Malaya Journal of Matematik, Vol. 9, No. 1, 409-414, 2021

https://doi.org/10.26637/MJM0901/0069

Dynamical system induced by weighted tensor sum and tensor product operator with Lie group representations

G. S. Vigneswari[1], P. Chandra Kala[2]* and V. Anusuya[3]

Abstract

In this article we studied about Dynamical System induced by Weighted tensor sum and tensor product operator and multiplier representations of Lie groups.

Keywords

Weighted tensor sum and tensor product operator, Differentiable manifolds, Multiplier representations, Lie groups, Transformation groups, dynamical Systems, Operator valued mapping.

AMS Subject Classification

47B33, 22E05, 22E60, 47B37.

[1] *Research Scholar (Reg No. 18213152092033), S. T. Hindu College, Nagercoil-629002, Tamil Nadu, India.*

[2] *Department of Mathematics, Sree Devi Kumari Women's College, Kuzhithurai-629163, Tamil Nadu, India.*

[3] *Department of Mathematics, S. T. Hindu College, Nagercoil-629002, Tamil Nadu, India.*

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu.

*__Corresponding author__: [1] vigneshwari.gs.vg@gmail.com; [2]kala.mathchandra@gmail.com; [3] anusuyameenu@yahoo.com

Article History: Received **01** January **2021**; Accepted **09** February **2021**

Contents

1. Introduction

Let U_x be an open subset of $\mathbb{C}^n$ and let $H(U_x)$ denote the algebra of all complex valued anaytic functions on U_x. Let $\varphi : U_x \to U_x$ be an analytic map. Then $f_1 \circ \varphi \boxplus f_2 \circ \varphi \varepsilon H(U_x)$ for all $f_1, f_2 \in H(U_x)$. The map $f_1 \boxplus f_2 \to f_1 \circ \varphi \boxplus f_2 \circ \varphi$ is a linear transformation on $H(U_x)$ and we denote it by C_φ. Since $C_\varphi(f_1 \boxplus f_2) = f_1 \circ \varphi \boxplus f_2 \circ \varphi = C_\varphi f_1 \boxplus C_\varphi f_2 N$ is an algebra homomorphism with $C_\varphi(1) = 1$, where 1 is the constant one function on U_x. If φ is a diffiomorphism, then C_φ is invertible and $C_\varphi^{-1} = C_{\varphi^{-1}}$. Since $C_{\varphi\circ\psi}(f_1 \boxplus f_2) = f_1 \circ \varphi \circ \psi \boxplus f_2 \circ \varphi \circ \psi = C_\varphi C_\psi f_1 \boxplus C_\varphi C_\psi f_2$. $C_{\varphi\circ\psi} = C_\varphi C_\psi$ and hence $\varphi \to C_\varphi$ is not multiplicative . Let $\pi : U_x \to \mathbb{C}$ be a complex valued analytic map and $\varphi : U_x \to U_x$ be an analytic map. Then define the map $C_\varphi \boxplus M_{\pi_g}(f_1 \boxplus f_2) : H(U_x) \boxplus H(U_x) \to H(U_x) \boxplus H(U_x)$ as $(C_\varphi \boxplus M_{\pi_g})(f_1 \boxplus f_2) = C_\varphi f_1 \boxplus M_{\pi_g} f_2$. Then certainly $C_\varphi \boxplus M_{\pi_g}$ is a linear transformation on $H(U_X) \boxplus H(U_x)$ and it is called the tensor sum operator and weighted transormation on $H(U_x)$ induced by φ and π. If $\varphi(x) = x$, then M_{π_g} is the multiplication transformation $M_\pi f = \pi.f$ and if $\pi = 1$, then $C_\varphi \boxplus M_{\pi_g}$ is the tensor sum operator on C_φ and M_{π_g}. The class multiplication transformations and the class of tensor sum transformations on $H(U_x)$ are contained in the class of weighted tensor sum operator. If $H(U_x)$ has topology on it and C_φ and M_π are continuous, then they are called tensor sum operator and multiplication operator respectively. These operator have been studied extensively during last five decades or so and play significant roles in study of dynamical systems , semi-groups of operators, Cauchy problems and Wavelet theory for details we refer to [34,69,79].

In this article we present an application of weighted tensor sum and tensor product operator and multiplier representation of Lie groups. This establishes a connection between classical mathematics and modern mathematics.

Definition 1. By an $n-$dimensional differentiable (real) manifold, we mean a Housdorff topological space which is connected and each point has a neighbourhood holomorphic to some open subset of $\mathbb{R}^n$ similarly, we can define $n-$dimensional differentiable complex manifold.

Definition 2. Let X be an $n-$dimensional differentiable manifold and let $C^\infty(X)$ be the algebra of all $C^\infty-$functions on

X. Then a tangent vector at a point $p \in X$ is a linear map $T_P : C^\infty(X) \to \mathbb{R}$ such that $T_P(f_1 f_2) = f_1(P)T_P f_2 + f_2(P)T_P f_1$, for all $f_1, f_2 \in C^\infty(X)$

The set $T_P(X)$ of all tangent vectors at P is called the tangent space at P and the disjoint union of the tangent spaces of X. i.e., $TX = \cup_{P\in X} T_P X$ is called the tangent bundle of the differentiable manifold X.

A vector field on X is a smooth map $V : X \to TX$ such that the image of P denoted by V_P, lies in $T_P X$, the tangent space at P. The vector field is smooth if for every $f \in C^\infty$, the function $Vf : X \to \mathbb{R}$ defined by $Vf(P) = V_P(f)$ is smooth on X. The set of all smooth vector fields on X is denoted by $\chi(X)$, which is also a vector space.

Definition 3. Let X and Y be two differentiable manifolds and $\varphi : X \to Y$ be a smooth map. Then the composition transformation $C_\varphi : C^\infty(Y) \to C^\infty(X)$ is given by $C_\varphi(f) = f \circ \varphi$. It is clear that for every $f \in C^\infty(Y)$, $f \circ \varphi \in C^\infty(X)$ and C_φ is algebra homomorphism . This homomorphism is some times called pull-back map it is denoted as φ^*. Suppose $p \in X$ and $\varphi(p) = q \in Y$. Then define the map $\psi_p : T_p(X) \to T_q(Y)$ as $\psi_p f = T_p(C_\varphi f) = T_p(\varphi^* f)$. This map ψ_p is called differential map and denoted as $d\varphi_p$.

Definition 4. A Lie group is non-empty set G with a binary operation satisfying the following conditions.
1. G is a group (with identity enement e)
2. G is smooth manifold
3. G is a topological group. In particular, the group operation $G \times G \to G$ and the inverse map $G \to G$ are smooth.

By a Lie transformation group, we mean the triple (X, G, π), where G is a Lie group. X is a differentiable manifold and $\pi : X \times G \to X$ is a map satisfying the following conditions.
1. $\pi(x, e) = x$, for all $x \in X$
2. $\pi(x, gh) = \pi(\pi(x, g), h)$
3. π is analytic in x and g. The map π is called an action of G on X or a motion on X induced by G.

We say that G acts on X effectively if for every $g \in G, g \neq e$, there exists $p \in X$ such that $\pi(p, g) \neq p$ and π is said to act freely on X if for every $g \in G$, $g \neq e$ and for every $p \in X$, $\pi(p, g) \neq p$.

2. Weighted tensor sum operator induced by a Lie transformation group

Let (U_x, G, π) be a Lie tansformation group, where U_x is an open subset of C^n, G is a Lie group and π is an action of G on U_x. ie., $\pi : U_x \times G \to U_x$ is a continuous map such that $\pi(x, e) = x$ and $\pi(x, gh) = \pi(\pi(x, g), h)$ for all $x \in U_x$ and $g, h \in G$, where e is the identity element of G. For $g \in G$, let the map $\pi_g : U_x \to U_x$ be defined as $\pi_g(x) = \pi(x, g)$, for every $x \in U_x$. The map π_g is a diffiomorphism and $\pi_g^{-1} = \pi_{g^{-1}}$. This π_g induces the tensor sum operator C_{π_g} on $H(U_x)$. We denote this transformation by C_g. Clearly $C_{gh} = C_g C_h$. Let $C_G = \{C_g : g \in G\}$. Then C_g is a group under tensor sum operator. The map $g \to C_g$ is a homomorphism from G to C_G. Define $\pi_G : B(H(U_x)) \times C_G \to B(H(U_x))$ as $\pi_G(A, C_g) = AC_g$, where $B(H(U_x))$ is the vector space of all linear transformation on $H(U_x)$. Then π_G is an action of C_G on $B(H(U_x))$. Similarly, G acts on $H(U_x)$ with action π' induced by tensor sum operator. ie., $\pi' : H(U_x) \times G \to H(U_x)$ is defined as $\pi'(f_1 \boxplus f_2) = C_g(f_1 \boxplus f_2) = C_g f_1 \boxplus C_g f_2$. Thus a Lie group action on U_x gives rise to an action of G on $H(U_x)$ gives rise to an action of G on $H(U_x)$ and an action on $B(H(U_x))$ induced by the tensor sum operator.

For $x \in U_x$, the map $\pi^x : G \to U_x$ as $\pi^x(g) = \pi(x, g)$, for every $g \in G$. The range of $\pi^x = \{\pi(x, g) : g \in G\}$ and it is called the orbit of x. We denote it by O^x and the collection $\{O^x : x \in U_x\}$ is called the orbit space.

Let $H_b(U_x)$ denote the space of all bounded analytic functions and $H_0(U_x)$ denote the space of analytic functions vanishing at infinity. Then $H_b(U_x)$ and $H_0(U_x)$ are Banach spaces of analytic functions with norm defined as
$||f_1 \boxplus f_2|| = \sup\{||f_1(x)|| + ||f_2(x)|| : x \in X\}$
Then it has basis of closed absolutely neighbourhoods of the origin of the form
$B = \{f_1 \boxplus f_2 : f_1, f_2 \in B(H(U_x)) : ||f_1 \boxplus f_2|| \leq 1\}$
Let $v : U_x \times G \to \mathbb{C}$ be an analytic map and let $v_h(x) = v(x, h)$ for $x \in X$. For $h, g \in G$, define the map $v_h C_g : H(U_x) \to H(U_x)$ as $(v_h C_g(f_1 \boxplus f_2)) = v_h f_1 \circ \pi_g \boxplus v_h f_2 \circ \pi_g$.
Then $v_h C_g$ is weighted tensor sum operator on $H(U_x)$ induced by the pair (h, g). Consider the map $\hat{T} : G \to B(H(U_x))$ defined by $\hat{T}(g) = v_h C_g$. We see later that $\hat{T}$ is a homomorphism under certain condition on v. Thus every element g of G gives rise to a weighted tensor sum operator and transformations with weighted function v_h. In general, every element $(h, g) \in G \times G$ gives rise to weighted tensor sum operator $v_h C_g$. Thus the mapping $T' : G \times G \to B(H(U_x))$ defined as $T'(h, g) = v_h C_g$. Clearly $T'(h, g) = v_h C_g = T(g)$ and $T'(h, e) = M_{v_h}$(multiplication transformation induced by v_h).

Theorem 1. *Let (U_x, G, π) be a Lie transformation group and $v : U_x \times G \to \mathbb{C}$ be an analytic map. Then*

(i) C_g is a coninuous operator on $H_b(U_x)$ and $C_{gh} = C_g C_h$.

(ii) $v_h C_g$ is continuous on $H_b(U_x)$ if v_h is bounded.

(iii) C_g is invertible , for every $g \in G$.

(iv) $v_h C_g$ is invertible if $v_h \neq 0$.

(v) The map $\varphi : G \to B(H(U_x))$ defined by $\varphi(g) = C_g$ is a homomorphism and hence it is a representation of G on $H(U_x)$.

Proof.
(i). Since $||C_g(f_1 \boxplus f_2)|| = ||C_g(f_1)|| \boxplus ||C_g(f_2)||$
$= \sup_{x\in U_x}\{||(f_1 \circ \pi_g)(x)|| \boxplus ||(f_2 \circ \pi_g)(x)||$ for every $x \boxplus x \in U_x \boxplus U_x\}$
$\leq \sup_{x\in U_x} ||f_1(\pi_g(x))|| \boxplus ||f_2(\pi_g(x))||$

$\leq \sup_{x\in U_x} ||(f_1(x)|| \boxplus \sup_{x\in U_x} ||(f_2(x)||$
$\leq ||f_1|| \boxplus ||f_2||$
$\leq ||f_1 \boxplus f_2||$
≤ 1
Therefore C_g is continuous.
(ii). $||v_hC_g(f_1 \boxplus f_2)|| = ||v_hC_g(f_1)|| \boxplus ||v_hC_g(f_2)||$
$= \sup_{x\in U_x}\{||v_h(x)(f_1 \circ \pi_g)(x)|| \boxplus ||v_h(x)(f_2 \circ \pi_g)(x)||\}$
$\leq \sup_{x\in U_x} ||v_h(x)f_1(\pi_g(x))|| \boxplus \sup_{x\in U_x} ||v_h(x)(f_2(\pi_g(x))||$
$\leq \sup||v_h(x)|| \sup||f_1(x)|| \boxplus \sup||v_h(x)|| \sup||f_2(x)||$
$\leq C||f_1(x)|| \boxplus C||f_2(x)||$
$\leq C||f_1 \boxplus f_2||$, for all $x \in U_x$
$\leq ||f_1 \boxplus f_2||$
≤ 1
where C is a positive constant.
(iii). $C_g^{-1} = C_{g^{-1}}$
Since $C_{g^{-1}}C_g(f_1 \boxplus f_2) = C_{g^{-1}}C_g(f_1) \boxplus C_{g^{-1}}C_g(f_2)$
$= C_{g^{-1}}(f_1 \circ \pi_g) \boxplus C_{g^{-1}}(f_2 \circ \pi_g)$
$= f_1 \circ \pi_g \circ \pi_{g^{-1}} \boxplus f_2 \circ \pi_g \circ \pi_{g^{-1}}$
$= f_1 \boxplus f_2$
(iv). *ker* $v_hC_g = \{f_1 \boxplus f_2 : v_hC_g(f_1 \boxplus f_2) = v_hC_gf_1 \boxplus v_hC_gf_2 = 0\} = \{0\}$
For $v_hC_g(f_1 \boxplus f_2) = 0$
$\Rightarrow v_hC_g(f_1) \boxplus v_hC_g(f_2) = 0$
$\Rightarrow v_h(f_1 \circ \pi_g)(x) \boxplus v_h(f_2 \circ \pi_g)(x) = 0$
$\Rightarrow f_1(x) \boxplus f_2(x) = 0$
$\Rightarrow f_1 \boxplus f_2(x) = 0$
Since $v_h(x) \neq 0$, for every $x \in U_x$,
Let $f_1, f_2 \in H(U_x)$ and let $f_1' = \frac{f_1 \circ \pi_{g^{-1}}}{v_h \circ \pi_{g^{-1}}}$ and $f_2' = \frac{f_2 \circ \pi_{g^{-1}}}{v_h \circ \pi_{g^{-1}}}$.
Then $f_1', f_2' \in H(U_x)$ and $v_hC_gf_1' = f_1$ and $v_hC_gf_2' = f_2$.
Hence v_hC_g is invertible.
(v). $\varphi_(gh)(f_1 \boxplus f_2) = \varphi_(gh)(f_1) \boxplus \varphi_(gh)(f_2)$
$= C_{gh}f_1 \boxplus C_{gh}f_2$
$= f_1 \circ \pi_{gh} \boxplus f_2 \circ \pi_{gh}$
$= f_1 \circ \pi_h \circ \pi_g \boxplus f_2 \circ \pi_h \circ \pi_g$
$= C_gC_hf_1 \boxplus C_gC_hf_2$
$= C_gC_h(f_1 \boxplus f_2)$
$= \varphi_g\varphi_h(f_1 \boxplus f_2)$ □

Note 2. If $A(U_x)$ denote the vector space of all complex - valued functions on U_x which are analytic in some neighbourhood of 0, then most of results reported, so for are true in case of $A(U_x)$. Clearly $H(U_x)$ is contained in $A(U_x)$.

Definition 5. An analytic mapping $v : U_x \times G \to \mathbb{C}$ is called a co-cycle over G if $v(x,e) = 1$ and $v_{gh} = v_g.\ \ v_h \circ \pi_g$ for every g and h in G. A co-cycle v is called co-boundary if $v_g(x) = \frac{\beta(\pi_g(x))}{\beta(x)}$, i.e., $v_g = \frac{\beta \circ \pi_g}{\beta}$, for some $0 \neq \beta \in H(U_x)$. In case v is a co-boundary, the weighted tensor sum operator v_gC_g is given by
$v_gC_g(f_1 \boxplus f_2) = v_gC_g(f_1) \boxplus v_gC_g(f_2)$
$= \beta^{-1}C_g(\beta f_1) \boxplus \beta^{-1}C_g(\beta f_2)$

In the following theorem, we present a representation of Lie group G in terms of the weighted tensor sum operator.

Theorem 3. *Let (U_x, G, π) be a Lie transformation group and let v be a co-cycle over G. Then v_gC_g is a weighted tensor sum operator on $H(U_x)$ and the mapping $\hat{T} : G \to B(H(U_x))$ given by $\hat{T}(g) = v_gC_g$ is a multiplier representation of G, where $B(H(U_x))$ is the algebra of all linear transformation on $H(U_x)$*

Proof. We have already seen that v_hC_g is a weighted tensor sum operator on $H(U_x)$. We shall show that the mapping $\hat{T}$ is a representation.
(i)Since $[\hat{T}(g)(f_1 \boxplus f_2)](x) = [\hat{T}(g)f_1]x \boxplus [\hat{T}(g)f_2](x)$
$= [v_gC_gf_1](x) \boxplus [v_gC_gf_2](x)$for every $f_1, f_2 \in H(U_x)$, $g \in G$ and $x \in U_x$
We have $[\hat{T}(e)(f_1 \boxplus f_2)](x) = [\hat{T}(e)f_1](x) \boxplus [\hat{T}(e)f_2](x)$
$= v_e(C_ef_1)(x) \boxplus v_e(C_ef_2)(x)$
$= v_e(x)(C_ef_1)(x) \boxplus v_e(x)(C_ef_2)(x)$
$= v(x,e)f_1(\pi(x,e)) \boxplus v(x,e)f_2(\pi(x,e))$
$= f_1(x) \boxplus f_2(x)$
$= (f_1 \boxplus f_2)(x)$
Hence $\hat{T}(e) = 1$.
(ii) Let g and h be in G. Then for $f_1, f_2 \in H(U_x)$
$\hat{T}(gh)(f_1 \boxplus f_2) = \hat{T}(gh)(f_1) \boxplus \hat{T}(gh)(f_2)$
$= v_{gh}c_{gh}f_1 \boxplus v_{gh}c_{gh}f_2$
$= v_g.v_h \circ \pi_g.f_1 \circ \pi_h \circ \pi_g \boxplus v_g.v_h \circ \pi_g.f_2 \circ \pi_h \circ \pi_g$
$= v_g.(v_h.f_1 \circ \pi_h) \circ \pi_g \boxplus v_g.(v_h.f_2 \circ \pi_h)$
$= v_g(\hat{T}(h)f_1) \circ \pi_g \boxplus v_g(\hat{T}(h)f_2 \circ \pi_g)$
$= \hat{T}(g)\hat{T}(h)f_1 \boxplus \hat{T}(g)\hat{T}(h)f_2$
$= \hat{T}(g)\hat{T}(h)(f_1 \boxplus f_2)$
Thus $\hat{T}(gh) = \hat{T}(g)\hat{T}(h)$. This shows that $\hat{T}$ is a homomorphism and hence a multiplier representation of G induced by co-cycle v. □

Note 4. If v is a co-boundary over G, then representation $\hat{T}$ is given by
$[\hat{T}(g_1 \boxplus g_2)](x) = \hat{T}(g_1) \boxplus \hat{T}(g_2) = \frac{\beta \circ \pi_{g_1}}{\beta}C_{g_1} \boxplus \frac{\beta \circ \pi_{g_2}}{\beta}C_{g_2}$,
which is also a weighted tensor sum operator.

Definition 6. Let G be a local Lie group and U_x be a open subset of $\mathbb{C}^n$. Suppose π is a mapping form $U_x \times G \to \mathbb{C}^n$. Then G acts on U_x as a local Lie transformation group if

1. $\pi(x,g)$ is analytic in x and g.
2. $\pi(x,e) = x$ for all $x \in U_x$
3. $\pi(x,gh) = \pi(\pi(x,g),h)$ if $\pi(x,g) \in U_x$

It is evident that π_g is locally an injection for g in small neighbourhood of e. We denote this local Lie transformation group by triple $(U_x, G, \pi)_l$
Let $(U_x, G, \pi)_l$ be a (local) Lie transformation group acting on an open neighbourhood U_x of $\mathbb{C}^n, 0 \in U_x$ and let $A(U_x)$ be the vector space of all complex-valued functions on U_x analytic in a neighbourhood of 0. Then a (local) multilplier representation $\hat{T}$ of G on $A(U_x)$ with multiplier v, consists of a mapping $\hat{T}(g)$ on $A(U_x)$ defined for $g \in G, f_1, f_2 \in A(U_x)$

by
$[\hat{T}(g)(f_1 \boxplus f_2)](x) = [\hat{T}(g)f_1]x \boxplus [\hat{T}(g)f_2](x)$
$= v_g(x)f_1(\pi_g(x)) \boxplus v_g(x)f_2(\pi_g(x)); x \in U_x, \pi_g(x) \in U_x$ and
$\pi_g(x) = \pi(x,g)$
In the following theorem we generalise Theorem *2* to give a multiplier representation of local Lie group in terms of weighted tensor sum operator.

Theorem 5. *Let $(U_x, G, \pi)_l$ be a local Lie transformation group and let $v : U_x \times G \to \mathbb{C}$ be a co-cycle over G. Let $A(U_x)$ be the vector space of all complex-valued functions on U_x which are analytic on a neighbourhood of 0. Then $f_1 \boxplus f_2 \to v_g.C_g f_1 \boxplus v_g.C_g f_2$ is a weighted tensor sum operator on $A(U_x)$ and $g \to v_g C_g$ is a (local) multiplier representation of G on $A(U_x)$.*

Proof. The proof runs parallel to the proof of Theorem *2*. □

Note 6. $(C_g(f_1 \boxplus f_2))(x) = f_1(\pi_g(x)) \boxplus f_2(\pi_g(x))$, whenever $\pi_g(x) \in U_x$, since π_g is locally an injection, $f_1 \circ \pi_g \boxplus f_2 \circ \pi_g \in A(U_x)$.

Definition 7. Let G be a local Lie group. Then the Lie algebra $L(G)$ of the local Lie group G it is the set of all tangent vectors at e equipped with the operations of vector addition and Lie product. Let $\hat{T}$ be a multiplier representation of G and let $f_1 \boxplus f_2 \in A(U_x)$. If $\alpha \in L(G)$, then $\hat{T}_{exp\ \alpha t}(f_1 \boxplus f_2) \in A(U_x)$ for sufficiently small values of $|t|$.

The generalised Lie derivative $D_\alpha(f_1 \boxplus f_2)$ of an analytic function $f_1 \boxplus f_2$ under 1-parameter group $exp\ \alpha t, t \in \mathbb{R}$ is the analytic function.
$D_\alpha(f_1 \boxplus f_2)(x) = D_\alpha f_1(x) \boxplus D_\alpha f_2(x)$
$= \frac{d}{dt}\left[\hat{T}_{exp\ \alpha t} f_1\right](x) \boxplus \frac{d}{dt}\left[\hat{T}_{exp\ \alpha t} f_2\right](x)$ at $t = 0$.
In case $v = 1$, the generalised Lie derivative becomes the ordinary Lie derivative.

We shall cite the following example from [*4*], to illustrate Theorem *3*.

Example 7. Let $G = SL(2)$, the Lie group of 2×2 matrices with determinant 1. Let U_x be an open subset of $\mathbb{C}^n$ and v be the cocycle over G defined by $v(z,g) = (bz+d)^{2n}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$ and $2n$ is not a non-negative integer. If π is the map from $U_x \times G \to \mathbb{C}$ defined by $\pi(z,g) = \frac{az+c}{bz+d}$, then the multiplier representation of $SL(2)$ with v as multiplier is given by the following weighter tensor sum operator.
$[\hat{T}_g(f_1 \boxplus f_2)](z) = v_g(z)f_1(\pi_g(z)) \boxplus v_g(z)f_2(\pi_g(z))$
$= v(z,g)(C_g f_1)(z) \boxplus v(z,g)(C_g f_2)(z)$
$= v(z,g)f_1(\pi(z,g)) \boxplus v(z,g)f_2(\pi(z,g))$
$= (bz+d)^{2n} f_1\left(\frac{az+c}{bz+d}\right) \boxplus (bz+d)^{2n} f_2\left(\frac{az+c}{bz+d}\right)$
$= (bz+d)^{2n}\left[f_1\left(\frac{az+c}{bz+d}\right) \boxplus f_2\left(\frac{az+c}{bz+d}\right)\right]$
$= (bz+d)^{2n}(f_1 \boxplus f_2)\left(\frac{az+c}{bz+d}\right)$

3. Dynamical System Induced by Weighted tensor sum operator and Lie group representations

Let X be a Complex Banach space and U_x be a balanced open subset of Banach space x and y. Let $\pi_g : U_x \times \mathbb{C}$ defined by $\pi_g(x) = e^{gh(x)}$ for all $g \in \mathbb{R}$ and $x \in U_x$, where $h \in B(H(U_x))$ and $\|h\|_\infty = sup\{\|h(x)\| : x \in U_x\}$. Also $\varphi_t : R \to R$ is defined $\varphi_t(w) = t + w$ the self-map.

Theorem 8. *Let (U_x, G, π) be a Lie transformation group and $v : U_x \times G \to \mathbb{C}$ be an analytic map. Then*

(i) $C_g \boxplus M_{\pi_g}$ is a continuous operator on $H_b(U_x) \boxplus H_b(U_x)$ and $C_{gh} \boxplus M_{\pi_g} \le 1$.

(ii) $v_h(C_g \boxplus M_{\pi_g})$ is continuous on $H_b(U_x) \boxplus H_b(U_x)$ if v_h is bounded.

(iii) $C_g \boxplus M_{\pi_g}$ is invertible for every $g \in G$.

(iv) $v_h(C_g \boxplus M_{\pi_g})$ is invertible if $v_h \ne 0$.

(v) The map $\varphi : G \to B(H(U_x)) \boxplus B(H(U_x))$ defined by $\varphi(g) = C_g \boxplus M_{\pi_g}$ is a homomorphism and hence it is a representation of G on $H(U_x) \boxplus H(U_x)$.

Proof. (i) Since $\|C_g f_1 \boxplus M_{\pi_g} f_2\| = \sup \|C_g f_1\| + \|M_{\pi_g} f_2\|$ for every $t \in \mathbb{R}$
$\le \sup \|f_1 \circ \pi_g(x)\| + \|\pi_g(x)\| \|f_2(x)\|$ for every $x \boxplus x \in U_x \boxplus U_x$
$\le \sup \|f_1(x)\| \left\|e^{gh(x)}\right\| \|f_2(x)\|$
$\le \sup \|f_1(x)\| \left\|e^{|g| \|h\|_\infty}\right\| \|f_2(x)\|$
$\le \sup \|f_1(x)\| \|f_2(x)\|$
$\le \|(f_1 \boxplus f_2)(x)\|$
≤ 1
Therefore $C_g \boxplus M_{\pi_g}$ is continuous at the origin. Hence proved.
(ii) Since $\|v_h(C_g f_1 \boxplus M_{\pi_g} f_2)\| = \sup \|v_h(x)\| \left(\|C_g f_1\| + \|M_{\pi_g} f_2\|\right)$ for every $t \in \mathbb{R}$
$\le \sup \|v_h(x)\| \|f_1 \circ \pi_g(x)\| \cdot \|\pi_g(x)\| \|f_2(x)\|$
$\le \sup \|v_h(x)\| \|f_1(\pi_g(x))\| \left\|e^{gh(x)}\right\| \|f_2(x)\|$
$\le \sup \|v_h(x)\| \|f_1(x)\| \left\|e^{|g| \|h\|_\infty}\right\| \|f_2(x)\|$
$\le c \|f_1(x)\| \|f_2(x)\|\ as\ t \to 0$
$\le c \|(f_1 \boxplus f_2)(x)\| \le 1$ where c is positive constant.
(iii) $C_g^{-1} = C_{g^{-1}}$.
Since $\left(C_g^{-1}(C_g f_1 \boxplus M_{\pi t} f_2)\right) = \left(C_g^{-1} C_g f_1\right) \boxplus \left(C_g^{-1} M_{\pi_t} f_2(x)\right)$
$= \left(f_1 \circ \pi_g \circ \pi_{g^{-1}}\right) \boxplus \pi_g(x)\left(f_2 \circ \pi_{g^{-1}}\right)$
$= f_1 \boxplus f_2$
(iv) $ker\ v_h(C_g \boxplus M_{\pi_g}) = \{(f_1 \boxplus f_2) : \|v_h(C_g f_1 \boxplus M_{\pi_g} f_2)\| = 0\} = \{0\}$
For $\|v_h(C_g f_1 \boxplus M_{\pi_g} f_2)\| = 0$
$\Rightarrow\ v_h\left((C_g f_1(x)) \boxplus (M_{\pi g f_2(x)})\right) = 0$
$\Rightarrow\ v_h((f_1 \circ \pi_g(x)) \boxplus (\pi_g(x) f_2(x))) = 0$
$\Rightarrow\ v_h(f_1 \circ \pi_g \boxplus f_2 \circ \pi_g) = 0$
$\Rightarrow\ v_h((f_1 \boxplus f_2) = 0$
Since $v_h(x) \ne 0$, for every $x \in U_x$.

Let $f_1 \boxplus f_2 \in H(U_x) \boxplus H(U_x)$ and let $f_1' = \frac{f_1 \circ \pi_{g^{-1}}}{v_h \circ \pi_{g^{-1}}}$ and $f_2' = \frac{f_2 \circ \pi_{g^{-1}}}{v_h \circ \pi_{g^{-1}}}$.
Then $f_1' \boxplus f_2' \in H(U_x) \boxplus H(U_x)$ and $v_h(C_g f_1' \boxplus M_{\pi_g} f_2') = f_1 \boxplus f_2$.
Hence $v_h(C_g \boxplus M_{\pi_g})$ is invertible.
(v) $\varphi(gh)(f_1 \boxplus f_2) = (C_{gh} \boxplus M_{\pi_g})(f_1 \boxplus f_2)$
$= C_{gh} f_1 \boxplus M_{\pi_{gh}} f_2$
$= f_1 \circ \pi_{gh} \boxplus f_2 \circ \pi_{gh}$
$= f_1 \circ \pi_h \circ \pi_g \boxplus f_2 \circ \pi_h \circ \pi_g$
$= C_g C_h f_1 \boxplus C_g C_h f_2 = \varphi(g)\varphi(h)(f_1 \boxplus f_2)$ □

Definition 8. An analytic mapping $v : U_x \times G \to \mathbb{C}$ is called a co-cycle over G if $v(x,e) = 1$ and $v_{gh} = v_g \cdot v_h \circ \pi_g$ for every g and h in G. A Cocycle v is called co-boundary if $v_g(x) = \frac{\beta(\pi_g(x))}{\beta(x)}$ i.e., $v_g = \frac{\beta_0 \pi_g}{\beta}$ for some $0 \neq \beta \in H(U_x)$. In case v is a co-boundary, the weighted tensor sum operator $v_g(C_g \boxplus M_{\pi_g})$ is given be $v_g C_g f_1 \boxplus v_g M_{\pi_g} f_2 = \beta^{-1} C_g(\beta f_1) \boxplus \beta^{-1} M_{\pi_g}(\beta f_2)$.

In the following theorem we present a representation of Lie group G in terms of the Dynamical system and weighted tensor sum operator.

Theorem 9. *Let (U_x, G, π) be a Lie transformation group and let v be a co-cycle over G. Then $v_g(C_g \boxplus M_{\pi_g})$ is a dynamical system and weighted tensor sum operator on $H(U_x) \boxplus H(U_x)$ and the mapping $\hat{T} : G \to B(H(U_x)) \boxplus B(H(U_x))$ given by $\hat{T}(g) = v_g(C_g \boxplus M_{\pi_g})$ is a multiplier representation of G, where $B(H(U_x)) \boxplus B(H(U_x))$ is the algebra of all linear transformation on $H(U_x) \boxplus H(U_x)$.*

Proof. We have already seen that $v_g(C_g \boxplus M_{\pi_g})$ is a dynamical system and weighted sum operator on $H(U_x) \boxplus H(U_x)$. We shall show that the mapping $\hat{T}$ is a representation. Since
$\left[(\hat{T}(g)(f_1 \boxplus f_2)\right](x) = v_(g)\left[C_g f_1 \boxplus M_{\pi_g} f_2)(x)\right]$
$= v_g\left[f_1 \circ \pi_g(x)\right] \boxplus \left[\pi_g f_2(x)\right]$
$= v_g\left[f_1(x) \boxplus f_2(x)\right]$
$= v_g(x)(f_1 \boxplus f_2)(x)$for every $f_1 \boxplus f_2 \in H(U_x) \boxplus H(U_x), g \in G$ *and* $x \in U_x$ *as* $t \to 0$
We have $\left[\hat{T}(e)(f_1 \boxplus f_2)\right](x) = (v_e(C_e f_1 \boxplus M_{\pi_e} f_2))(x)$
$= v_e(x)\left[C_e f_1(x) \boxplus M_{\pi_e} f_2(x)\right]$
$= v_e(x)\left[f_1 \circ \pi_e(x) \boxplus \pi_e(x) f_2(x)\right]$
$= v_e(x)\left[f_1(\pi(x,e)) \boxplus f_2 \circ \pi_e(x)\right]$
$= v(x,e)(f_1 \boxplus f_2)(x)\ for\ every\ x \in U_x$
$= (f_1 \boxplus f_2)(x)$ Hence $\hat{T}(e) = 1$.
2. Let g and h be in G. Then for $f_1 \boxplus f_2 \in H(U_x) \boxplus H(U_x)$,
$\hat{T}(gh)(f_1 \boxplus f_2)(x) = v_{gh}(C_{gh} f_1 \boxplus M_{\pi_{gh}} f_2)$
$= v_g v_h \circ \pi_g\left((f_1 \circ \pi_{gh}) \boxplus (f_2 \circ \pi_{gh})\right)$
$= v_g \cdot v_h \circ \pi_g \cdot f_1 \circ \pi_h \circ \pi_g \boxplus v_g \cdot v_h \cdot \pi_g \cdot f_2 \circ \pi_h \circ \pi_g$
$= v_g \cdot (v_h \cdot f_1 \circ \pi_h) \circ \pi_g \boxplus v_g \cdot (v_h \cdot f_2 \circ \pi_h) \circ \pi_g$
$= v_g \cdot (\hat{T}(h) f_1) \circ \pi_g \boxplus v_g \cdot (\hat{T}(h) f_2) \circ \pi_g$
$= \hat{T}(g)\hat{T}(h)(f_1 \boxplus f_2)$
Thus $\hat{T}(gh) = \hat{T}(g)\hat{T}(h)$. This shows that $\hat{T}$ is a homomorphism and hence a multiplier representation of G induced by co-cycle v. □

Note 10. If v is a co-boundary over G, then representation $\hat{T}$ is given by $\hat{T}(g) = \frac{\beta \circ \pi_g}{\beta}(C_g \boxplus M_{\pi_g})$, which is also a dynamical system and weighted tensor sum operator.

Theorem 11. *Let $(U_x, G, \pi)_l$ be a local Lie transformation group and let $v : U_x \times G \to \mathbb{C}$ be a co-cycle over G. Let $A(U_x)$ be the vector space of all complex valued functions on U_x which are analytic on a neighbourhood of 0. Then $f_1 \boxplus f_2 \to v_g.(C_g f_1 \boxplus M_{\pi_g} f_2)$ is a weighted tensor sum operator on $A(U_x)$ and $g \to v_g(C_g \boxplus M_{\pi_g})$ is a local multiplier representation of G on $A(U_x)$.*

Proof. The proof runs parallel to the proof of Theorem *4*. □

Note 12. $(C_g f_1 \boxplus M_{\pi_g} f_2)(x) = C_g f_1(x) \boxplus M_{\pi_g} f_2(x)$, whenever $\pi_g(x) \in U_x$, since π_g is locally an injection.

Theorem 13. *Let (U_x, G, π) be a Lie transformation group and $v : U_x \times G \to \mathbb{C}$ be an analytic map. Then*

1. *$M_{\pi_g} \boxplus C_g$ is a continuous operator on $H_b(U_x) \boxplus H_b(U_x)$ an $M_{\pi_g} \boxplus C_g \leq 1$.*
2. *$v_h(M_{\pi_g} \boxplus C_g)$ is continuous on $H_b(U_x) \boxplus H_b(U_x)$ if v_h is bounded.*
3. *$M_{\pi_g} \boxplus C_g$ is invertible for every $g \in G$.*
4. *$v_h(M_{\pi_g} \boxplus C_g)$ is invertible if $v_h \neq 0$*
5. *The map $\varphi : G \to B(H(U_x)) \boxplus B(H(U_x))$ defined by $\varphi(g) = M_{\pi_g} \boxplus C_g$ is a homomorphism and hence it is a representation of G on $H(U_x) \boxplus H(U_x)$*

Proof. The proof runs similar to the proof of the Theorem *8*. □

Theorem 14. *Let (U_x, G, π) be a Lie transformation group and let v be a co-cycle over G. Then $v_g(M_{\pi_g} \boxplus C_g)$ is a weighted tensor sum operator on $H(U_x) \boxplus H(U_x)$ and the mapping $\hat{T} : G \to B(H(U_x))$ given by $\hat{T}(g) = v_g(M_{\pi_g} \boxplus C_g)$ is a multiplier representation of G, where $B(H(U_x)) \boxplus B(H(U_x))$ is the algebra of all linear transformation on $H(U_x) \boxplus H(U_x)$.*

4. Dynamical system induced by weighted tensor product operator and Lie group representations

Theorem 15. *Let (U_x, G, π) be a Lie transformation group and $v : U_x \times G \to \mathbb{C}$ be an analytic map. Then*

1. *$C_g \otimes M_{\pi_g}$ is a continuous operator on $H_b(U_x) \otimes H_b(U_x)$ and $C_g \otimes M_{\pi_g} \leq 1$.*
2. *$v_h(C_g \otimes M_{\pi_g})$ is continuous on $H_b(U_x) \otimes H_b(U_x)$ if v_h is bounded.*
3. *$C_g \otimes M_{\pi_g}$ is invertible for every $g \in G$.*

4. *$v_h(C_g \otimes M_{\pi_g})$ is invertible if $v_h \neq 0$.*

5. *The map $\varphi : G \to B(H(U_x)) \otimes B(H(U_x))$ defined by $\varphi(g) = C_g \otimes M_{\pi_g}$ is a homomorphism and hence it is a representation of G on $H(U_x) \otimes H(U_x)$.*

Theorem 16. *Let (U_x, G, π) be a Lie transformation group and let v be a co-cycle over G. Then $v_g(C_g \otimes M_{\pi_g})$ is a dynamical system and weighted tensor product operator on $H(U_x) \otimes H(U_x)$ and the mapping $\hat{T} : G \to B(H(U_x)) \otimes B(H(U_x))$ given by $\hat{T}(g) = v_g(C_g \otimes M_{\pi_g})$ is a multiplier representation of G, where $B(H(U_x)) \otimes B(H(U_x))$ is the algebra of all linear transformation on $H(U_x) \otimes H(U_x)$.*

Theorem 17. *Let $(U_x, G, \pi)_l$ be a local Lie transformation group and let $v : U_x \times G \to \mathbb{C}$ be a co-cycle over G. Let $A(U_x)$ be the vector space of all complex valued functions on U_x which are analytic on a neighbourhood of 0. Then $f_1 \otimes f_2 \to v_g.(C_g f_1 \otimes M_{\pi_g} f_2)$ is a weighted tensor product operator on $A(U_x)$ and $g \to v_g(C_g \otimes M_{\pi_g})$ is a local multiplier representation of G on $A(U_x)$.*

Theorem 18. *Let (U_x, G, π) be a Lie transformation group and $v : U_x \times G \to \mathbb{C}$ be an analytic map. Then*

1. *$M_{\pi_g} \otimes C_g$ is a continuous operator on $H_b(U_x) \otimes H_b(U_x)$ an $M_{\pi_g} \otimes C_{gh} \leq 1$.*

2. *$v_h(M_{\pi_g} \otimes C_g)$ is continuous on $H_b(U_x) \otimes H_b(U_x)$ if v_h is bounded.*

3. *$M_{\pi_g} \otimes C_g$ is invertible for every $g \in G$.*

4. *$v_h(M_{\pi_g} \otimes C_g)$ is invertible if $v_h \neq 0$*

5. *The map $\varphi : G \to B(H(U_x)) \otimes B(H(U_x))$ defined by $\varphi(g) = M_{\pi_g} \otimes C_g$ is a homomorphism and hence it is a representation of G on $H(U_x) \otimes H(U_x)$*

References

[1] Chernoff. P and Marsden, On continuity and smoothness of group actions, *Bull. Amer. Math. Sac.*, 76(1970), 1044-1049

[2] H. W. Guggerheimer, *Differential Geometry*, McGraw Hill Book Company, India (1963).

[3] Hohn M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics Springer, New York, 2013.

[4] F. Jafari, Z. Slodkowski and T. Tonev, *Semigroups of operations on Hardy spaces and co-cycle of flows*, IMA Preprint series 2220, University of Minhessota, July 2008.

[5] J. S. Manhos, Weighted composition operators on weighted locally convex spaces of analytic function, *Southeast Asian Bull. Math.,* 29(1)(2005), 127-140

[6] W. Miller, Jr, *Lie theory and special functions*, Academic Press, New York, London, 1968.

[7] Senthilkumar and P. Chandra Kala, Operators and dynamical system in measurable function spaces, *Global Journal of Mathematical Sciences: Theory and Practical*, 4(2)(2012), 107–114.

[8] Senthilkumar and P. Chandra Kala, Tensor sum and dynamical systems, *Acta Mathematica Scientia*, 4(2)(2012), 115-120.

[9] Senthilkumar, P. Chandra Kala and T. Prasad, Tensor product operators induced Dynamical system on weighted locally convex space, *Jordan journal of Mathematics and Statistics*, 6(2013), 169-181.

[10] J. H. Shipro, *Composition Operators Classical Function Theory, Universal : Tracts in Mathematics*, springer, Newyork , NY, USA, 1993.

[11] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North Holland Mathematics studies, 1993.

[12] R. K. Singh and J. S. Manhas, Multiplication operators and Dynamical systems, *J. Austral. Math .Soc. (series A)*, 53(1992), 92–102.

[13] R. K. Singh and J. S. Manhas, Composition operator on weighted spaces of Vector-valued continuous function, *Acta. Sci. Math*, 1988, 1–10.

[14] R. K. Singh and D. C. Kumar, Weighted composition operators on functional Hilbert spaces, *Bull. Aust. Math. Soc.*, 31(1985), 117–126.

[15] R. K. Singh and J. S. Manhos, Multiplication operators on weighted spaces of vector valued continuous functions, *J. Aust. Math. Soc., (Series A)*, 50(1991), 98–107.

[16] R. K. Singh and J. S. Manhas, Weighted composition operators and weighted spaces of cross-sections, *Acta Scientiarum Mathematicarum*, 59(1994), 3-4, 1994.

[17] R. K. Singh and J. S. Manhas, Composition operators and Multiplication operators on weighted spaces of analytical functions, Research article, *International Journal of Mathematics and Mathematical Sciences*, 2007.

[18] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North-Holland Publishing Co., Amsterdom, London, N. Y. 1993.

[19] R. K. Singh and W. H. Summers, Composition operators on weighted spaces of continuous functions, *J. Austral. Math Soc. Series A*, 45 (1988), 303-319.

[20] R. K. Singh, V. Sahal and V. K. Pandey: Weighted substitution operators on $L^p(\mathbb{R})$ and wavelet theory, *Mathematics Newsletter (Ramanujam Math Soc)*, 21(2011), 2011, 6–16.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆