On the embedding of Γ-semigroup Amalgam

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Abstract

Γ semigroup is introduced as a generalization of semigroups by M. K Sen and Saha. In this paper we describe amalgam of two Γ- semigroups and discuss the embeddability of this amalgam. Further we obtained a necessary condition for the embeddability of completely α-regular Γ-semigroup amalgam.

Keywords

Γ- semigroups , Semigroup Amalgam , Γ-monomorphisms , free Γ-product.

AMS Subject Classification

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1. Preliminaries

In the following first we briefly recall amalgams of groups, semigroups and free product of semigroup amalgams, then we recall Γ- semigroups, the amalgam of Γ- semigroups and their properties as described by M K Sen and Saha [9].

A group amalgam consists of three groups G, K and H, called core of the amalgam and embeddings $m : H \to G$ and $n : H \to K$. The amalgam is said to be embeddable if we can find another group T which contains $G \cup K$ containing H. If $G \cup K = H$ in T, then amalgam is called strongly embeddable. Schreierr (1927) in [3] and [10] has shown that group amalgam is always embeddable. But all semigroup amalgams are not always embeddable.

Definition 1.1. *(cf.[6]) A semigroup amalgam $\mathscr{K} = [\{S_i; i \in I\} : U : \{\theta_i; i \in I\}]$ consists of a semigroup U, the core of the amalgam, a family of mutually disjoint semigroups $\{S_i : i \in I\}$ and a family of monomorphisms $\theta_i : U \to S_i$ $i \in I$.*

We say that the amalgam $\mathscr{K}$ is embedded in another semigroup T if there exists monomorphism $\mu : U \to T$ and

for each $i \in I$ a monomorphism $\mu_i : S_i \to T$ with

1. $\theta_i \mu_i = \mu$ for each $i \in I$.

2. $S_i \mu_i \cap S_j \mu_j = U \mu$ for all $i, j \in I$ such that $i \neq j$.

The amalgam is called weakly embeddable if it satisfies condition (1) only and is strongly embeddable if both conditions are satisfied. We say that the embedding of the amalgam is possible whenever such a semigroup T exists.

A free product is an operation that takes two algebraic objects and construct a new object in the same category. For a family of semigroups $\{S_i : i \in I\}$, the collection of all finite strings of the form $(a_1, a_2, \cdots, a_n)$ for $a_i \in \{\cup S_i; i \in I \}$ is a semigroup with operation juxtaposition and is known as free product of semigroups denoted by $\Pi^* S_i$. The free product $\Pi^*_U S_i$ of the amalgam $\mathscr{A} = [U; S_i, \phi_i]$ is defined as the quotient semigroup of the ordinary free product $\Pi^* S_i$ in which for each i and j in I the image $\phi_i(u)$ of the image u of U in S_i is identified its image $\phi_j(u)$ in S_j. Howie [6] showed that there always exist a mapping μ_i from each S_i of the amalgam to the amalgamated free product with the following properties:

1. each μ_i is one-one .

2. $\mu_i(S_i) \cap \mu_j(S_j) \subseteq \mu(U)$ for all i, j in I such that $i \neq j$.

If the above two conditions hold, then we say that the amalgam $\mathscr{A}$ is *naturally embedded in its free product* and based on this [6] [Theorem 8.2.4] proved that the semigroup amalgam is embeddable in a semigroup if and only if it is naturally embedded in its free product.

Definition 1.2. *(cf.[9]) Let S,Γ be two non-empty sets. Then S is called a Γ-semigroup if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a,\gamma,b) \to a\gamma b$ satisfying the condition $(a\gamma)b\mu c = a\gamma(b\mu c)$ for all $a,b,c \in S$ and $\gamma,\mu \in \Gamma$.*

Let S be a Γ-semigroup, A and B are two subsets of S, then the set $\{a\gamma b : a \in A, b \in B \gamma \in \Gamma\}$ is denoted by $A\Gamma B$. A nonempty subset $A \subset S$ of a Γ-semigroup is called a Γ-subsemigroup of S if $A\Gamma A \subset A$. For S,T be two Γ-semigroups, a mapping $f : S \to T$ such that and $f(a\gamma b) = f(a)\gamma f(b)$ for each $a,b \in S$ and $\gamma \in \Gamma$ is called a Γ-semigroup homomorphism.

Next we generalize amalgams of semigroups to amalgams of Γ-semigroups in terms of Γ-mappings.

Definition 1.3. *Let U_1, S_1, S_2 be mutually disjoint non-empty sets and $\Gamma_0, \Gamma_1, \Gamma_2$ such that U is a Γ_0-semigroup called core of the amalgam, S_1 is a Γ_1-semigroup, S_2 is Γ_2-semigroup. Then the collection $\mathscr{A} = [(U,\Gamma_0),(S_1,\Gamma_1),(S_2,\Gamma_2),f_1,f_2]$ constitutes a Γ-semigroup amalgam where $f_i = (f'_i, f''_i)$ are Γ-monomorphisms such that $f'_i : U \to S_i$ and $f''_i : \Gamma_0 \to \Gamma_i$ for $i = \{1,2\}$.*

A Γ-semigroup amalgam $[(U,\Gamma_0),(S_1,\Gamma_1),(S_2,\Gamma_2),f_1,f_2]$ is said to be embeddable in another Γ-semigroup (T,Γ_T) if there exist mappings (Γ-monomorphisms) $g = (g',g'') : (U,\Gamma_0) \to (T,\Gamma_T)$ and $g_i = (g'_i, g''_i) : (S_i,\Gamma_i) \to (T,\Gamma_T)$ where $g_i : S_i \to T$ and $g''_i : \Gamma_i \to \Gamma_T$ for $i = 1,2$ and $g' : U \to T$, $g'' : \Gamma_0 \to \Gamma_T$ such that the following holds true:

[1] : the diagram

$$(U,\Gamma_0) \xrightarrow{f_1} (S_1,\Gamma_1)$$
$$\downarrow{f_2} \quad \searrow{f} \quad \downarrow{g_1} \quad \text{commutes and}$$
$$(S_2,\Gamma_2) \xrightarrow{f_2} (T,\Gamma_T)$$

[2] : $g_1((S_1,\Gamma_1)) \cap g_2((S_2,\Gamma_2)) = g((U,\Gamma_0))$

The second condition can be restated as:

- whenever $g_1(s_1\gamma_1 s'_1) = g_2(s_2\gamma_2 s'_2)$ (for $s_1, s'_1 \in S_1$, $s_2, s'_2 \in S_2$, $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$) there exists $u \in U$ and $\gamma_0 \in \Gamma_0$ such that $f'_i(u) = s_i$ and $f''_i(\gamma_0) = \gamma_i$ for $i = 1,2$.

If the amalgam satisfies condition [1] then it is called weakly Γ-embeddable and if the amalgam satisfies both conditions, then it is called strongly Γ-embeddable.

2. Free product of Γ-semigroups

Free product of an indexed family of pairwise disjoint semigroups was introduced by J M Howie [6] and the notion of free Γ-semigroup was introduced by M K Sen [9]. In the following we define free product of a family of mutually disjoint Γ-semigroups $\{S_i : i \in I\}$ and a family $\{\Gamma_i : i \in T\}$ which are also mutually disjoint.

Definition 2.1. *Let $\{\Gamma_i : i \in T\}$ be a family of disjoint non-empty sets, $\{S_i : i \in I\}$ be a family of mutually disjoint Γ-semigroups and $x \in \cup\{S_i : i \in I\}$. Then there is a unique l in I such that $x \in S_l$ and l is called pointer of x, denoted by $l = \zeta(x)$. Similarly for an element $\gamma \in \cup\{\Gamma_i : i \in I\}$, there is a unique m in I such that $\gamma \in \Gamma_m$ and m is called pointer of γ.*

Consider Δ as the collection of all elements of the form $(x_1,\gamma_1,x_2,\gamma_2,\cdots,\gamma_{m-1},x_m)$ where $m \geq 1$ is an integer, $x_n \in \cup\{S_i : i \in I\}$ for $n = 1,2,3,\cdots,m$, $\gamma_k \in \cup\{\Gamma_i : i \in I\}$ for $k = 1,2,3,\cdots,m-1$ and $\zeta(x_n) \neq \zeta(x_{n+1}) \neq \zeta(\gamma_n)$ for $n = 1,2,3,\cdots,m-1$ of all finite strings. Define the product of two elements in Δ as follows: $\mathbf{a} = (a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$ and $\mathbf{b} = (b_1,\alpha_1,b_2,\alpha_2,\cdots,\alpha_{n-1},b_n)$ are elements in Δ and $\gamma \in \Gamma$, the product of $\mathbf{a}$ and $\mathbf{b}$ in Δ is :

$$\mathbf{a}\,\gamma\,\mathbf{b} = \begin{cases} (a_1,\cdots,a_m\gamma b_1,\cdots,b_n) & \text{if } \zeta(a_m = \zeta(\gamma) = \zeta(b_1) \\ (a_1,\cdots,a_m,\gamma,b_1,\cdots,b_n) & \text{otherwise} \end{cases}$$

Theorem 2.2. *Let $\Delta = \{(x_1,\gamma_1,x_2,\gamma_2,\cdots,\gamma_{m-1},x_m)\}$ where $m \geq 1$ is an integer, $x_n \in \cup\{S_i : i \in I\}$ for $n = 1,2,3,\cdots,m$, $\gamma_k \in \cup\{\Gamma_i : i \in I\}$ for $k = 1,2,3,\cdots,m-1$. Then Δ is a Γ-semigroup*

Proof:
For associativity, we have to consider the following cases. Let

$$\mathbf{a} = (a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$$
$$\mathbf{b} = (b_1,\alpha_1,b_2,\alpha_2,\cdots,\alpha_{n-1},b_n)$$
$$\mathbf{c} = (c_1,\beta_1,c_2,\beta_2,\cdots,\beta_{p-1},c_p)$$

in Δ and α,β in $\cup\{\Gamma_i : i \in I\}$.

When $\zeta(a_m) = \zeta(\alpha) = \zeta(b_1)$ and $\zeta(b_n) = \zeta(\beta) = \zeta(c_1)$
$\mathbf{a}\,\alpha\,\mathbf{b} = (a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m\alpha b_1,\alpha_1,\cdots,\alpha_{n-1},b_n)$
$\mathbf{b}\,\beta\,\mathbf{c} = b_1,\alpha_1,\cdots,\alpha_{n-1},b_n\beta c_1,\beta_1,\cdots,\beta_{p-1},c_p)$

$$(a\alpha b)\beta c = (a_1,\cdots,a_m\alpha b_1,\cdots,b_n\beta c_1,\cdots,c_p)$$
$$= a\alpha(b\beta c)$$

In a similar way one can see the associativity follows in other situations also. Hence Δ is a Γ-semigroup.

Free product of disjoint Γ-semigroups

Here we are considering the case of two Γ-semigroups over the same Γ and the case of a family of disjoint Γ-semigroups over the same Γ is similar in nature.

Consider non-empty sets S_1, S_2, Γ and let S_1 and S_2 are Γ-semigroups over the same Γ. As earlier, the set of all elements of the form $(a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$ where $a_1,a_2,\cdots,a_m \in S_1 \cup S_2$ and $\gamma_1,\gamma_2,\cdots,\gamma_{m-1} \in \Gamma$ is a Γ-semigroup and is called free-Γ-semigroup product over the alphabet $S_1 \cup S_2$ relative to Γ. We denote it by $S_i^*\Gamma$. The empty word is the word which has a no letters. Operation on $S_i^*\Gamma$ is defined as same as that we defined earlier.
For

$$a = (a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$$
$$b = (b_1,\alpha_1,b_2,\alpha_2,\cdots,\alpha_{n-1},b_n)$$

in $S_i^*\Gamma$ and $\gamma \in \Gamma$

$$\mathbf{a}\,\gamma\,\mathbf{b} = \begin{cases} (a_1,\cdots,a_m\gamma b_1,\cdots,b_n) & \text{if } \zeta(a_m = \zeta(b_1) \\ (a_1,\cdots,a_m,\gamma,b_1,\cdots,b_n) & \text{if } \zeta(a_m \neq \zeta(b_1) \end{cases}$$

It is straight forward to check associativity for the various cases arise, and so we can see that $S_i^*\Gamma$ is a Γ-semigroup.

Remark 2.3. *There are elements of $S_i^*\Gamma$ which are strings (a_i) of length one where $a_i \in S_1 \cup S_2$. In fact $S_i^*\Gamma$ is generated by strings of length one, since*
$(a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$
$= (a_1),(\gamma_1),(a_2),(\gamma_2),\cdots,(\gamma_{m-1}),(a_m)$
for every $(a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$ in $S_i^*\Gamma$. So we exclude the brackets in writing a string of length one.

A vital property of free products is the following.

Proposition 2.4. *Let $\Delta = S_i^*\Gamma$ be the free-Γ-product of two Γ-semigroups over the same Γ. Then for each $i \in \{1,2\}$ there exists a Γ-monomorphism $\theta_i : (S_i,\Gamma) \to \Delta$ given by, $\theta_i(a\gamma b) = (a\gamma b)$ associating the elements $a,b \in S_1 \cup S_2$ and $\gamma \in \Gamma$ with the one letter word $(a\gamma b)$. If T is a Γ-semigroup over the same Γ for which there is a Γ-morphism $\psi_i : S_i \to T$ for each $i = 1,2$ then there is a unique morphism $\lambda : \Delta \to T$ with the property that the diagram*

$$S_i \xrightarrow{\theta_i} \Delta$$
$$\downarrow{\psi_i} \quad \swarrow{\lambda}$$
$$T$$

commutes for every $i = 1,2$.

Proof:
Since each element $a\gamma b$ in $S_1 \cup S_2$ is also a word $(a\gamma b)$ in Γ, we can consider $S_1 \cup S_2$ as a Γ-subsemigroup of Δ. Given T and Γ-morohisms ψ_i are given $(i = 1,2)$, define $\lambda : \Delta \to T$ by,
$\lambda((a_1,\gamma_1,a_2,\cdots,\gamma_{m-1},a_m))$
$= \psi_{\zeta(a_1)}(a_1),\gamma_1,\psi_{\zeta(a_2)}(a_2),\cdots,\gamma_{m-1},\psi_{\zeta(a_m)}(a_m).$
The expression on the right side is the product of elements of T, hence λ maps Δ into T. Further λ is a Γ-morphism, for

$$\mathbf{a} = (a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$$
$$\mathbf{b} = (b_1,\alpha_1,b_2,\alpha_2,\cdots,\alpha_{n-1},b_n)$$

elements of Δ and $\gamma \in \Gamma$, then if $\zeta(a_m) \neq \zeta(b_1)$,

$$\begin{aligned} \lambda(a\gamma b) &= \lambda((a_1,\gamma_1,\cdots,\gamma_{m-1},a_m,\gamma,b_1,\alpha_1,\cdots,b_n)) \\ &= \psi_{\zeta(a_1)}(a_1),\cdots,\gamma,\psi_{\zeta(b_1)}(b_1),\cdots\psi_{\zeta(b_n)}(b_n) \\ &= \lambda(a)\gamma\lambda(b) \end{aligned}$$

The above diagram commutes since

$$\begin{aligned} \lambda\theta_i(a\gamma b) &= \lambda(\theta_i(a\gamma b)) \\ &= \lambda((a\gamma b)) \\ &= \psi_{\zeta(a)}(a)\gamma\psi_{\zeta(b)}(b) \\ &= \psi_i(a\gamma b) \end{aligned}$$

hence $\lambda\theta_i = \psi_i$.

The uniqueness of the λ follows from the way that Δ is created by words of length one. If λ is to make the diagram commute then it is compulsory to have $\psi_i(a\gamma b) = \lambda((a\gamma b))$ for every $i = 1,2$. Then, if λ is to be a Γ- homomorphism we must have
$\lambda(a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m)$
$= \lambda(a_1),\gamma_1,\lambda(a_2),\gamma_2,\cdots,\gamma_{m-1},\lambda(a_m)$
$= \psi_{\zeta(a_1)}(a_1)\gamma_1\psi_{\zeta(a_2)}(a_2)\gamma_2\cdots\gamma_{m-1}\psi_{\zeta(a_m)}(a_m)$
for every $a_1,\gamma_1,a_2,\gamma_2,\cdots,\gamma_{m-1},a_m$ in Δ.
ie., Δ must be exactly what we defined it.

Proposition 2.5. *Let S_1 and S_2 be two Γ-semigroups , Δ be the free product of S_1 and S_2 and let H be a Γ-semigroup such that*

- *there exists a Γ-monomorphism $f_i : S_i \to H$ for $i = 1,2$.*

- *if T is a Γ-semigroup and if there exist Γ-monomorphism $g_i : S_i \to T$ for $i = 1,2$ then there exists a unique Γ-monomorphism $\delta : H \to T$ such that the diagram*

$$S_i \xrightarrow{f_i} H$$
$$\downarrow{g_i} \quad \swarrow{\delta}$$
$$T$$

commutes for every $i = 1,2$.

then H is isomorphic to Δ.

Proof
From the property of Δ in Proposition 2.4, when $T = \Delta$ and $\psi_i = \phi_i$ for $i = 1,2$ there is a unique Γ-morphism (identity map 1_Δ) $\Delta \to \Delta$ making the diagram

$$S_i \longrightarrow \Delta$$
$$\downarrow{\theta_i} \quad \swarrow$$
$$\Delta$$

commutes for every $i = 1,2$.
By uniqueness 1_Δ is the only Γ-morphism from Δ into Δ having the property. Similarly the identity map 1_H is the only Γ-morphism from H into H with the property that the diagram

$$S_i \xrightarrow{f_i} H$$
$$\downarrow{f_i} \quad \swarrow$$
$$H$$

commutes for every $i = 1,2$.
Now apply Proposition 2.4 with $T = H$ and $\psi_i = f_i$ for $i = 1,2$ then we have a morphism λ from Δ to H such that

$$S_i \xrightarrow{\theta_i} \Delta \qquad (2.1)$$
$$\downarrow{f_i} \quad \swarrow{\lambda}$$
$$H$$

commutes for every $i = 1, 2$.

Then by the assumed property of H with $T = \Delta$ and $g_i = \theta_i$ we obtain a Γ-morphism $\delta : H \to \Delta$ such that the diagram

$$S_i \xrightarrow{f_i} H$$
$$\downarrow{\theta_i} \quad \swarrow{\delta}$$
$$\Delta$$

(2.2)

commutes for every $i = 1, 2$.

It follows that

$$\delta \lambda \theta_i = \delta f_i = \theta_i$$
$$\lambda \delta f_i = \lambda \theta_i = f_i$$

that is, if we track together, the dragrams 2.1 and 2.2 in both of the possible ways we obtain commutative diagrams

$$S_i \xrightarrow{\theta_i} \Delta \quad S_i \xrightarrow{f_i} H$$
$$\downarrow{\theta_i} \swarrow{\delta\lambda} \quad \downarrow{f_i} \swarrow{\lambda\delta}$$
$$\Delta \qquad\qquad H$$

for $i = 1, 2$

by the uniqueness , we get,

$$\delta\lambda = 1_\Delta \text{ and } \lambda\delta = 1_H$$

thus δ and λ are mutually inverse Γ-isomorphisms and $H \simeq \Delta$ as required.

We denote by Γ **-Sgrp** the category of Γ-semigroups which has the Γ-semigroups as objects and the Γ-homomorphisms of Γ-semigroups as arrows.

Then by the above two propositions we can conclude that Δ is the unique coproduct in the sense of category theory.

3. Congruences and Free Γ-product

In this section we discuss certain congruences on Γ-semigroups.

Definition 3.1. *(cf.[1]) Let S be a Γ-semigroup. An equivalence relation ρ on S is called congruence if $x\rho y$ implies that $(x\gamma z)\rho(y\gamma z)$ and $(z\gamma x\rho(z\gamma y)$ for all $x, y, z \in S$ and $\gamma \in \Gamma$.*

Let ρ be a congruence relation on (S, Γ).
Then $S/\rho = \{\rho(x) : x \in S\}$ is the set of all equivalence classes of elements of S with respect to ρ.

Theorem 3.2. *(cf.[12]) Let ρ be a congruence relation on Γ-semigroup (S, Γ). Then S/ρ is a Γ-semigroup.*

Theorem 3.3. *(cf.[4]) Let $(\phi, g) : (S_1, \Gamma_1) \to (S_2, \Gamma_2)$ be a homomorphism. Define the relation $\rho_{(\phi,g)}$ on (S_1, Γ_1) as follows:*

$$x\rho_{(\phi,g)}y \iff \phi(x) = \phi(y).$$

Then $\rho_{(\phi,g)}$ is a congruence on (S_1, Γ_1).

Theorem 3.4. *(cf.[12]) Let S and T be Γ-semigroups under the same Γ and $\phi : S \to T$ be a Γ-homomorphism. Then there is a Γ-homomorphism $\psi : s/ker\phi \to T$ such that $im\phi = im\psi$ and the diagram*

$$S \xrightarrow{\phi} T$$
$$\downarrow{(ker\phi)^\#} \quad \swarrow{\psi}$$
$$S/ker\phi$$

commutes where $(ker\phi)^\#$ is the natural mapping from S into $S/ker\phi$ defined by $(ker\phi)^\#(x) = xker\phi$ for all $x \in S$.

Corollary 3.5. $S/ker\phi \cong im\phi$.

Free Γ-product of amalgam

Free Γ- product $A = [(U, \Gamma_0), (S_1, \Gamma_1), (S_2, \Gamma_2), f_1, f_2]$ of the amalgam written as Δ_U is defined as the quotient semigroup of the ordinary free Γ- product Δ in which for each $i = 1, 2$; the image $f_1(u\gamma_0 u')$ of an element $u\gamma_0 u'$ of (U, Γ_0) in (S_1, Γ_1) is identified with its image $f_2(u\gamma_0 u')$ in (S_2, Γ_2).

More precisely, denote θ_i the natural Γ-monomorphism from S_i to $\Delta = S_i^* \Gamma$, then we define $\Delta_U = S_i^* \Gamma/\rho$ where ρ is the congruence on Δ generated by the subset

$$\mathbf{R} = \{(\theta_1 f_1(u\gamma_0 u'), \theta_2 f_2(u\gamma_0 u')) : u\gamma_0 u' \in U\} \quad (3.1)$$

of $\Delta \times \Delta$.

It is clear that for each $i = 1, 2$ there is a morphism $\mu_i = \rho^\# \theta_i$ from $S_i \to \Delta_U$. It is also clear from the definition of ρ that we have a commutative diagram

$$U \xrightarrow{f_1} S_1$$
$$\downarrow{f_2} \qquad \downarrow{\mu_1}$$
$$S_2 \xrightarrow{\mu_1} \Delta_U$$

So there exists a Γ-morphism $\mu : U \to \Delta_U$ such that $\mu = \mu_1 f1 = \mu_2 f_2$.
Since $\mu(U) \leqslant \mu_i(S_i)$,
we necessarily have $\mu(U) \leqslant \mu_1(S_1) \cap \mu_2(S_2)$.
Hence the Γ-amalgam is embedded in its free Γ product if and only if

- each μ_i is one-one

- $\mu_1(S_1) \cap \mu_2(S_2) \leqslant \mu(U)$

If the above two conditions hold, we say that the amalgam is naturally Γ-embedded in its free Γ product.

Theorem 3.6. *The Γ-semigroup amalgam $\mathscr{A} = [(U, \Gamma_0), (S_1, \Gamma_1), (S_2, \Gamma_2), f_1, f_2]$ is embeddable in a Γ-semigroup if and only if it is naturally Γ-embedded in its free Γ product.*

Before moving to the proof, we have to make use of the following result.

Proposition 3.7. *If A is a Γ semigroup amalgam, the the free Γ product Δ_U of the amalgam is the pushout of the diagram $\{U \to S_1\}_{i=1,2}$. That is*

(1) *There exists for each $i = 1,2$ a Γ-morphism $\mu_i : S_i \to \Delta_U$ such that the diagram $\{U \to S_i \to \Delta_U\}_{i=1,2}$ commutes. That is, $\mu_1 f_1 = \mu_2 f_2$.*

(2) *If V is a Γ-semigroup for which morphisms $g_i : S_i \to V$ exist such that $g_1 f_1 = g_2 f_2$, then there exists a unique Γ- morphism $\delta : \Delta_U \to V$ such that the diagram*

$$
\begin{array}{ccc}
S_i & \xrightarrow{\mu_i} & \Delta_U \\
{\scriptstyle g_i}\downarrow & \swarrow{\scriptstyle \delta} & \\
V &
\end{array}
$$

commutes for each $i = 1,2$.

Proof

We have already observed property [1].
To see property [2], notice that by Proposition 2.4 there is a unique morphism $\lambda : \Delta \to v$ such that

$$
\begin{array}{ccc}
S_i & \xrightarrow{\theta_i} & \Delta \\
{\scriptstyle g_i}\downarrow & \swarrow{\scriptstyle \lambda} & \\
V &
\end{array}
$$

is a commutative diagram for each $i = 1,2$.
Now, for all $u, u' \in U$ and $\gamma_0 \in \Gamma$

$$
\begin{aligned}
\lambda \theta_1 f_1(u\gamma_0 u') &= g_1 f_1(u\gamma_0 u') \\
&= g_2 f_2(u\gamma_0 u') \\
&= \lambda \theta_2 f_2(u\gamma_0 u')
\end{aligned}
$$

Hence $(\theta_1 f_1(u\gamma_0 u'), \theta_2 f_2(u\gamma_0 u')) \in \lambda \circ \lambda^{-1}$. So from equation (3.1) $\mathbf{R} \subseteq \lambda \circ \lambda^{-1}$, hence, since $\lambda \circ \lambda^{-1}$ is a congruence

$$
\rho = R^{\#} \subseteq \lambda \circ \lambda^{-1}
$$

By Theorem 3.4 it follows that the morphism $\lambda : \Delta \to V$ factors through $\Delta_U = S/\rho$,
that is, there is a unique morphism $\delta : \Delta_U \to V$ such that the diagram

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\rho^{\#}} & \Delta_U \\
{\scriptstyle \lambda}\downarrow & \swarrow{\scriptstyle \delta} & \\
V &
\end{array}
\quad commutes.
$$

By the definition of g_i and from the commutativity of the above two diagrams, for $i = 1,2$

$$
g_i = \lambda \theta_i = \delta \rho^{\#} \theta_i = \delta \mu_i
$$

hence the diagram commutes.
Now we are going back to the proof of the theorem.
Proof:

One way is obvious and suppose that A is embeddable in a Γ-semigroup T, so that there exists Γ-monomorphisms $g_i = (g_i', g_i'') : (S_i, \Gamma_i) \to (T, \Gamma_T)$ for $i = 1,2$ and $g = (g', g'') : (U, \Gamma_0) \to (T, \Gamma_T)$ such that $f_1 g_1 = f_1 g_2 = g$ and such that $g_1(S_1) \cap g_2(S_2) = g(\lambda)$.
By Proposition 3.7 there exists a unique Γ-morphism $\delta : \Delta_U \to T$ such that

$$
\begin{array}{ccc}
S_i & \xrightarrow{\mu_i} & \Delta_U \\
{\scriptstyle g_i}\downarrow & \swarrow{\scriptstyle \delta} & \\
T &
\end{array}
$$

is a commutative diagram for each $i = 1,2$.
For $s\gamma s', s_1 \gamma_1 s_1' \in (S_1, \Gamma_1)$, if $\mu_1(s\gamma s') = \mu_1(s_1 \gamma_1 s_1')$ then Suppose

$$
\begin{aligned}
\delta(\mu_1(s\gamma s')) &= \delta(\mu_1(s_1\gamma_1 s_1')) \text{ so} \\
g_1(s\gamma s') &= g_1(s_1\gamma_1 s_1') \\
s\gamma s' &= s_1\gamma_1 s_1'
\end{aligned}
$$

thus μ_1 is a Γ-monomorphism.

Similarly we get μ_2 is a Γ-monomorphism.
Suppose $x \in \mu_1 S_1 \cap \mu_2 S_2$, without loss of generality, assume $x = \mu_1(s_1 \gamma_1 s_1') = \mu_2(s_2 \gamma_2 s_2')$ where $s_1, s_1' \in S_1$, $s_2, s_2' \in S_2$, $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$.
Then $\delta(x) = \delta(\mu_1(s_1 \gamma_1 s_1')) = g_1(s_1 \gamma_1 s_1') \in g_1(S_1)$. Similarly, $\delta(x) \in g_2(S_2)$. Thus $\delta(x) \in g_1(S_1) \cap g_2(S_2) = g(U)$ and so there exist $u = u\gamma_0 u' \in U$ such that $\delta(x) = g(U)$.
That is

$$
\begin{aligned}
g_1(s_1\gamma_1 s_1') &= \delta\mu_1(s_1\gamma_1 s_1') \\
&= \delta(x) \\
&= g(u) \\
&= g_1 f_1(u\gamma_0 u')
\end{aligned}
$$

similarly

$$
\begin{aligned}
g_2(s_2\gamma_2 s_2') &= \delta\mu_2(s_2\gamma_2 s_2') \\
&= \delta(x) \\
&= g(u) \\
&= g_2 f_2(u\gamma_0 u')
\end{aligned}
$$

Since g_i is Γ-monomorphism it follows that $f_1(u\gamma_0 u') = s_1 \gamma_1 s_1'$ and $f_2(u\gamma_0 u') = s_2 \gamma_2 s_2'$ and so $x = \mu_1(s_1 \gamma_1 s_1') = \mu_1(f_1(u\gamma_0 u'))$ and $x = \mu_2(s_2 \gamma_2 s_2') = \mu_2 f_2(u\gamma_0 u') \in \mu(U)$.
Thus A is naturally Γ-embedded in Δ_U as required.

4. On Amalgam of Completely α-Regular Γ-Semigroups

In the following we define completely α-regular Γ-semigroups and provide necessary condition to the embeddability of their amalgam.

Definition 4.1. *An element $a \in S$ is called α-regular if there exist elements $x \in S$ and $\alpha \in \Gamma$ such that $a = a\alpha x\alpha a$. If every element of a Γ-semigroup is α-regular in S then S is called α-regular Γ-semigroup.*

Definition 4.2. *For an α-regular Γ-semigroup S, an element $b \in S$ is called α- inverse of an element $a \in S$ if $a = a\alpha b\alpha a$ and $b = b\alpha a\alpha b$ for all $a, b \in S$ and $\alpha \in \Gamma$.*

Definition 4.3. *An α−regular semigroup S is called Γ-inverse Γ-semigroup if every element in S has a unique α-inverse in S.*

Definition 4.4. *An element a is completely α-regular if there exist an element x in S such that $a = a\alpha x\alpha a$ and $a\alpha x = x\alpha a$ for all $a, x \in S$ and $\alpha \in \Gamma$. If all the elements in a Γ-semigroup S is completely α-regular then S is called completely α-regular Γ-semigroup.*

Theorem 4.5. *A Γ-semigroup amalgam $(U; S_1, S_2, \phi_1, \phi_2)$, in which S_1 and S_2 are completely α-regular Γ-semigroups over the same Γ is embeddable only if U is also completely α-regular.*

Proof:
Consider an amalgam $A = (U; S_1; S_2; \phi_1; \phi_2)$ in which S_1 and S_2 are both completely α-regular Γ-semigroups. Suppose A is embeddable, say in a Γ-semigroup T and consider an element $u \in U$.
Let us denote $\phi_1(u) = s_1$ and $\phi_2(u) = s_2$.
Since S_1 and S_2 are completely α-regular, there exist inverses $s_1^{-1} \in S_1$ and $s_2^{-1} \in S_2$ of s_1 and s_2 respectively.
Let $\psi_i : S_i \to T, i \in 1, 2$ be the embedding Γ-monomorphisms, then

$$\psi_1(s_1) = \psi_1\phi_1(u) = \psi_2\phi_2(u) = \psi_2(s_2)$$

. We can calculate in T:

$$\begin{aligned}
\psi_2(s_2^{-1}) &= \psi_2(s_2^{-1}\gamma s_2\gamma s_2^{-1}) \\
&= \psi_2(s_2^{-1})\gamma\psi_2(s_2)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1})\gamma\psi_1(s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1})\gamma\psi_1(s_1\gamma s_1^{-1}\gamma s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1})\gamma\psi_1(s_1)\gamma\psi_1(s_1^{-1})\gamma\psi_1(s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1})\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_1(s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1}\gamma s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_1(s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1}\gamma s_2)\gamma\psi_1(s_1^{-1}\gamma s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1}\gamma s_2)\gamma\psi_1(s_1\gamma s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1}\gamma s_2)\gamma\psi_1(s_1)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2^{-1}\gamma s_2)\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2\gamma s_2^{-1})\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2\gamma s_2^{-1}\gamma s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_2(s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_1(s_1)\gamma\psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})
\end{aligned}$$

$$\begin{aligned}
&= \psi_1(s_1\gamma s_1^{-1})\gamma\psi_2(s_2^{-1}) \\
&= \psi_1(s_1^{-1}\gamma s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_1(s_1)\gamma\psi_2(s_2^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2)\gamma\psi_2(s_2^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2\gamma s_2^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}\gamma s_2) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})\gamma\psi_2(s_2) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})\gamma\psi_1(s_1) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})\gamma\psi_1(s_1\gamma s_1^{-1}\gamma s_1) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})\gamma\psi_1(s_1)\gamma\psi_1(s_1^{-1})\gamma\psi_1(s_1) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1})\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1})\gamma\psi_1(s_1) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2^{-1}\gamma s_2)\gamma\psi_1(s_1^{-1}\gamma s_1) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2\gamma s_2^{-1})\gamma\psi_1(s_1\gamma s_1^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2\gamma s_2^{-1})\gamma\psi_1(s_1)\gamma\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2\gamma s_2^{-1})\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2\gamma s_2^{-1}\gamma s_2)\gamma\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_2(s_2)\gamma\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\gamma\psi_1(s_1)\gamma\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1}\gamma s_1\gamma s_1^{-1}) \\
&= \psi_1(s_1^{-1})
\end{aligned}$$

Now, using condition (2) of embeddability, there exists $u' \in U$ such that $\phi_i(u') = s_i^{-1}$ for $i \in \{1, 2\}$.
Then, because $s_1\gamma s_1^{-1}\gamma s_1 = s_1$ implies $\phi_1^{-1}(s_1\gamma s_1^{-1}\gamma s_1) = \phi_1^{-1}(s_1)$, we have $u\gamma u'\gamma u = u$ due to injectivity of ϕ_1. Similarly we can conclude that $u'\gamma u\gamma u' = u'$ and $u\gamma u' = u'\gamma u$. Thus U is completely Γ-regular.

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