that G is $\perp$-continuous.

Now, we show that G is $\perp$-preserving. For each $i, j \in W$ with $i \perp j$ and $t \in I$, we have

$$(Gi)(t) = c + \int_0^t Y(s, i(s))ds \geq 1.$$

It follows that $[(Gi)(t)][(Gj)(t)] \geq (Gj)(t)$ and so $(Gi)(t) \perp (Gj)(t)$. Then, G is $\perp$-preserving.

Next, we claim that G is generalized orthogonal F-contraction. Let $i, j \in W$ with $i \perp j$. Suppose that $G(i) \neq G(j)$. For each $s \in I$, we have

$$|i(s) - j(s)| \leq \zeta(s)e^{D(s)}.$$

and so

$$d(i, j) = \sup_{s \in I} e^{-D(s)}|i(s) - j(s)| \leq \zeta(s).$$

It follows that

$$1 + \tau d(i, j) \leq 1 + \tau \zeta(s).$$

From (ii), for each $t \in I$, we obtain

$$
\begin{aligned}
|(Gi)(t) - (Gj)(t)| &\leq \int_0^t |Y(s, i(s)) - Y(s, j(s))|ds \\
&\leq \int_0^t \frac{|\zeta(s)|}{1 + \tau \zeta(s)}|i(s) - j(s)|ds \\
&\leq \int_0^t \frac{|\zeta(s)|}{1 + \tau d(i, j)}|i(s) - j(s)| \\
&\quad e^{-D(s)}e^{D(s)}ds \\
&\leq \frac{d(i, j)}{1 + \tau d(i, j)} \int_0^t |\zeta(s)|e^{D(s)}ds \\
&\leq \frac{d(i, j)}{1 + \tau d(i, j)}\left(e^{D(t)} - 1\right) \\
&\leq \frac{d(i, j)}{1 + \tau d(i, j)}\left(e^{D(t)} - 1\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
e^{-D(t)}|(Gi)(t) - (Gj)(t)| &\leq e^{-D(t)}\left(e^{D(t)} - 1\right) \\
\frac{d(i, j)}{1 + \tau d(i, j)} & \\
= \left(1 - e^{-D(t)}\right) & \frac{d(i, j)}{1 + \tau d(i, j)} \\
\leq \left(1 - e^{-\|\zeta\|_1}\right) & \frac{d(i, j)}{1 + \tau d(i, j)}.
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
d(Gi, Gj) &\leq \frac{d(i, j)}{1 + \tau d(i, j)} \\
\frac{1 + \tau d(i, j)}{d(i, j)} &\leq \frac{1}{d(Gi, Gj)} \\
\tau + \frac{1}{d(i, j)} &\leq \frac{1}{d(Gi, Gj)} \\
\tau + \frac{1}{d(i, j)} &\leq \frac{1}{d(Gi, Gj)} \\
\tau - \frac{1}{d(Gi, Gj)} &\leq -\frac{1}{d(i, j)}.
\end{aligned}
$$

Taking a function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(c) = -\frac{1}{z}$ for all $z > 0$, it follows that G is generalized orthogonal F-contraction. By Theorem 3.3, G has a unique fixed point and hence the differential Eq. (5.1) has a unique positive solution. $\qquad\square$

6. Conclusion

In this paper, we proved fixed point theorems for generalized orthogonal F-contraction and F-Suzuki contraction mappings on O- complete metric space.

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