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Decomposition of wheel graphs into stars, cycles and paths

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Abstract

Let $G=(V,E)$ be a finite graph with n vertices. The Wheel graph W_n is a graph with vertex set $\{v_0,v_1,v_2,\ldots,v_n\}$ and edge-set consisting of all edges of the form v_iv_{i+1} and v_0v_i where $1\leq i\leq n$, the subscripts being reduced modulo n. Wheel graph of $(n+1)$ vertices denoted by W_n. Decomposition of Wheel graph denoted by $D(W_n)$. A star with 3 edges is called a claw S_3. In this paper, we show that any Wheel graph can be decomposed into following ways.

$$D(W_n)=\begin{cases}(n-2d)S_3,\ d=1,2,3,\ldots & \text{if } n\equiv 0\ (mod\ 6)\\ [(n-2d)-1]\,S_3 \text{ and } P_3,\ d=1,2,3\ldots & \text{if } n\equiv 1\ (mod\ 6)\\ [(n-2d)-1]\,S_3 \text{ and } P_2,\ d=1,2,3,\ldots & \text{if } n\equiv 2\ (mod\ 6)\\ (n-2d)S_3 \text{ and } C_3,\ d=1,2,3,\ldots & \text{if } n\equiv 3\ (mod\ 6)\\ (n-2d)S_3 \text{ and } P_3,\ d=1,2,3\ldots & \text{if } n\equiv 4\ (mod\ 6)\\ (n-2d)S_3 \text{ and } P_2,\ d=1,2,3,\ldots & \text{if } n\equiv 5\ (mod\ 6)\end{cases}.$$

Keywords

Wheel Graph, Decomposition, claw.

AMS Subject Classification

05C70.

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Contents

1. Introduction

Graph theory is proved to be tremendously useful in modeling the essential features of system with finite components. Graphical models are used to represent telephone network, railway network, communication problems, traffic network etc. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and the relations by edges [5].

A Wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. A Wheel graph with n vertices can also be defined as the 1-skeleton of an n-gonal pyramid. Wheel graph of $(n+1)$ vertices denoted by W_n. Number of edges of Wheel graph with $(n+1)$ vertices is $2n$. A star S_n is the complete bipartite graph K_1,n; A tree with one internal vertex and n edges.

All graphs considered here are finite and undirected without loops, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [8] and to study about the decomposition of graphs into paths, stars and cycles is referred to [1], [2], [3] and [4].

As usual C_n denotes the cycle of length n, P_{n+1} denotes the path of length n, S_3 denotes the claw and $D(W_n)]$ denotes decomposition of Wheel graph.

2. Basic Definitions

In this section, we see some basic definitions of graph decomposition and Wheel graph.

Let $L = \{H_1, H_2, ..., H_r\}$ be a family of subgraphs of G. An $L-decomposition$ of G is an edge- disjoint decomposition of G into positive integer α_i copies of H_i, where $i \in \{1, 2, ..., r\}$. Furthermore, if each $H_i \quad (i \in \{1, 2, ..., r\})$ is isomorphic to a graph H, we say that G has an $H-decomposition$. It is easily seen that $\sum_{i=1}^{r} \alpha_i\, e(H_i) = e(G)$ is one of the obvious necessary conditions for the existence of a $\{H_1, H_2, ..., H_r\}-decomposition$ of G. For convenience, we call the equation, $\sum_{i=1}^{r} \alpha_i\, e(H_i) = e(G)$, a *necessary sum condition*.

Let $G = (V, E)$ be a finite graph with n vertices. The Wheel graph W_n is a graph with vertex set $\{v_0, v_1, v_2, ..., v_n\}$ and edge-set consisting of all edges of the form $v_i v_{i+1}$ and $v_0 v_i$ where $1 \leq i \leq n$, the subscripts being reduced modulo n. Wheel graph of $(n+1)$ vertices denoted by W_n.

Obviously Wheel graph W_n has every vertex of degree 3 except the internal vertex. Internal vertex has degree n.

3. Examples of Wheel graphs

In this section, we see some examples of Wheel Graphs.

W_6 and W_8 are represented in Figure 1.1 and Figure 1.2 respectively.

4. Decomposition of Wheel Graph W_n

In this section, we characterize the theorem of decomposition of Wheel graph W_n into claws, cycles and paths.

Theorem 4.1. *Any Wheel $W_n, n \geq 3$ can be decomposed into C_n and S_n.*

Proof. Proof is immediate from the definition of the Wheel graph. □

Theorem 4.2. *Any Wheel graph $W_n, n \geq 3$ can be decomposed into following ways.*

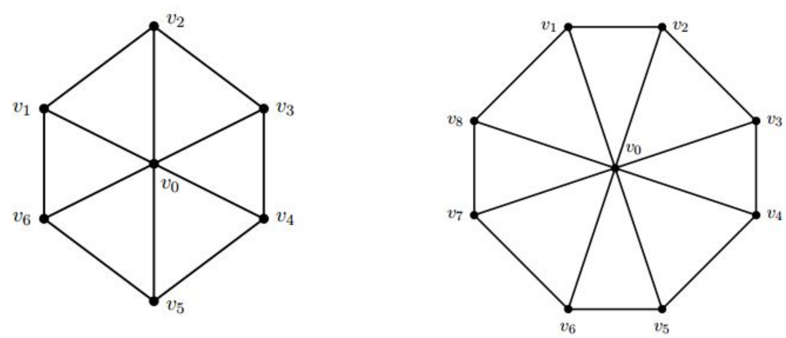


Figure 1.1 W_6 Figure 1.2 W_8

Figure 1. *Examples of Wheel Graphs*

$$D(W_n) = \begin{cases} (n-2d)S_3,\ d = 1,2,3,... \\ \quad \text{if } n \equiv 0 \ (mod\ 6) \\ [(n-2d)-1]\, S_3 \text{ and } P_3,\ d = 1,2,3... \\ \quad \text{if } n \equiv 1 \ (mod\ 6) \\ [(n-2d)-1]\, S_3 \text{ and } P_2,\ d = 1,2,3,... \\ \quad \text{if } n \equiv 2 \ (mod\ 6) \\ (n-2d)S_3 \text{ and } C_3,\ d = 1,2,3,... \\ \quad \text{if } n \equiv 3 \ (mod\ 6) \\ (n-2d)S_3 \text{ and } P_3,\ d = 1,2,3... \\ \quad \text{if } n \equiv 4 \ (mod\ 6) \\ (n-2d)S_3 \text{ and } P_2,\ d = 1,2,3,... \\ \quad \text{if } n \equiv 5 \ (mod\ 6) \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, v_2, ..., v_n\}$ and

$$E(W_n) = \left\{e_1, e_2, ..., e_n, e_1^{'}, e_2^{'}, ..., e_n^{'}\right\}$$

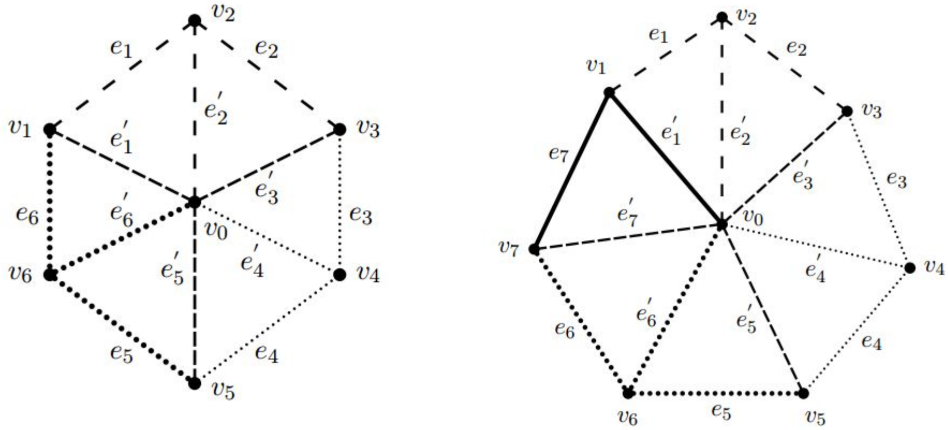
where e_i and $e_i^{'}$, $1 \leq i \leq n$ are edges in outer and inner cycles.

Case 1. $n \equiv 0$ (mod 6).
Then $n = 6d, d = 1, 2, 3, ...$
To Prove: W_n decomposed into $(n-2d)S_3$. That is to prove that W_n decomposed into $4dS_3$, since $n-2d = 6d-2d = 4d$.
Let $E_i = \left\{e_i, e_{i+1}^{'}, e_{i+1}\right\}$ where $i = 1, 3, 5, ...,$ $n-1$ and $E_j = \left\{e_i^{'}, e_{i+2}^{'}, e_{i+4}^{'}\right\}$ where $i = 1, 7, 13, ..., n-5$.
The edge induced subgraph $< E_i >$ forms $3d$ claws and the edge induced subgraph $< E_j >$ forms d claws. Hence W_n decomposed into $3d + d = 4d$ claws.

Case 2. $n \equiv 1$ (mod 6).
Then $n = 6d+1, d = 1, 2, 3, ...$
To Prove: W_n decomposed into $[(n-2d)-1]$ S_3 and P_3. That is to prove that W_n decomposed into $4dS_3$ and P_3, since $(n-2d)-1 = (6d+1-2d)-1 = 4d$.

Let $E_i = \left\{e_i, e'_{i+1}, e_{i+1}\right\}$ where $i = 1,3,5,...,$
$n-2$, $E_j = \left\{e'_i, e'_{i+2}, e'_{i+4}\right\}$ where $i = 3,9,15,$
$..., n-4$ and $E_k = \left\{e'_1, e_n\right\}$.
The edge induced subgraph $< E_i >$ forms $3d$ claws, the edge induced subgraph $< E_j >$ forms d claws and the edge induced subgraph $< E_k >$ forms a path P_3 of length 2. Hence W_n decomposed into $3d + d = 4d$ claws and a path P_3.

Illustration: Decomposition of Wheel graph on case 1 and case 2 explained through the following Figure-2.



$n \equiv 0 \ (\text{mod } 6) \quad W_6$ $\qquad$ $n \equiv 1 \ (\text{mod } 6) \quad W_7$

Figure 2.

Figures 2. represents decomposition of W_6 into 4 claws and decomposition of W_7 into 4 claws and a path P_3 respectively.

All edges of the claws and path differentiated in the above Figure-2.

Case 3. $n \equiv 2$ (mod 6).
Then $n = 6d + 2, d = 1,2,3,....$
To Prove: W_n decomposed into $[(n-2d) - 1]$
S_3 and P_2. That is to prove that W_n decomposed into $(4d + 1)S_3$ and P_2, since $(n-2d) - 1 = (6d + 2 - 2d) - 1 = 4d + 1$.
Let $E_i = \left\{e_i, e'_{i+1}, e_{i+1}\right\}$ where $i = 1,3,5,...,$
$n-1$, $E_j = \left\{e'_i, e'_{i+2}, e'_{i+4}\right\}$ where $i = 1,7,13,$
$19, ..., n-7$ and $E_k = \left\{e'_{n-1}\right\}$.
The edge induced subgraph $< E_i >$ forms $3d + 1$ claws, the edge induced subgraph $< E_j >$ forms d claws and the edge induced subgraph $< E_k >$ forms a path P_2 of length 1. Hence W_n decomposed into $3d + 1 + d = 4d + 1$ claws and a path P_2.

Case 4. $n \equiv 3$ (mod 6).
Then $n = 6d - 3, d = 1,2,3,....$
To Prove: W_n decomposed into $(n-2d)S_3$ and C_3. That is to prove that W_n decomposed into $(4d - 3)S_3$ and C_3, since $n - 2d = 6d - 3 - 2d = 4d - 3$.
Let $E_i = \left\{e_i, e'_{i+1}, e_{i+1}\right\}$ where $i = 1,3,5,...,$
$n-2$, $E_j = \left\{e'_i, e'_{i+2}, e'_{i+4}\right\}$ where $i = 3,9,15,$
$..., n-6$ and $E_k = \left\{e'_1, e'_n, e_n\right\}$.
The edge induced subgraph $< E_i >$ forms $3d - 2$ claws, the edge induced subgraph $< E_j >$ forms $d - 1$ claws and the edge induced subgraph $< E_k >$ forms one cycle C_3 of length 3. Hence W_n decomposed into $3d - 2 + d - 1 = 4d - 3$ claws and a cycle C_3.

Illustration: Decomposition of Wheel graph on case 3 and case 4 explained through the following Figure-3.

The above two figures represents decomposition of W_8 into 5 claws and a path P_2 and decomposition of W_9 into 5 claws and a cycle C_3 respectively.

All edges of the claws and path differentiated in the above Figure-3.

Case 5.
$n \equiv 4$ (mod 6).
Then $n = 6d - 2, d = 1,2,3,....$
To Prove: W_n decomposed into $(n-2d)S_3$ and P_3. That is to prove that W_n decomposed into $(4d - 2)S_3$ and P_3, since $n - 2d = 6d - 2 - 2d = 4d - 2$.
Let $E_i = \left\{e_i, e'_{i+1}, e_{i+1}\right\}$ where $i = 1,3,5,...,$
$n-1$, $E_j = \left\{e'_i, e'_{i+2}, e'_{i+4}\right\}$ where $i = 3,9,15,$
$..., n-7$ and $E_k = \left\{e'_1, e'_{n-1}\right\}$.

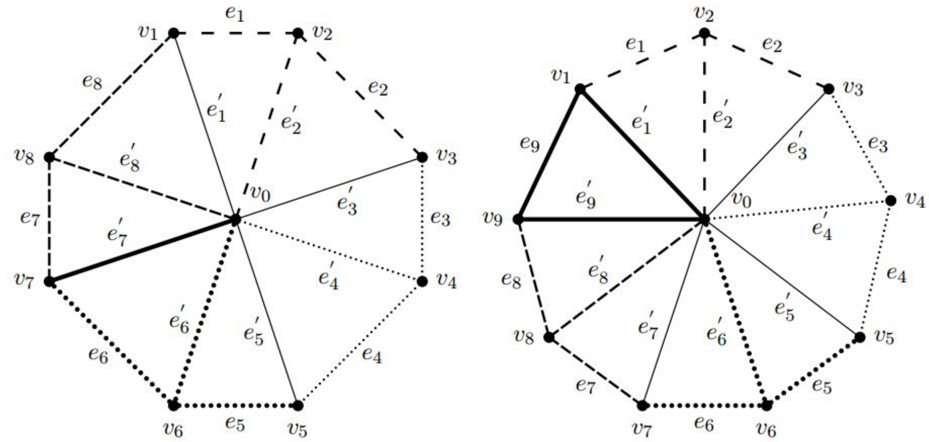

$n \equiv 2 \pmod 6$ W_8 $n \equiv 3 \pmod 6$ W_9

Figure 3.

The edge induced subgraph $< E_i >$ forms $3d-1$ claws, the edge induced subgraph $< E_j >$ forms $d-1$ claws and the edge induced subgraph $< E_k >$ forms a path P_3 of length 2. Hence W_n decomposed into $3d-1+d-1 = 4d-2$ claws and a path P_3.

Case 6.
$n \equiv 5 \pmod 6$.
Then $n = 6d-1, d = 1,2,3,....$
To Prove: W_n decomposed into $(n-2d)S_3$ and P_2. That is to prove that W_n decomposed into $(4d-1)S_3$ and P_2, since $n-2d = 6d-1-2d = 4d-1$.

Let $E_i = \left\{e_i, e^{'}_{i+1}, e_{i+1}\right\}$ where $i = 1,3,5,...,$ $n-2$, $E_j = \left\{e^{'}_i, e^{'}_{i+2}, e^{'}_{i+4}\right\}$ where $i = 1,7,13,$ $...,n-4$ and $E_k = \{e_n\}$.
The edge induced subgraph $< E_i >$ forms $3d-1$ claws, the edge induced subgraph $< E_j >$ forms d claws and the edge induced subgraph $< E_k >$ forms a path P_2 of length 1. Hence W_n decomposed into $3d-1+d = 4d-1$ claws and a path P_2. □

Illustration: Decomposition of Wheel graph on case 5 and case 6 explained through the following Figure-4.

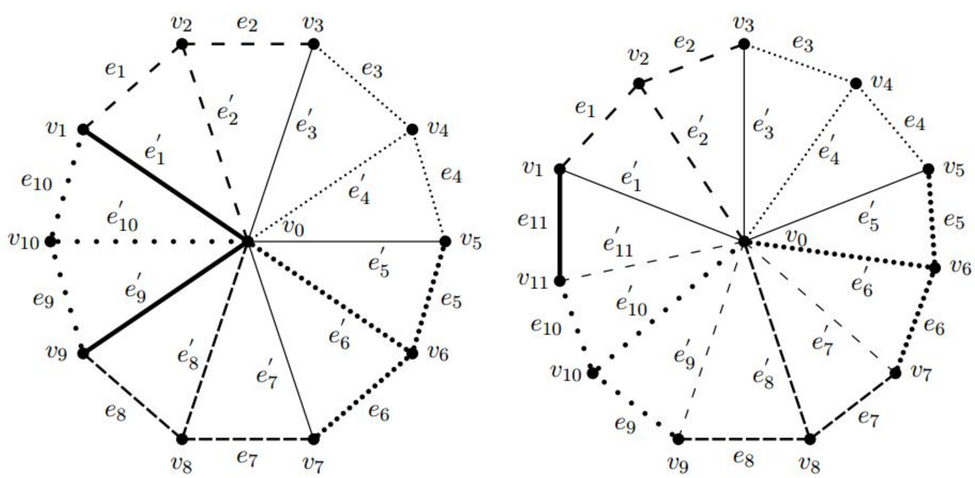

$n \equiv 4 \pmod 6$ W_{10} $n \equiv 5 \pmod 6$ W_{11}

Figure 4.

Figures 4 represents decomposition of W_{10} into 6 claws and a path P_3 and decomposition of W_{11} into 7 claws and a path P_2 respectively.

All edges of the claws and path differentiated in the above Figure-4.

Remark 4.3. *In the above theorem Case 1 guarantees that there is a claw decomposition for W_n.*

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