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Planar automaton accepting Le-diagram, alternative tableaux

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Abstract

Here we focus on the construction of the planar automatons accepting two types of tableaux, namely, Le-diagrams and alternative tableaux. Then we associate a quadratic algebra Q with the constructed planar automaton and then by redefining the transition function by giving distinct labeling for distinct terms of the rewriting rule we get Q-tableaux. Finally, we give the proof of the equivalence between the tableaux accepting a planar automaton and Q-tableaux obtained, where the equivalence is given by Xavier Viennot in [8].

Keywords

Planar automaton, Le-diagram, alternative tableaux.

AMS Subject Classification

52B35.

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Contents

1. Introduction

Planar automaton is a concept which formalizes the idea of recognizing certain planar figures drawn on a 2D lattice. It reads planar figure consisting of finite cells, where each cell takes a state depending on the states of its neighbors. A quadratic algebra Q is defined by some relations and generators [8]. Quadratic algebra associates some combinatorial objects (called Q-tableaux) with it. A Q-tableau is a Young (Ferrers) diagram, with some kind of labels in the cells of the diagram.

The Le-diagram(Permutation tableau) was introduced by Postnikov [1] in relation with some positivity problems in algebraic geometry and Le-diagrams of length $n+1$ are in one-to-one correspondence with alternative tableaux of length n introduced by Xavier Viennot [7]. Also, permutation tableaux are in bijection with permutations in S_n, which was first given by E. Steingrimsson and L. Williams [4]. Later several papers have been published giving different bijections between these two; see [2], [5]. In [8], Xavier Viennot is giving a bijection between permutations in S_n and alternative tableaux of size $n-1$.

Here we do our work on Le-diagrams and alternative tableaux, i.e, we construct planar automatons accepting these two tableaux and then associate a quadratic algebra Q corresponding to each planar automaton constructed and then by redefining transition function we obtain Q-tableaux. At the end we give the proof of the following proposition given by Xavier Viennot in [8].

Proposition 1.1. *The Q-tableaux related to a quadratic algebra Q is equivalent to the tableaux T accepted by a planar automaton* $P = (\mathscr{L}, \mathscr{B}, \mathscr{A}, \theta, w, uv)$ *with P satisfying the con-*

dition that $s \neq t$ implies $\theta(s,B,A) \neq \theta(t,B,A)$, where $B \in \mathscr{B}$, $A \in \mathscr{A}$ and $s,\ t \in \mathscr{L}$.

In the second section we are giving basic definitions that we use in the coming section. In sections 3.1 and 4.1 we construct planar automaton and then associate quadratic algebra Q for Le-diagrams and alternative tableaux respectively. In section 3.2 and 4.2 we obtain Q-tableaux corresponding the quadratic algebras in sections 3.1 and 4.1. In section 4 we give proof of Proposition 1.1.

2. Basic Definitions

Definition 2.1. *Let n be an integer. A partition of n is $\lambda = (\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where λ_i's are positive integers. Then λ_i is called part of the partition.*

Definition 2.2. *A Ferrers diagram is a finite collection of boxes or cells, arranged in left-justified rows, with the length of the rows $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ (where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) from bottom to top. Also $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ gives the partition λ of n (the total number of cells of the diagram). Then Ferrers diagram is said to be of shape λ.*

Note 2.3. *Here by first row and first column of the Ferrers diagram we mean the bottom row and left most column respectively.*

Definition 2.4. *In a Ferrers diagram if there exists horizontal or vertical extended lines then it is called an extended Ferrers diagram. That is, extended Ferrers diagram is a Ferrers diagram with possibly empty rows or columns.*

Note 2.5. *By empty row or column we mean that a row or column without any cell.*

Definition 2.6. *A Le-diagram is a Ferrers diagram F whose cells filled with 0 and 1 such that*

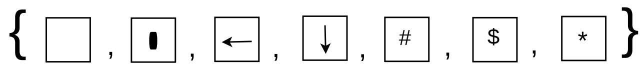
1. *in each column at least one 1*
2. $\begin{matrix} 1 \cdots 0 \\ \vdots \\ 1 \end{matrix}$ `is forbidden (not necessarily consecutive cells)`

For our convenience we take empty cell as entry 0 and doted cell as entry 1.

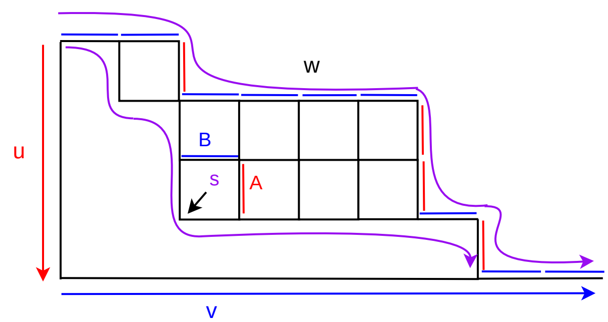
Definition 2.7. *[3] An alternative tableau is a Ferrers diagram F (with possibly empty rows and columns) with a partial filling of the cells with left arrows $\leftarrow$ and down arrows $\downarrow$, such that all cells left of a left arrow, or below a down arrow are empty. In other words, all cells pointed by an arrow must be empty.*

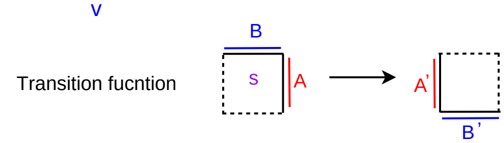
Definition 2.8. *Let A and $\mathscr{B}$ be the disjoint collection of some alphabets called vertical operators and horizontal operators respectively. Let $w \in (\mathscr{A} \cup \mathscr{B})^*$ be a word. Now construct a path by putting horizontal line for the horizontal operator and vertical line for the vertical operator one after the other corresponding to the alphabets in the word w.*

Definition 2.9. *[8] Let T be a Ferrers diagram with initial word $w \in (\mathscr{A} \cup \mathscr{B})^*$, such that each cell has certain labeling $s \in \mathscr{L}$, where $\mathscr{L}$ is a set of labels of the form*



Then A planar automaton is a reading of above Ferrers diagram with a transition function θ mapping $(s,B,A) \longrightarrow (B',A')$, where $s \in \mathscr{L}$, $B,\ B' \in \mathscr{B}$ and $A,\ A' \in \mathscr{A}$. It is denoted by $P = (\mathscr{L},\mathscr{B},\mathscr{A},\theta,w,uv)$, where uv is the final word of T. Then we call T as a tableau accepted by the planar automaton $P = (\mathscr{L},\mathscr{B},\mathscr{A},\theta,w,uv)$.





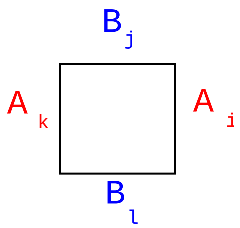
Definition 2.10. *A filtered algebra over a field $\mathbb{K}$ is an algebra $(\boldsymbol{A},\cdot)$ over $\mathbb{K}$ which has an increasing sequence $\{0\} \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_i \subseteq \ldots \subseteq \boldsymbol{A}$ of subspaces of $\boldsymbol{A}$ such that $\boldsymbol{A} = \bigcup_{i \in \mathbb{N}} F_i$ and that is compatible with the multiplication in the following sense: For all $m,\ n \in \mathbb{N}$, $F_m F_n \subseteq F_{n+m}$.*

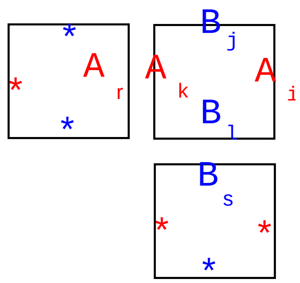
Definition 2.11. *A quadratic algebra is a filtered algebra generated by degree one elements, with defining relations of degree 2.*

Definition 2.12. *[8] Consider a quadratic algebra Q generated by $\mathscr{B} = \{B_j\}_{j \in J}$ and $\mathscr{A} = \{A_i\}_{i \in I}$ for every $j \in J$, $i \in I$ (where I and J are any index sets) with $\mathscr{A} \cap \mathscr{B} = \emptyset$, satisfying the commutation relation $B_j A_i = \sum_{kl} c_{ij}^{kl} A_k B_l$. A complete Q-tableau is a Ferrers diagram where each cell is labeled by the set R of rewriting rules with following compatibility condition: Set of rewriting rules $R = \{B_j A_i \to c_{ij}^{kl} A_k B_l :\ i,\ k \in I$ and $j,\ l \in J\}$ where $B_j A_i \to c_{ij}^{kl} A_k B_l$ is denoted as Compatibility condition is: for B_j, B_l, $B_s \in \mathscr{B}$ and A_i, A_k, $A_r \in \mathscr{A}$, where $j,\ l,\ s \in J$, $i,\ k,\ r \in I$, if $B_j A_i \longrightarrow A_k B_l$ and if*

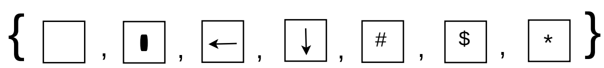
then $B_s = B_l$ and $A_r = A_k$.

Definition 2.13. *[8] Let Q be a quadratic algebra generated by $\mathscr{B} = \{B_j\}_{j \in J}$ and $\mathscr{A} = \{A_i\}_{i \in I}$ for every $j \in J$, $i \in I$ with $\mathscr{A} \cap \mathscr{B} = \emptyset$, satisfying the commutation relation $B_j A_i =$*





$\sum_{kl} c^{kl}_{ij} A_k B_l$. *Let* $R = \{B_j A_i \to c^{kl}_{ij} A_k B_l : i,\ k \in I$ *and* $j,\ l \in J\}$ *be the set of all rewriting rules of Q and let* $\mathscr{L}$ *be the set of all labels of the form*



consisting of at least as much distinct labels as the number of maximum terms in the commutation relation. Define $\varphi : R \to \mathscr{L}$ *such that the following condition holds: if for* $q,\ s \in J$ *and* $p,\ r \in I$, $(B_j A_i \to c^{kl}_{ij} A_k B_l) \neq (B_q A_p \to c^{rs}_{pq} A_r B_s)$ *and* $\varphi(B_j A_i \to c^{kl}_{ij} A_k B_l) = \varphi(B_q A_p \to c^{rs}_{pq} A_r B_s)$, *then* $(i, j) \neq (p, q)$. *Then Q-tableau is defined as the image of a complete Q-tableau under* φ.

In other words, for a commutation equation of the type $\alpha = \beta_1 + \beta_2 + \cdots + \beta_r$, we have to define $\varphi : R \to \mathscr{L}$ such that $\varphi(\alpha \to \beta_i) \in \mathscr{L}$ are distinct.

3. Construction of a planar automaton accepting Le-diagrams and obtaining corresponding Q-tableaux

3.1 Planar automaton accepting Le-diagrams

First we define the horizontal and vertical operators of a cell C satisfying the conditions of the Le-diagram in the following way

1. If there is no cell consisting of ● above the cell C then label the bottom horizontal edge of the cell C as D.

2. If there are one or more cell consisting of ● above the cell C and no cell consisting of ● below C label the bottom horizontal edge of the cell C as X_1.

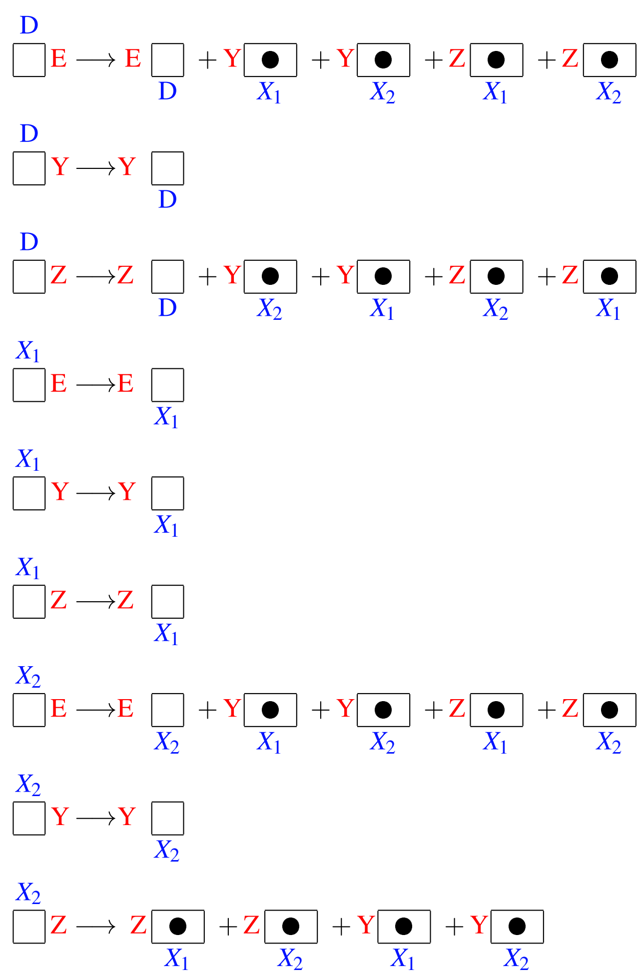
3. If there are one or more cell consisting of ● above and below the cell C then label the bottom horizontal edge of the cell C as X_2.

4. If there is no cell consisting of ● to the right of the cell C then label the right vertical edge of the cell C as E.

5. If there are one or more cell consisting of ● to the right of the cell C and no cell consisting of ● to the left of the cell C then label the right vertical edge of the cell C as Y.

6. If there are one or more cell consisting of ● to the right and the left of the cell C then label the right vertical edge of the cell C as Z.

Then the transition function is defined by:



Then the quadratic algebra Q is given by

$$
\begin{aligned}
DE &= ED + YX_1 + YX_2 + ZX_1 + ZX_2 \\
DY &= YD \\
DZ &= ZD + YX_2 + YX_1 + ZX_2 + ZX_1 \\
X_1E &= EX_1 \\
X_1Y &= YX_1 \\
X_1Z &= ZX_1
\end{aligned}
$$

$$X_2E = EX_2 + YX_1 + YX_2 + ZX_1 + ZX_2$$
$$X_2Y = YX_2$$
$$X_2Z = ZX_1 + ZX_2 + YX_1 + YX_2$$

Example 3.1. *Following is an example of Le-diagram.*

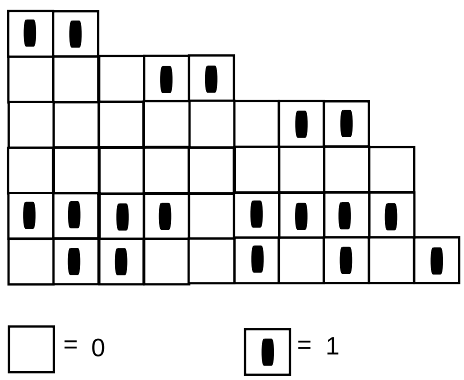


Figure 1

where empty cells denote the entry 0 *and doted cells denote the entry* 1.
By performing planar automaton to this Le-diagram to each cell step by step finally we get

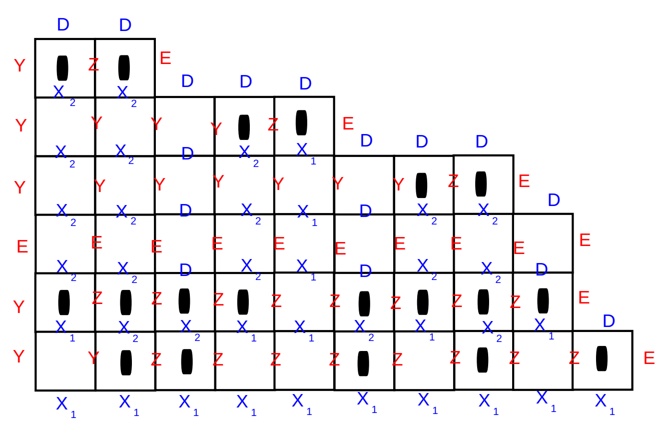
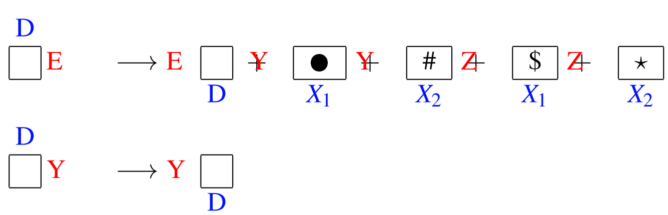
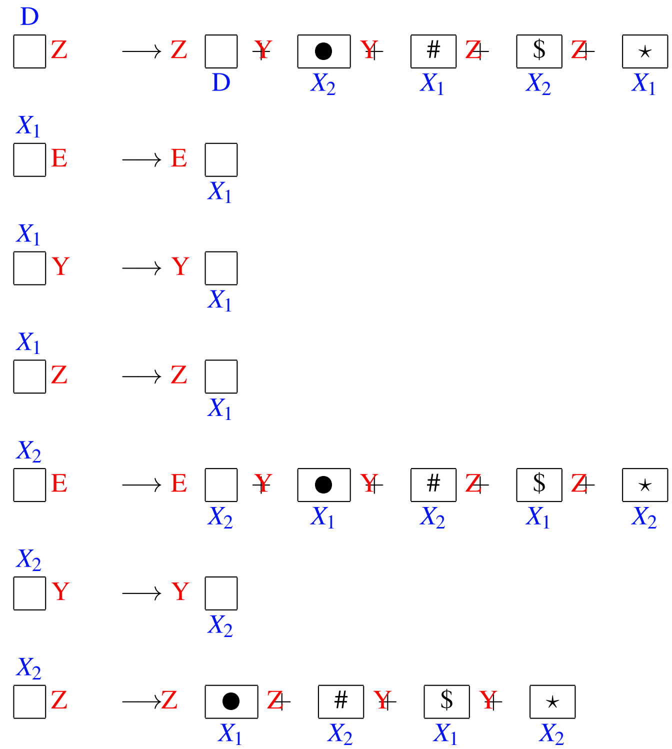


Figure 2

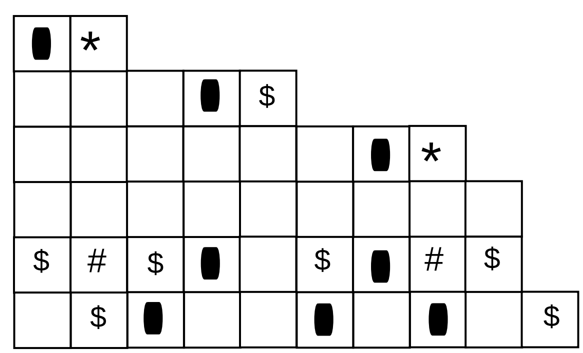
3.2 Q-tableaux of the quadratic algebra corresponding to Le-diagrams

Now to get Q-tableaux of the quadratic algebra Q of Le-diagram in section 3.1, we rewrite the transition function in section 3.1 by giving distinct labeling for the distinct rewriting rules of single commutation equation of the quadratic algebra as follows:





Example 3.2. *The Q-tableau corresponding to the complete Q-tableau in Figure 2 is given by*



4. Construction of a planar automaton accepting alternative tableaux and obtaining corresponding Q-tableaux

4.1 Planar automaton accepting alternative tableaux

First we define the horizontal and vertical operators of a cell C satisfying the conditions of the alternative tableau in the following way

1. If there is no cell consisting of ↓, ← above the cell C then label the bottom horizontal edge of the cell C as D.

2. If there is only one cell consisting of ↓ above the cell C and nn cell consisting of ↓, ← below the cell C then label the bottom horizontal edge of the cell C as X_1.

3. If there are one or more cell consisting of $\leftarrow$ above the cell C then label the bottom horizontal edge of the cell C as X_2.

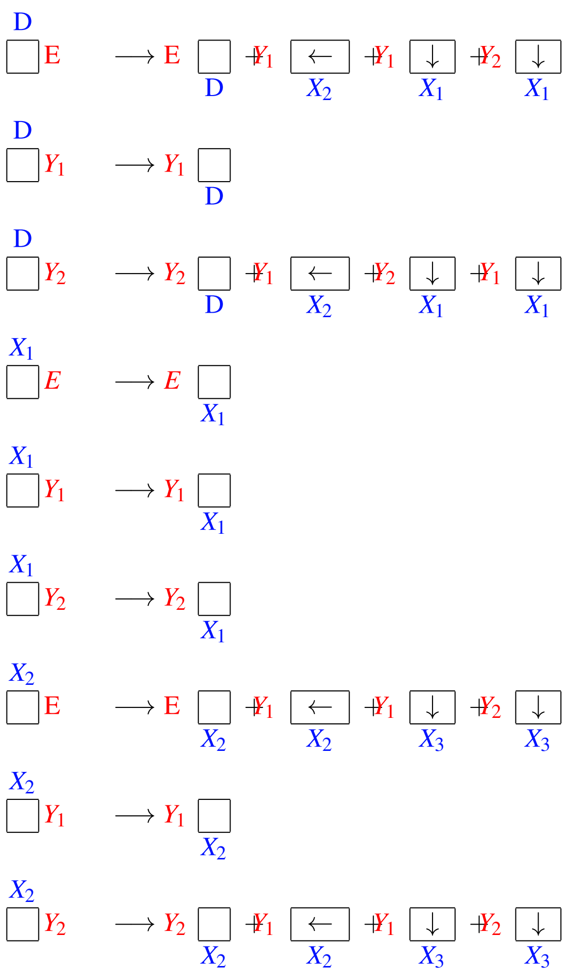
4. If there one or more cell consisting of $\leftarrow$ and only one $\downarrow$ above the cell C and nn cell consisting of $\downarrow$, $\leftarrow$ below the cell C then label the bottom horizontal edge of the cell C as X_3.

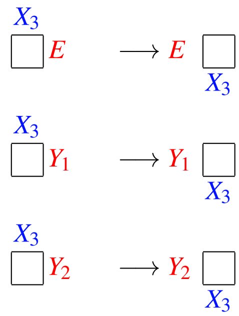
5. If there is no cell consisting of $\downarrow$, $\leftarrow$ to the right of the cell C then label the right vertical edge of the cell C as E.

6. If there is only one or no cell consisting of $\leftarrow$, there is no restriction on $\downarrow$ (it may be one or more or no) to the right of the cell C and no cell consisting of $\downarrow$, $\leftarrow$ to the left of the cell C then label the right vertical edge of the cell C as Y_1.

7. If there are one or more cell consisting of $\downarrow$ to the right of the cell C and only one $\leftarrow$, there is no restriction on $\downarrow$ (it may be one or more or no) to the left of the cell C then label the right vertical edge of the cell C as Y_2.

Then the transition function is defined by:





Then the quadratic algebra Q is given by

$$
\begin{aligned}
DE &= ED + Y_1X_2 + Y_1X_1 + Y_2X_1 \\
DY_1 &= Y_1D \\
DY_2 &= Y_2D + Y_1X_2 + Y_2X_1 + Y_1X_1 \\
X_1E &= EX_1 \\
X_1Y_1 &= Y_1X_1 \\
X_1Y_2 &= Y_2X_1 \\
X_2E &= EX_2 + Y_1X_2 + Y_1X_3 + Y_2X_3 \\
X_2Y_1 &= Y_1X_2 \\
X_2Y_2 &= Y_2X_2 + Y_1X_2 + Y_1X_3 + Y_2X_3 \\
X_3E &= EX_3 \\
X_3Y_1 &= Y_1X_3 \\
X_3Y_2 &= Y_2X_3
\end{aligned}
$$

Example 4.1. *Consider the following alternative tableau.*

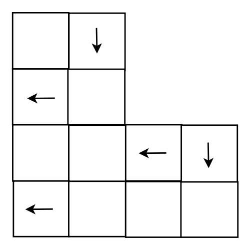
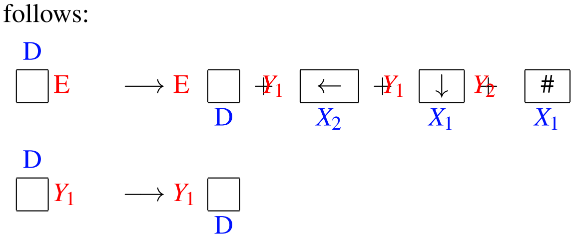


Figure 3

By performing planar automaton on this alternative tableau at each cell step by step finally we obtain

4.2 Q-tableaux of the quadratic algebra corresponding to alternative tableaux

Now to get Q-tableaux of the quadratic algebra in section 4.1 of alternative tableaux, we rewrite the transition function in section 4.1 by giving distinct labeling for distinct rewriting rules of single commutation equation of the quadratic algebra as follows:



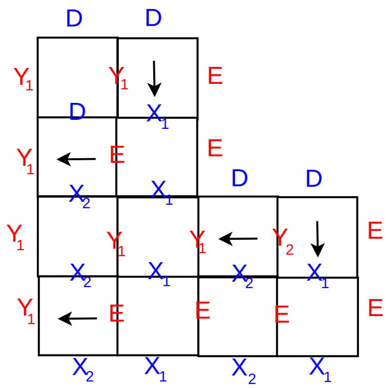
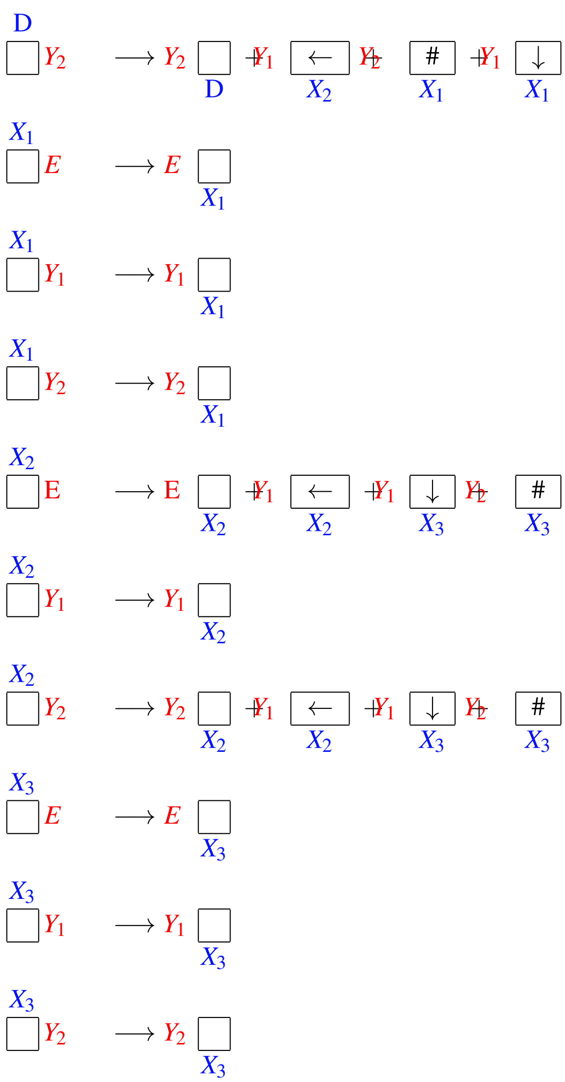
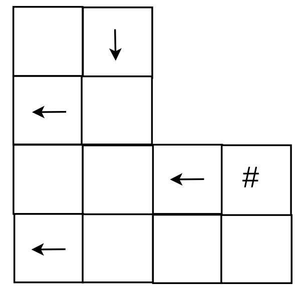


Figure 4



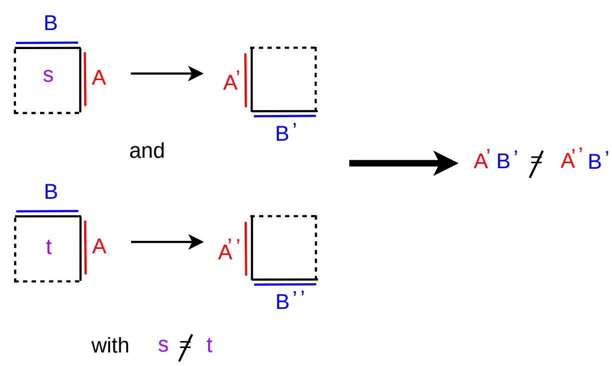
Example 4.2. *The Q-tableau corresponding to the complete Q-tableau in Figure 4 is given by*



5. Equivalence of tableaux accepting planar automaton with Q-tableaux

Now for the sake of completion we give the proof of Proposition 1.1 as follows:

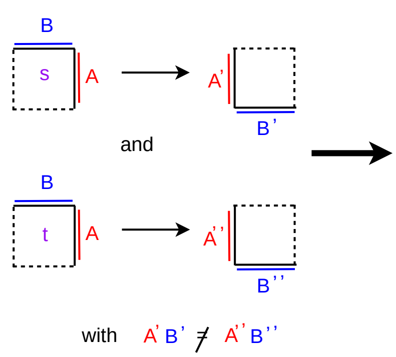
Proof. Let T be the tableaux accepted by a planar automaton $P = (\mathscr{L}, \mathscr{B}, \mathscr{A}, \theta, w, uv)$ such that $s \neq t$ implies that $\theta(s, B, A) \neq \theta(t, B, A)$, where $B \in \mathscr{B}$, $A \in \mathscr{A}$ and $s,\ t \in \mathscr{L}$. i.e, for B', $B'', \in \mathscr{B}$ and A', $A'' \in \mathscr{A}$



Then we can associate a quadratic algebra

$$BA = \sum_{s \in \mathscr{L}} A'B'$$

where $(B', A') = \theta(s, B, A)$. Now the set planar rewriting rules of the quadratic algebra Q is obtained from the transition function θ with an additional condition that if $\theta(s, B, A) \neq \theta(t, B, A)$, then $s \neq t$. i.e, for B', $B'', \in \mathscr{B}$ and A', $A'' \in \mathscr{A}$



Then we get Q-tableaux corresponding to the quadratic algebra using the set of rewriting rules just we described.

Conversely suppose that Q-tableaux corresponding to a quadratic algebra Q are given. Then clearly the set of rewriting rules of the quadratic algebra is the transition function θ for the planar automaton, which also satisfies the condition $s \neq t$ implies $\theta(s,B,A) \neq \theta(t,B,A)$. Then the Q-tableaux are exactly the tableaux T accepted by the planar automaton. Hence we can conclude that, since the set of planar rewriting of the quadratic algebra is equivalent to the transition function of the planar automaton, the Q-tableaux related to a quadratic algebra Q is equivalent to the tableaux T accepted by a planar automaton $P = (\mathscr{L}, \mathscr{B}, \mathscr{A}, \theta, w, uv)$ with P satisfying the condition that $s \neq t$ implies $\theta(s,B,A) \neq \theta(t,B,A)$. □

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