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Sum of fractional series through extended q-difference operators

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Abstract

In this paper, we define the extended q-difference operator, q-polynomial factorial and inverse of the extended q-difference operator and obtain the relation between shift operator and extended q-polynomial factorials. Also, we obtain the formula for some fractional series of arithmetic and geometric progressions in the field of Numerical Methods using the inverse of extended q-difference operator. Suitable examples are provided to illustrate the main results.

Keywords

Extended q-difference operator, Finite Series, Infinite Series, Polynomial factorial.

AMS Subject Classification

39A12, 39A70, 47B39, 39B60.

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Contents

1. Introduction

The theory of difference equations is based on the difference operator Δ defined as

$$\Delta u(k)=u(k+1)-u(k), k\in N=\{0,1,2,\cdots\}. \quad (1.1)$$

Also, many authors [1, 4, 5, 11] have suggested the definition of difference operator Δ_ℓ as

$$\Delta_\ell u(k)=u(k+\ell)-u(k), k\in[0,\infty), \ell\in(0,\infty), \quad (1.2)$$

and no significant results developed in the field of numerical methods. In 2006, by taking the definition of Δ as given in (1.2) and the theory of difference equations was developed in a different direction and many interesting results were obtained in the field of Numerical Methods [6]-[10].

In the field of approximation theory, the applications of q-calculus are new area in last 30 years. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupas in the year 1987. In 1997, Phillips considered another q-analogue of the classical Bernstein polynomials. Later several other researchers have proposed the q-extension of the well-known exponential-type operators which includes Baskakov operators, Szasz-Mirakyan operators, Meyer-Konig-Zeller operators, Bleiman, Butzer and Hahn operators, Picard operators, and Weierstrass operators. Also, the q-analogue of some standard integral operators of Kantorovich and Durrmeyer type was introduced, and their approximation properties were discussed [2].

In [12], while discussing the definition of $q-$derivative operator Δ_q as

$$\Delta_q u(k)=\frac{u(kq)-u(k)}{(q-1)k}, q\in(0,\infty),$$

and they didn't developed any significant results in Numerical Methods. But recently, V.Chandrasekar and K.Suresh have generalized the definition of Δ_q by $\Delta_{q(\ell)}$ as

$$\Delta_{q(\ell)}u(k)=\frac{u(kq)-\ell u(k)}{(q-\ell)k}$$

for the real valued function $u(k)$ and $\ell \in (0,\infty)$ and also obtained the several types of arithmetic-geometric progressions in the field of Numerical methods [3].

With this background, in this paper, we define the extended $q-$difference operator and derive the formula for fractional series in the field of Numerical Analysis using its inverse operator.

2. Preliminaries

In this section, we present some basic definitions and preliminary results for further subsequent discussions.

Definition 2.1. *If $u(k)$ is real valued function, then we define the extended $q-$difference operator $\Delta_{q(\ell)}$ as*

$$\Delta_{q(\ell)}u(k) = u((k+\ell)q) - u(k), q, \ell \in (0,\infty). \tag{2.1}$$

Lemma 2.2. *The relation between $\Delta_{q(\ell)}$ and $E^{q(\ell)}$ is*

$$E^{q(\ell)} = \Delta_{q(\ell)} + 1. \tag{2.2}$$

Proof. The shift operator $E^{q(\ell)}$ is defined by

$$E^{q(\ell)}u(k) = u((k+\ell)q), k \in [0,\infty). \tag{2.3}$$

The proof follows from (2.1) and (2.3). □

Lemma 2.3. *If $q, \ell \in N(1) = \{1,2,\cdots\}$, then*

$$1 + \Delta_{q(\ell)} = (1+\Delta)^{q(\ell)}. \tag{2.4}$$

Lemma 2.4. *If q and ℓ are positive reals and n is positive integer, then*

$$E^{nq(\ell)} = \sum_{r=0}^{n} nCr\Delta^{r}_{q(\ell)}. \tag{2.5}$$

Proof. Equation (2.5) follows by (2.2). □

The following two Lemma's are easily deductions from $\Delta_{q(\ell)}$.

Lemma 2.5. *Let $u(k)$ and $v(k) \neq 0$ be any two real valued functions. Then*

$$\Delta_{q(\ell)}[u(k)v(k)] = v((k+\ell)q)\Delta_{q(\ell)}u(k) + u(k)\Delta_{q(\ell)}v(k). \tag{2.6}$$

Lemma 2.6. *If $u(k)$ and $v(k) \neq 0$ are any two real valued functions, then*

$$\Delta_{q(\ell)}\left[\frac{u(k)}{v(k)}\right] = \frac{v(k)\Delta_{q(\ell)}u(k) - u(k)\Delta_{q(\ell)}v(k)}{v(k)v((k+\ell)q)}.$$

The following is the binomial theorem according to $\Delta_{q(\ell)}$.

Theorem 2.7. *If m and n are any two positive integers, then*

$$[(k+\ell)]^m = \frac{1}{q^{mn}}\left[\sum_{r=0}^{n} nCr\Delta^{r}_{q(\ell)}(k^m)\right]$$

Proof. The proof follows by operating both sides on $u(k) = k^m$ in (2.5) . □

Example 2.8. *If θ is in degrees taking only integer values in the anticlockwise direction then*

$$[sin(k+\theta)] = \frac{1}{q^n}\left[\sum_{r=0}^{n} nCr\Delta^{r}_{q(\ell)}sin(k)\right].$$

Proof. The proof follows by taking $\ell = \theta$ and operating on $u(k) = sin(k)$ in (2.5). □

3. Extended q-Polynomial Factorial

In this section, we define the extended q-polynomial factorial, relation between q-polynomial factorial and q-difference operator according to $\Delta_{q(\ell)}$.

Definition 3.1. *If n is positive integer, then we define the extended $q-$polynomial factorial is denoted by $k^{(n)}_{q(\ell)}$ is defined as*

$$k^{(n)}_{q(\ell)} = k\left(\frac{k-\ell}{q}\right)\left(\frac{k-2\ell}{q^2}\right)\cdots\left(\frac{k-(n-1)\ell}{q^{n-1}}\right). \tag{3.1}$$

Lemma 3.2. *If q and ℓ are positive reals and n is a positive integer, then*

$$\Delta_{q(\ell)}k^{(n)}_{q(\ell)} = k^{(n-1)}_{q(\ell)}\left[\frac{q^n-1}{q^{n-1}}k + C_{q(\ell)}\right], \tag{3.2}$$

where $C_{q(\ell)} = \frac{(q^n+(n-1))\ell}{q^{n-1}}$.

Proof. The proof follows from (2.1) and (3.1). □

Theorem 3.3. *If $k^n_{q(\ell)}$ is extended q-polynomial factorial and m, n are the any two positive integers then*

$$\Delta^m_{q(\ell)}k^n_{q(\ell)} = \frac{(q^n-1)}{q^{n-1}}\Delta^{m-1}_{q(\ell)}\left(k^{(n-1)}_{q(\ell)}k^{(1)}_{q(\ell)}\right) + \frac{(q^n+(n-1))\ell}{q^{n-1}}\Delta^{m-1}_{q(\ell)}k^{(n-1)}_{q(\ell)} \tag{3.3}$$

Proof. The proof follows by induction method on m and n. □

Theorem 3.4. *If q and ℓ are positive reals and n is a negative integer, then*

$$\Delta_{q(\ell)}\left[\frac{k+n\ell}{k+(n-1)\ell}\right] = \frac{(1-q)\ell k - q\ell^2}{[k+(n-1)\ell][(k+\ell)q+(n-1)\ell]} \tag{3.4}$$

Proof. (3.4) follows from (2.1) and using lemma 2.6. □

4. Inverse of Extended q-difference Operator

In this section, we define the inverse of extended q-difference operator and derived some interesting results using its inverse.

Definition 4.1. *The inverse of extended q-difference operator denoted by $\Delta^{-1}_{q(\ell)}$ is defined as if*

$$\Delta_{q(\ell)}v(k)=u(k) \text{ then } v(k)=\Delta^{-1}_{q(\ell)}u(k)+c_j \tag{4.1}$$

and the n^{th} order inverse operator denoted by $\Delta^{-n}_{q(\ell)}$ is defined as

$$if\ \Delta^{n}_{q(\ell)}v(k)=u(k) \text{ then } v(k)=\Delta^{-n}_{q(\ell)}u(k)+c_j,$$

where c_j is a constant, depends upon $k\in N_\ell(j), j=k-\left[\frac{k}{\ell}\right]\ell$.

Lemma 4.2. *If $u(k)$ and $v(k)\neq 0$ are any two real valued functions, then*

$$\begin{aligned}\Delta^{-1}_{q(\ell)}\left[u(k)v(k)\right]&=u(k)\Delta^{-1}_{q(\ell)}v(k)\\&\quad-\Delta^{-1}_{q(\ell)}\left[\Delta^{-1}_{q(\ell)}v\left((k+\ell)q\right)\Delta_{q(\ell)}u(k)\right]\end{aligned}$$

Proof. The proof follows from (2.6) and Definition 4.1. □

Theorem 4.3. *If k,ℓ and q are positive real values, then*

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}u\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)=\Delta^{-1}_{q(\ell)}u(k)-\Delta^{-1}_{q(\ell)}u\left(j_{q(\ell)}\right), \tag{4.2}$$

where $j_{q(\ell)}=\dfrac{k-\ell\sum\limits_{t=1}^{\left[\frac{k}{\ell}\right]}q^t}{q^{\left[\frac{k}{\ell}\right]}}$.

Proof. The proof follows from (4.1) and the relation

$$\Delta_{q(\ell)}\left[\sum_{r=0}^{\left[\frac{k}{\ell}\right]}u\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)\right]=u(k).$$

□

Lemma 4.4. *For $\lambda\neq 1$, $k\geq 2q\ell$ and $P(k)$ is any function of k then*

$$\begin{aligned}&\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\lambda^{\left[\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}\right)q\right]}P\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)\\&=\frac{\lambda^{kq}}{\lambda^{\Delta_{q(\ell)}k}-1}\left[1-\frac{\lambda^{\Delta_{q(\ell)}k}\Delta_{q(\ell)}}{\lambda^{\Delta_{q(\ell)}k}-1}\right.\\&\left.+\frac{\lambda^{2\Delta_{q(\ell)}k}\Delta^2_{q(\ell)}}{\left(\lambda^{\Delta_{q(\ell)}k}-1\right)^2}+\cdots\right]P(k)+c_j.\end{aligned}$$

Proof. If $F(k)$ is any function of k then

$$\begin{aligned}\Delta_{q(\ell)}\lambda^kF(k)&=\lambda^{((k+\ell)q)}F\left((k+\ell)q\right)-\lambda^kF(k)\\&=\lambda^{kq}\left[\lambda^{\ell q}E^{q(\ell)}-\lambda^{(1-q)k}\right]F(k)\\&=\lambda^{kq}P(k)\end{aligned}$$

where

$$P(k)=\left[\lambda^{\ell q}E^{q(\ell)}-\lambda^{(1-q)k}\right]F(k)\ (or)$$

$$\left(\lambda^{(1-q)k}\right)^{-1}\left[\frac{\lambda^{\ell q}E^{q(\ell)}}{\lambda^{(1-q)k}}-1\right]^{-1}P(k)=F(k).$$

Operating $\Delta^{-1}_{q(\ell)}$ on both sides of the equation

$$\Delta_{q(\ell)}\lambda^kF(k)=\lambda^{kq}P(k),$$

we get

$$\begin{aligned}\Delta^{-1}_{q(\ell)}\lambda^{kq}P(k)&=\lambda^kF(k)+c\\&=\lambda^k\left(\lambda^{(1-q)k}\right)^{-1}\left[\frac{\lambda^{\ell q}E^{q(\ell)}}{\lambda^{(1-q)k}}-1\right]^{-1}P(k)+c_j.\end{aligned}$$

The proof follows by (2.2), (4.1) and the Binomial theorem. □

5. Applications in Numerical Methods

In this section, we derived some fractional series using the inverse of extended q-difference operators with suitable examples are provided.

Theorem 5.1. *If q and ℓ are positive reals, then*

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\left[\frac{q^{(2)}\left[\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}\right)^2+\ell^2\right]+\left(2q^2-2q+1\right)\ell\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}\right)}{\left[\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+\ell\right)q-\ell\right]}\right]=k\Big|^{k}_{j_{q(\ell)}}. \tag{5.1}$$

Proof. (5.1) follows from (2.1) and lemma 4.2. □

Example 5.2. *Consider the fractional series*

$$\begin{aligned}F=&\frac{79002}{(27)(57)}+\frac{(70686)(3)}{(729)(17)}+\frac{(1284822)(9)}{(19683)(103)}\\&+\cdots+\frac{(1.32350526\times10^{11})(2187)}{(2.824295365\times10^{11})(43663)}\end{aligned}$$

Solution: Taking $k=65,\ell=8$, and $q=3$ in (5.1), we get

$$\begin{aligned}F=&65-\left[\frac{65-8\sum\limits_{t=1}^{8}3^t}{3^8}\right]\\&=65+11.98826398=76.98826398.\end{aligned}$$

Theorem 5.3. *If $k \in [0,\infty)$ and n is a negative integer, then*

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]} \left[\frac{(1-q)\ell\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right) - q\ell^2}{\left(\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}+\ell\right)q+(n-1)\ell\right)\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}+(n-1)\ell\right)} \right] = \frac{k+n\ell}{k+(n-1)\ell}\Big|_{j_{q(\ell)}}^{k}. \tag{5.2}$$

Proof. From (2.1) and (4.1), we have

$$\Delta_{q(\ell)}^{-1}\left[\frac{(1-q)\ell k - q\ell^2}{((k+\ell)q+(n-1)\ell)\,(k+(n-1)\ell)}\right] = \left[\frac{k+n\ell}{k+(n-1)\ell}\right]. \tag{5.3}$$

The proof follows from (4.2) and (5.3). □

Example 5.4. *Consider the fractional series*

$$F = \frac{-78}{(-13)(13)} + \frac{-39}{(-26)(-13)} + \frac{-39}{(-65)(-26)} + \cdots + \frac{(-39)(4096)}{(-319475)(-159731)}$$

Solution: Substituting $n=-10, k=46, \ell=3$, and $q=2$ in (5.2), we get

$$F = \frac{16}{13} - \left[\frac{\left(\frac{46-3\sum_{t=1}^{15} 2^t}{2^{15}}\right)+(-30)}{\left(\frac{46-3\sum_{t=1}^{15} 2^t}{2^{15}}\right)+(-33)}\right]$$
$$= 1.230769231 - 0.923073792$$
$$= 0.307695439.$$

Theorem 5.5. *If q and ℓ are positive reals and n is a positive integer, then*

$$\Delta_{q(\ell)}\left[\frac{k+n\ell}{k+(n+1)\ell}\right] = \frac{(q-1)\ell k + q\ell^2}{[k+(n+1)\ell]\,[(k+\ell)\,q+(n+1)\ell]} \tag{5.4}$$

Proof. By using (2.1) and lemma 2.6, we get (5.4). □

Theorem 5.6. *If $k \in [0,\infty)$ and $n \in N(1)$, then*

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]} \left[\frac{(q-1)\ell\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}\right) + q\ell^2}{\left(\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}+\ell\right)q+(n+1)\ell\right)\left(\frac{k-\ell\sum_{t=1}^{r} q^t}{q^r}+(n+1)\ell\right)} \right] = \frac{k+n\ell}{k+(n+1)\ell}\Big|_{j_{q(\ell)}}^{k}. \tag{5.5}$$

Proof. From (2.1) and (4.1), we have

$$\Delta_{q(\ell)}^{-1}\left[\frac{(q-1)\ell k + q\ell^2}{((k+\ell)q+(n+1)\ell)\,(k+(n+1)\ell)}\right] = \left[\frac{k+n\ell}{k+(n+1)\ell}\right].$$

The proof follows from (4.2) and the above relation. □

Example 5.7. *Consider the fractional series*

$$F = \frac{376}{9125} + \frac{(376)(3)}{35125} + \frac{(376)(9)}{210469} + \cdots + \frac{(376)(19683)}{(7.857849644\times 10^{11})}$$

Solution: Substituting $n=7, k=41, \ell=4$, and $q=3$ in (5.5), we get

$$F = \frac{69}{73} - \left[\frac{\left(\frac{41-4\sum_{t=1}^{10} 3^t}{3^{10}}\right)+28}{\left(\frac{41-4\sum_{t=1}^{10} 3^t}{3^{10}}\right)+32}\right]$$
$$= 0.945205479 - 0.846158555$$
$$= 0.099046924.$$

6. Conclusion

In this paper, an advance has been developed for some results on the solutions of extended q-difference equations governed by (4.2) along with the function $u(k)$ in the field of Numerical analysis. Also, by selecting large value for k and small positive value for q and ℓ one can find the sum of several fractional series easily using the Theorem mentioned above.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Theory, Methods and Applications, *Second Edition, Revised and Expanded, Marceldekker*, New York, 2000.

[2] G. Bangerezako, An Introduction to q-difference equations, *Bujumbura*, 2007 (Preprint).

[3] V. Chandrasekar and K. Suresh, Theory and Applications of Generalized q-Derivative Operator, *International Journal of Pure and Applied Mathematics*, 101(6)(2015), 963-973.

[4] S. N. Elaydi, An Introduction to Difference Equations, *Second Edition*, Springer, 1999.

[5] Walter G. Kelley and Allan C. Peterson, Difference Equations, An Introduction with Applications, *Academic Press*, 1991 .

[6] M. Maria Susai Manuel, G. Britto Antony Xavier, and E. Thandapani, Theory of Generalized Difference Operator and Its Applications, *Far East Journal of Mathematical Sciences*, 20(2)(2006), 163-171.

[7] M. Maria Susai Manuel, G. Britto Antony Xavier, and E. Thandapani, Qualitative Properties of Solutions of Certain Class of Difference Equations, *Far East Journal of Mathematical Sciences*, 23(3)(2006), 295-304.

[8] M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, Rotatory and Boundedness of Solutions of Certain Class of Difference Equations, *International Journal of Pure and Applied Mathematics*, 33(3)(2006), 333-343.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Generalized Bernoulli Polynomials Through Weighted Pochhammer Symbols, *Far East Journal of Applied Mathematics*, 26(3)(2007), 321-333.

[10] M. Maria Susai Manuel and G. Britto Antony Xavier, Recessive, Dominant and Spiral Behaviours of Solutions of Certain Class of Generalized Difference Equations, *International Journal of Differential Equations and Applications*, 10(4)(2007), 423-433.

[11] R. E. Mickens, Difference Equations, *Van-Nostrand Reinhold Company*, New York, 1990.

[12] Thomas Ernst, A Method for q-Calculus, *Journal of Nonlinear Mathematical Physics*, 10(4)(2003), 487-525.

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