Malaya Journal of Matematik, Vol. 9, No. 1, 480-485, 2021

https://doi.org/10.26637/MJM0901/0081

On $\mathscr{G}$-connected and $\mathscr{G}$-Lindeloff spaces in generalized topology

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Abstract

In this paper the notion of generalized Lindeloff property introduced using generalized topological structure. Also discuss Some properties of generalized connectedness and subspace GT.

Keywords

Generalized Topology, generalized continuity, generalized neighborhoods generalized connectedness, $\mathscr{G}$-Lindeloff, subspace GT.

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Article History: Received **26** November **2020**; Accepted **02** February **2021**

Contents

1. Introduction

In 2002 Császár[3]Introduced the notion of generalized topological spaces (GTS) and generalized continuity in his paper named 'Generalized topology, generalized continuity'.

The purpose of present paper is to introduce $\mathscr{G}-$ Lindeloff and study some basic properties of this structure. In section 2, collect all preliminaries and basic definitions useful for subsequent sections. In section 3 we discuss some properties of generalized connectedness and some basic theorems. In section 4 introduce the concept of $\mathscr{G}-$ Lindeloff spaces and discuss some basic theorems.

2. Preliminaries

Definition 2.1 ([4]). *Let X be a set and $\exp(X)$ its power set. According to Császár, a subset $\mathscr{G}$ of $\exp(X)$ is called generalized topology (GT) on X and $(X,G.)$ is called a generalized to pological space (GTS) if $\mathscr{G}$ has the following properties.*

1. *$\Phi \in \mathscr{G}$.*
2. *$\mathscr{G}$ is closed under arbitrary union.*

Definition 2.2 ([4]). *A GTG is called strong if $X \in \mathscr{G}$.*

Definition 2.3 ([4]). *A subset A is called A is called $\mathscr{G}$-open if $A \in \mathscr{G}$. A subset B is called $\mathscr{G}$-closed if $X \setminus B$ is $\mathscr{G}$-open. The generalized topology is denoted by $\mathscr{G}$-topology.*

Definition 2.4 ([4]). *Let $(X,\mathscr{G})$ and $\left(Y,\mathscr{G}^1\right)$ are two generalized topological spaces, $f : X \to Y$ be a function. Then f is called $\left(\mathscr{G},\mathscr{G}^1\right)$ continuous on X, if for any $\mathscr{G}^1$-open set O in $Y, f^{-1}(O)$ is $\mathscr{G}$ open in X.*

Definition 2.5 ([4]). *The function f called a $\mathscr{G}$-homeomorphism from X to Y, if both f and f^{-1} are $\mathscr{G}$-continuous. If we have a $\mathscr{G}$-homeomorphism between X and Y we say that they are $\mathscr{G}$ homeomorphic and denoted by $X \cong_{\mathscr{G}} Y$.*

Definition 2.6 ([5]). *Let $(X,\mathscr{G})$ be s GTS. A collection U of subsets of X is said to be a $\mathscr{G}$-cover of X if the union of elements U equals X.*

Definition 2.7 ([5]). *Let $(X,\mathscr{G})$ be a GTS.A $\mathscr{G}$-subcover of a $\mathscr{G}$-cover τ is a sub collection μ of $\mathscr{U}$ which itself is a $\mathscr{G}$-cover. If the elements of t are $\mathscr{G}$-open then we say that U is a $\mathscr{G}$-open cover.*

Definition 2.8 ([5]). *If every $\mathscr{G}$-open cover of X has a finite $\mathscr{G}$-sub cover then we say that X is $\mathscr{G}$-compact.*

Theorem 2.9 ([5]). *$\mathscr{G}$-continuous image of $\mathscr{G}$-compact set is $\mathscr{G}$-compact.*

Definition 2.10 ([2]). *The intersection of $\mathscr{G}$-closed set containing A is called the $\mathscr{G}-$ closure of A and is denoted by $C_{\mathscr{G}}(A)$.*

The fundamental reference for the topological ring and their properties is [6].

Definition 2.11 ([7]). *Let $(X,\mathscr{G})$ be a $\mathscr{G}$-topological space and B be a sub collection of $\mathscr{G}$ is called a base for $\mathscr{G}$-topological space, if every $\mathscr{G}$-open set can be expressed as union of some members of B.*

Definition 2.12 ([5]). *Let $(X,\mathscr{G})$ be a GTS. A point $x \in X$ is called a $\mathscr{G}$-cluster point of $A \subseteq X$ if $U \cap (A \backslash \{x\}) \neq \phi$ for each U in $\mathscr{G}$ with x in U.*

Theorem 2.13 ([8]). *$B \subseteq P(X)$ is a base for a GT$\mathscr{G}$ if and only if whenever U is a $\mathscr{G}$-open set and $x \in U$, then there exist a $B \in B$ such that $x \in B \subseteq B$.*

Definition 2.14 ([7]). *Let $(X,\mathscr{G}_X)$ and $(Y,\mathscr{G}_Y)$ be $\mathscr{G}$-topological spaces. The product $\mathscr{G}$-topology on $X \times Y$ is the $\mathscr{G}G$-topology having as a basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where $U \in \mathscr{G}G_X$ and $V \in \mathscr{G}G_Y$.*

Definition 2.15 ([1]). *Let $(X,\mathscr{G}G)$ be a GTS. X is called $\mathscr{G}-$ connected if there are no nonempty disjoint $\mathscr{G}$-open subsets U,V of X such that $U \cup V = X$.*

Definition 2.16 ([9]). *Let $(X,\mathscr{G})$ be a GTS and Y be a subset of X then $\mathscr{G}_Y = \{U \cap Y : U \in \mathscr{G}\}$ is a GT on Y, it is called the subspace GT on Y.*

3. More results on generalized connectedness in generalized topology

Theorem 3.1. *The $\mathscr{G}$-connectedness is preserved under g-continuous functions.*

Proof. Let $f : X \to Y$ is a $\mathscr{G}$-continuous onto function. Assume that X is $\mathscr{G}$-connected. To show that Y is $\mathscr{G}$-connected. Assume the contradiction that there are two disjoint $\mathscr{G}$-open subsets U,V of Y such that $U \cup V = Y$. Then $X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Thus X can be expressed as union of two disjoint $\mathscr{G}$-open subsets of X. Which is a contradiction. □

Theorem 3.2. *Suppose X and Y are $\mathscr{G}$-homeomorphic GTS. Then X and Y are either both $\mathscr{G}$-connected or both not $\mathscr{G}$-connected.*

Theorem 3.3. *A union of two intersecting $\mathscr{G}$-connected sub spaces is $\mathscr{G}$-connected. That is, suppose $X = U \cup V$, where U,V are both $\mathscr{G}$-connected, and $U \cap V \neq \emptyset$. Then X is $\mathscr{G}-$ connected.*

Proof. Assume X is not $\mathscr{G}-$ connected, $X = A \cup B$, where A and B are nonempty, disjoint and $\mathscr{G}$-open subsets. Pick a point $v \in U \cap V$. We can assume $v \in A$. Now consider $U \cap A$ and $U \cap B$. They are $\mathscr{G}$-open subsets in the subspace GT $\mathscr{G}_U$. Also $U \cap A$ contains v and so is nonempty. If $U \cap B$ is non-empty, we would get U is not $\mathscr{G}-$ connected. So we must have $U \cap B$ is empty. So $U \subseteq A$. Using the same argument we get $V \subseteq B$. But then $X = U \cup V$ is contained in A, and B must be empty. Contradiction. □

Theorem 3.4. *Let $(X,\mathscr{G})$ be a GTS, then X is $\mathscr{G}$-connected if and only if there are no two nonempty subsets A,B of X such that $X = AUB$ and $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$.*

Proof. Assume X is $\mathscr{G}$-connected. If possible there exist two nonempty subsets A,B of X such that $X = AUB$ and $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$. Taking complements and apply DeMorgan's law on both sides of the equation $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$, implies that $C_{\mathscr{G}}(A)^c \cup C_{\mathscr{G}}(B)^c = X$.

Also $C_{\mathscr{G}}(A)^c \cap C_{\mathscr{G}}(B)^c = \phi$, for if there exist $z \in X$ suchthat $z \in C_{\mathscr{G}}(A)^c \cap C_{\mathscr{G}}(B)^c$, which implies that $z \notin C_{\mathscr{G}}(A)$ and $z \notin C_{\mathscr{G}}(B) \Longrightarrow z \notin C_{\mathscr{G}}(A) \cup C_{\mathscr{G}}(B) \supseteq A \cup B = X \Longrightarrow z \notin X$. This become a contradiction. Hence X can be expressed as union of two disjoint nonempty $\mathscr{G}$-open sets, this is a contradiction to the fact that X is $\mathscr{G}$-connected.

Conversely assume there are no two nonempty subsets A,B of X such that $X = AUB$ and $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$. If possible X is not $\mathscr{G}-$ connected, then there exist two nonempty disjoint $\mathscr{G}$-open sets A,B of X such that $X = AUB$. So $A = X - B$ is $\mathscr{G}$-closed. Similarly B is G-closed. Hence $C_{\mathscr{G}}(A) = A$ and $C_{\mathscr{G}}(B) = B$. So there are no two nonempty subsets A,B of X such that $X = AUB$ and $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$. which is a contradiction. Hence X is $\mathscr{G}$-connected. □

Definition 3.5. *Let $(X,\mathscr{G})$ be a GTS, $x_0 \in X$ and $N \subseteq X$. Then x_0 is said to be a generalized interior point of N, if there is a G-open set V such that $x_0 \in V \subseteq N$. The set of all generalized interior points of N is called the generalized interior of N and is denoted by $\mathscr{G}\,\mathrm{int}(N)$.*

Example 3.6. *Consider $X = \mathbb{R}$ and $\mathbb{Z}$ be the set of all integers. If $\mathscr{G} = \{U \subseteq \mathbb{R} : U \subseteq \mathbb{R} - \mathbb{Z}\}$, then $\mathscr{G}$ is a GT on X and $\frac{1}{2}$ is a generalized interior point of the set of rational numbers $\mathbb{Q}$.*

Theorem 3.7. *Let $(X,\mathscr{G})$ be a GTS and $A \subseteq X$. Then $\mathscr{G}$ int (A) is the union of all $\mathscr{G}$-open sets contained in A. It is also the largest $\mathscr{G}$-open subset of X contained in A.*

Proof. Let $\mathscr{M}$ be the family of all $\mathscr{G}$-open sets contained in A.$\mathscr{M}$ is nonempty because $\phi \in \mathscr{M}$. Let $V = \bigcup_{\mathscr{G} \in \mathscr{M}} \mathscr{G}$. Clam: $V = \mathrm{Gint}(A)$. Now $x \in V$ then $x \in \mathscr{G}$ for some $\mathscr{G} \in \mathscr{M}$. That is $x \in \mathscr{G} \subseteq A \Rightarrow x \in \mathscr{G}int(A)$. conversely let $x \in \mathscr{G}\,\mathrm{int}(A)$. Then there is a $\mathscr{G}$-open set H such that $x \in H \subseteq A$. Then $H \in \mathscr{M}$ and so $H \subseteq V$. So $x \in V$. Suppose G is any $\mathscr{G}$-open set contained in A. Then $\mathscr{G} \in \mathscr{M}$ and so $\mathscr{G} \subseteq \mathscr{G}\,\mathrm{int}(A)$. Hence $\mathscr{G}\,\mathrm{int}(A)$ is the largest $\mathscr{G}$-open set contained in A. □

Definition 3.8. *Let A be a subset of a GTS X. Then its generalized boundary is the set $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(X - A)$. It is denoted by $\mathscr{G}\partial A$.*

Remark 3.9. *1. The generalized boundary of a set is always a $\mathscr{G}$-closed set*

2. The generalized boundary of a set is same as The generalized boundary of its complement.

Theorem 3.10. *Let A be a subset of a GTS X then $C_{\mathscr{G}}(A)$ is the disjoint union of $\mathscr{G}$ int (A) with the generalized boundary of A.*

Proof. **Claim:** $C_{\mathscr{G}}(A) = \mathscr{G}int(A) \cup \mathscr{G}\partial A$. We have

$$\begin{aligned}&\mathscr{G}int(A) \cup \mathscr{G}\partial A\\ &= \mathscr{G}\,\mathrm{int}(A) \cup (C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(X-A))\\ &= [\mathscr{G}\,\mathrm{int}(A) \cup (C_{\mathscr{G}}(A))] \cap [\mathscr{G}\,\mathrm{int}(A) \cup (C_{\mathscr{G}}(X-A))]\\ &= (C_{\mathscr{G}}(A)) \cap X\\ &= C_{\mathscr{G}}(A)\end{aligned}$$

□

Theorem 3.11. *Let $(X,\mathscr{G})$ be a GTS and $A \subseteq X$. Then A is $\mathscr{G}$-closed if and only if it contains its generalized boundary and A is $\mathscr{G}$-open if and only if it disjoint from its generalized boundary.*

Proof. Assume A is $\mathscr{G}-$ closed. Then $C_{\mathscr{G}}(A) = A$. We have $\mathscr{G}\partial A = C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(X-A) \subseteq C_G(A) =$ A. conversely suppose $\mathscr{G}\partial A \subseteq$ A. It is enough to prove that $C_{\mathscr{G}}(A) \subseteq A$. From above theorem $C_{\mathscr{G}}(A) = \mathscr{G}\,\mathrm{int}(A) \cup \mathscr{G}\partial A \subseteq A$. Hence $C_{\mathscr{G}}(A) = A$. So A is $\mathscr{G}-$ closed. Assume A is $\mathscr{G}$-open. Then

$$\begin{aligned}\mathscr{G}\partial A \cap A &= [C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(X-A)] \cap A\\ &= [C_{\mathscr{G}}(A) \cap (X-A)] \cap A\\ &= \phi\end{aligned}$$

Conversely assume $\mathscr{G}\partial A \cap A = \phi$. Then $\mathscr{G}\partial A \subseteq X - A$. But the generalized boundary of A is same as the generalized boundary of $(X-A)$. Hence by above part $(X-A)$ is $\mathscr{G}$-closed. So A is $\mathscr{G}$-open. □

Theorem 3.12. *Let $(X,\mathscr{G})$ be a GTS and $A \subseteq X$. Then A is both $G-$ closed and G-open (i.e $\mathscr{G}$-clopen) if and only if $\mathscr{G}\partial A = \phi$.*

Proof. A is both G-closed and $\mathscr{G}$-open implies $\mathscr{G}\partial A \subseteq$ Aand $\mathscr{G}\partial A \cap A = \phi$. Hence $\mathscr{G}\partial A = \phi$. Conversely assume $\mathscr{G}\partial A = \phi$. So $G\partial A \subseteq A$ and $\mathscr{G}\partial A \cap A = \phi$. Hence by above theorem, A is both $\mathscr{G}$-closed and $\mathscr{G}$-open. □

Example 3.13. *Consider $X = \mathbb{R}$ and $\mathbb{Z}$ be the set of all integers. If $\mathscr{G} = \{U \subseteq \mathbb{R} : U \subseteq \mathbb{R} - \mathbb{Z}\}$, then G is a GT on X. Note that the generalized boundary of $\mathbb{Z}$ is $C_{\mathscr{G}}(\mathbb{Z}) \cap C_{\mathscr{G}}(\mathbb{R}-\mathbb{Z}) = \mathbb{Z} \cap \mathbb{R} = \mathbb{Z}$.*

Theorem 3.14. *Let $(X,\mathscr{G})$ be a GTS and A,B are subsets of X. Then the given statements are equivalent:*

1. *$AUB = X$ and $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$*
2. *$AUB = X$ and $A \cap B = \phi$ and A,B are both $\mathscr{G}-$ closed in X.*
3. *$B = X - A$ and A is both $\mathscr{G}$-open and $\mathscr{G}$-closed in X.*
4. *$B = X - A$ and the generalized boundary of A is empty.*
5. *$AUB = X$ and $A \cap B = \phi$ and A,B are both $\mathscr{G}$-open in X.*

Proof. $(1) \implies (2)$
$C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi \Rightarrow A \cap B = \phi$ as $A \subseteq C_{\mathscr{G}}(A)$ and $B \subseteq C_{\mathscr{G}}(B)$. Also $C_{\mathscr{G}}(A) \subseteq X - C_{\mathscr{G}}(B) \subseteq X - B = A$. So $C_{\mathscr{G}}(A) = A$ as $A \subseteq C_{\mathscr{G}}(A)$. Hence A is $\mathscr{G}-$ closed in X. Similarly B is $\mathscr{G}$-closed in X.
$(2) \Rightarrow (3)$
By assumption A,B are both $\mathscr{G}$-closed in X and , which implies $X - A$ and $X - B$ are $\mathscr{G}$-open in X. But $X - A = B$ and $X - B = A$.
$(3) \Rightarrow (4)$
$B = X - A$ and both A,B are $\mathscr{G}-$ closed in X implies that the generalized boundary of $A = C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(X-A) = A \cap B = \phi$.
$(4) \Rightarrow (5)$
$B = X - A \Rightarrow AUB = X$ and $A \cap B = \phi$. By theorem 3.11A is $\mathscr{G}$-open and $\mathscr{G}-$ closed in X. Since $B = X - A$, both A and B are $\mathscr{G}$-open in X.
$(5) \Rightarrow (1)$
Assume $AUB = X$ and $A \cap B = \phi$ and A,B are both $\mathscr{G}$-open in X. Then $A = X - B$ and $B = X - A$. Hence A,B are both $\mathscr{G}$-open and $\mathscr{G}-$ closed in X. So $C_{\mathscr{G}}(A) = A$ and $C_{\mathscr{G}}(B) = B$ Hence $C_{\mathscr{G}}(A) \cap C_{\mathscr{G}}(B) = \phi$. □

Corollary 3.15. *If X is not a strong GTS, then X must be $\mathscr{G}$-connected.*

Proof. If X is not a strong GTS, then X is not $\mathscr{G}$-open. So X cannot be expressed as disjoint union of two nonempty $\mathscr{G}$-open sets in X. □

Theorem 3.16. *Let $(X,\mathscr{G})$ be a GTS. Then the following statements are equivalent:*

1. *X is $\mathscr{G}$-connected.*
2. *X cannot be written as the disjoint union of two nonempty $\mathscr{G}$-closed subsets of X*
3. *Every nonempty proper subset of X has a nonempty generalized boundary.*
4. *X cannot be written as the disjoint union of two nonempty $\mathscr{G}$-open subsets of X*

Definition 3.17. *Two subsets of A and B of a GTS X are said to generalized separated (or $\mathscr{G}-$ separated) if $C_{\mathscr{G}}(A) \cap B = \emptyset$ and $A \cap C_{\mathscr{G}}(B) = \phi$.*

Example 3.18. *Let $X = \{1,2,3\}$ and $\mathscr{G} = \{X, \phi, \{1,2\}, \{3\}\}$ is a GT on X. Then $\{1,2\}$ and $\{3\}$ are $\mathscr{G}-$ separated.*

Example 3.19. *Consider $X = \mathbb{R}$ and $\mathbb{Z}$ be the set of all integers. If $\mathscr{G} = \{U \subseteq \mathbb{R} : U \subseteq \mathbb{R} - \mathbb{Z}\}$, then $\mathscr{G}$ is a GT on X. Then $\mathbb{Q} - \mathbb{Z}$ and $\mathbb{Z}$ are* not*$\mathscr{G}-$ separated. Because $\mathbb{Q}^c$ is $\mathscr{G}$-open so $\mathbb{Q}$ is $\mathscr{G}$-closed. Hence the smallest $\mathscr{G}$-closed set containing $\mathbb{Q} - \mathbb{Z}$ is $\mathbb{Q} \cdot \mathbb{Q} \cap \mathbb{Z} \neq \emptyset$.*

Remark 3.20.
1. *Note that A and B are $\mathscr{G}-$ separated if and only if they are disjoint $\mathscr{G}-$ closed subsets of $A \cup B$ with subspace GT in $A \cup B$.*
2. *A GTS is $\mathscr{G}$-connected if it is not the union of two nonempty $\mathscr{G}$-separated subsets.*

Theorem 3.21. *Let X be a GTS and C be a $\mathscr{G}$-connected subset of X. Suppose $C \subseteq A \cup B$ where A and B are $\mathscr{G}-$ separated subsets of X. Then either $C \subseteq A$ or $C \subseteq B$*

Proof. Let $\mathscr{G} = C \cap A$ and $H = C \cap B$. Then G, H are $\mathscr{G}$-closed in $A \cup B$. Also $\mathscr{G} \cap H = \phi$. But C be a $\mathscr{G}$-connected subset of X. So either $\mathscr{G} = \phi$ or $H = \phi$. In the first case $C \subseteq B$ while in the second, $C \subseteq A$ □

Theorem 3.22. *Let C be a collection of $\mathscr{G}-$ connected subsets of X such that no two members of $\mathscr{C}$ are $\mathscr{G}-$ separated. Then $U_{C \in \mathscr{C}} C$ is also $\mathscr{G}-$ connected.*

Proof. Let $M = U_{C \in \mathscr{C}}$ C. If M is not $\mathscr{G}$-connected, then we can write M as $A \cup B$ where A, B are nonempty and $\mathscr{G}-$ separated subsets of X. By above proposition, for each $C \in \mathscr{C}$ either $C \subseteq A$ or $C \subseteq B$. Claim: $C \subseteq A$ for all $C \in \mathscr{C}$ or $C \subseteq B$ for all $C \in \mathscr{C}$. If not then there exist $C, D \in \mathscr{C}$ such that $C \subseteq A$ and $D \subseteq B$. But, A, B are $\mathscr{G}-$ separated subsets of X. Hence their subsets are also $\mathscr{G}$-separated subsets contradicting the hypothesis. Thus all members of $\mathscr{C}$ are contained in A or all are contained in B. So $M = A$ or $M = B$, contradicting that A, B are both nonempty. □

Theorem 3.23. *Let $\mathscr{C}$ be a collection of $\mathscr{G}-$ connected subsets of X of a space X and suppose K is a $\mathscr{G}$-connected subset of X (not necessarily a member of $\mathscr{C}$) such that $C \cap K \neq \phi$ for all $C \in \mathscr{C}$. Then $(U_{c \in C} C) \cup K$ is $\mathscr{G}$-connected.*

Proof. Let $M = (U_{C \in \mathscr{C}} C) \cup K$. Let $\mathscr{H} = \{K \cup C : C \in \mathscr{C}\}$. Clearly $M = \bigcup_{H \in \mathscr{H}} H$. By above theorem each member of $\mathscr{H}$ is $\mathscr{G}$-connected since it is a union of two $\mathscr{G}$-connected sets which intersect. Now any two members of $\mathscr{H}$ have at least points of K in common and so not $\mathscr{G}$-separated. So by above theorem M is $\mathscr{G}$-connected. □

Theorem 3.24. *Let X_1, X_2 be generalized topological spaces and $X = X_1 \times X_2$ with $\mathscr{G}-$ product topology. Then X is $\mathscr{G}$-connected.*

Proof. If either X_1 or X_2 is empty then so is X and the result hold trivially. So assume both X_1 and X_2 are non empty. Fix a point $y_1 \in X_1$. Then the set $\{y_1\} \times X_2$ is $\mathscr{G}$-homeomorphic to X_2 and hence is $\mathscr{G}$-connected. Call it K. For each $x \in X_2$, the set $X_1 \times \{x\}$ is similarly $\mathscr{G}$-connected and its intersection with K is nonempty. Also note that $X_1 \times X_2 = (\cup_{x \in X_2} X_1 \times \{x\}) \cup K$. So by above theorem $X_1 \times X_2$ is $\mathscr{G}-$ connected. □

Remark 3.25. *In an ordinary topological space closure of a connected set is connected. But in generalized topological space the $\mathscr{G}-$ closure of a $\mathscr{G}$-connected set need not be $\mathscr{G}$-connected.*

Example 3.26. *Let $X = \{a, b, c, d\}$ and*

$$\mathscr{G} = \{X, \emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}, \{c,d\}, \{b,c,d\}\}.$$

Let $A = \{a,b,c\}$. Then A is $\mathscr{G}-$ connected and $C_{\mathscr{G}}(A) = X$. since $X = \{a,b\} \cup \{c,d\}$, we see that X is not $G-$ Connected.

Definition 3.27. *A generalized component ($\mathscr{G}-$ component) of a GTS is a maximally $\mathscr{G}$-connected subset which is not properly contained in any $\mathscr{G}$-connected sub set of that GTS.*

Remark 3.28. *In ordinary topological spaces components are closed, but in generalized topological spaces $\mathscr{G}-$ components need not be $\mathscr{G}-$ closed.*

Example 3.29. *Let $X = \{a, b, c, d\}$ and*

$$\mathscr{G} = \{\emptyset, \{a,b\}, \{b,c\}, \{a,b,c\}\}.$$

Let $A = \{a,b,c\}$. Then A is $\mathscr{G}$-connected and a $\mathscr{G}-$ component of X. But A is not $\mathscr{G}-$ closed.

Theorem 3.30. *Any two distinct $\mathscr{G}$-components are mutually disjoint.*

Proof. Let C, C' be two $\mathscr{G}-$ components. If $C \cap C'$ is nonempty then by theorem 3.23$C \cup C'$ be $\mathscr{G}$-connected. But $C \subseteq C \cup C'$ and $C' \subseteq C \cup C'$. So by maximality of C and C' we get $=$ $C \cup C' = C'$. Thus two distinct $\mathscr{G}-$ components are mutually disjoint. □

Theorem 3.31. *Every nonempty $\mathscr{G}$-connected subset is contained in a unique $\mathscr{G}$-component.*

Proof. Let A be a nonempty $\mathscr{G}$-connected subset of a GTS X. Let $\mathscr{C}$ be the collection of all $\mathscr{G}$-connected subsets of X containing A and let $M = U_{C \in \mathscr{C}} C$. Then any two members of C intersect, so by theorem 3.22, M is $\mathscr{G}-$ connected. Clearly $A \subseteq M$. We claim M is a

$\mathscr{G}$-component. For suppose N is a $\mathscr{G}$-connected subset of X containing M. Then $N \in \mathscr{C}$ and so $N \subseteq M$. Hence $M = N$. In other words, M is a maximally $\mathscr{G}$-connected subset of X. Thus every nonempty subset is contained in a $\mathscr{G}$-component. Such a $\mathscr{G}$-component is unique since two distinct $\mathscr{G}$-components are disjoint. □

Theorem 3.32. *Let B be a base for a GT $\mathscr{G}$ on a set X and let$Y \subseteq X$. Let $B_Y = \{B \cap Y : B \in B\}$. Then B_Y is a base for the subspace GT $\mathscr{G}_Y$ on Y.*

Proof. Here we use theorem 2.14. Let $y \in Y$ and U be a $\mathscr{G}$-open set in Y containing y. Then $U = H \cap Y$ for some $\mathscr{G}$-open set in H in X. So $y \in H$ and hence there exist $B \in B$ such that $y \in B \subseteq H$. Then $y \in B \cap Y \subseteq H \cap Y = U$ and $B \cap Y \in B_Y$. Hence by theorem2.14, B_Y is a base for the subspace GT $\mathscr{G}_Y$ on Y. □

4. $\mathscr{G}$-Lindeloff Spaces

Definition 4.1. *Let $(X,\mathscr{G})$ be a GTS. Then X is said to be $\mathscr{G}-$ Lindeloff space iff each $\mathscr{G}$-cover of X has a countable $\mathscr{G}$-open subcover.*

Theorem 4.2. *If $(X,\mathscr{G})$ is a countable GTS. Then X is $\mathscr{G}-$ Lindeloff .*

Proof. Let $X=\{x_1,x_2,\ldots,x_n,\ldots\}$. Let $\mathscr{F}$ be a $\mathscr{G}$-open covering of X. Then each element of X belongs to at least one of the element of $\mathscr{F}$, say $x_1 \in F_1, x_2 \in F_2,\ldots,x_n \in F_n,\ldots$ where each $F_i \in \mathscr{F}$ for $i=1,2,3,\ldots,n,\ldots$. Then the collection $\{F_1,F_2,\ldots,F_n,\ldots\}$ is a countable sub collection of $\mathscr{F}$ and which is a $\mathscr{G}$-open cover of X. Hence X is $\mathscr{G}-$ Lindeloff. □

Theorem 4.3. *Let $(X,\mathscr{G})$ be a GTS, where $\mathscr{G}=\{U\subseteq X : X\backslash U$ is either countable or all of $X\}$, then X is $\mathscr{G}-$ Lindeloff.*

Proof. Let $\mathscr{F}$ be a $\mathscr{G}$-open cover of X. Then the complement of each member of $\mathscr{F}$ is either countable or all of X. Let $\mathscr{G}$ be a non empty arbitrary member of $\mathscr{F}$, then $X\backslash\mathscr{G}$ is countable. Let $X\backslash\mathscr{G}=\{x_1,x_2,\ldots,x_n,\ldots\}$. Since $\mathscr{F}$ is a $\mathscr{G}$-open cover of X, each $x_i, i=1,2,3,\ldots,n,\ldots$ belongs to at least one member of $\mathscr{F}$, say $x_1\in F_1, x_2\in F_2,\ldots,x_n\in F_n,\ldots$, where $F_i\in\mathscr{F}$ for $i=1,2,3,\ldots,n,\ldots$. Then the collection $\{\mathscr{G},F_1,F_2,\ldots,F_n,\ldots\}$ is a countable sub collection $\mathscr{F}$ which is a $\mathscr{G}$-open cover of X. Hence X is $\mathscr{G}$-Lindeloff . □

Theorem 4.4. *Let $(X,\mathscr{G})$ be a GTS. If $F_1,F_2,\ldots,F_n,\ldots$ are $\mathscr{G}-$ Lindeloff subset of X then $\bigcup_{n=1}^{\infty}F_n$ is is $\mathscr{G}-$ Lindeloff. i.e Countable union of $\mathscr{G}-$ Lindeloff sets is is $\mathscr{G}-$ Lindeloff.*

Proof. Let U and V be any two $\mathscr{G}-$ Lindeloff subsets of X. Let $\mathscr{F}$ be a $\mathscr{G}$-open cover of $U\cup V$. Then $\mathscr{F}$ is also a $\mathscr{G}$-open cover of both U and V. So by hypothesis there exist countable sub collection of $\mathscr{F}$ of $\mathscr{G}$-open sets, say, $\{U_1,U_2,\ldots,U_n,\ldots\}$ and $\{V_1,V_2,\ldots,V_n,\ldots\}$ covering U and V respectively. Clearly the collection $\{U_1,U_2,\ldots,U_n,\ldots,V_1,V_2,\ldots V_n,\ldots\}$ is a countable collection of $\mathscr{G}$-open sets covering $U\cup V$. By induction every countable union of $\mathscr{G}$ Lindeloff sets is is $\mathscr{G}-$ Lindeloff. □

Theorem 4.5. *Let $(X,\mathscr{G})$ be a GTS and $A\subseteq X$. Then A is a $\mathscr{G}-$ Lindeloff subset of X if and only if $(A,\mathscr{G}_A)$ is $\mathscr{G}-$ Lindeloff*

Proof. Assume that A is a $\mathscr{G}-$ Lindelof f subset of X. Let $\mathscr{F}$ be a $\mathscr{G}$-open cover of $(A,\mathscr{G}_A)$. So each member $\mathscr{G}$ of $\mathscr{F}$ is of the form $H\cap A$ for some $H\in\mathscr{G}$. For each $\mathscr{G}\in\mathscr{F}$, fix $D(\mathscr{G})\in\mathscr{G}$ such that $\mathscr{G}=D(\mathscr{G})\cap A$. Then the family $\{D(\mathscr{G}):\mathscr{G}\in\mathscr{F}\}$ is a $\mathscr{G}$-open cover of A by members of $\mathscr{G}$. Since A is a $\mathscr{G}-$ Lindeloff subset of X, this $\mathscr{G}$-open cover has a countable $\mathscr{G}-$ open sub cover, say, $\{D(\mathscr{G}_i):i=1,2,\ldots\}$ where $\mathscr{G}_i\in\mathscr{F}$, for all $i=1,2,\ldots$, Then $\{\mathscr{G}_1,\mathscr{G}_2,\ldots,\mathscr{G}_n,\ldots\}$ is a countable sub cover of $\mathscr{F}$. Hence $(A,\mathscr{G}_A)$ is $\mathscr{G}-$ Lindeloff.

Conversely assume $(A,\mathscr{G}_A)$ is $\mathscr{G}-$ Lindeloff f. Let $\mathscr{F}$ be a $\mathscr{G}$-open cover of A by members in $\mathscr{G}$. Then $\{\mathscr{G}\cap A:\mathscr{G}\in\mathscr{F}\}$ is a $\mathscr{G}$-open cover of A by members in $\mathscr{G}_A$. By assumption this $\mathscr{G}$-cover has a countable $\mathscr{G}$-open sub cover, say, $\{G_i\cap A:i=1,2,\ldots\}$ where $\mathscr{G}_i\in\mathscr{F}$ for $i=1,2,\ldots$. Clearly $\{\mathscr{G}_1,\mathscr{G}_2,\ldots,\mathscr{G}_n,\ldots\}$ is a countable subfamily of $\mathscr{F}$, covering the set A. Thus A is a $\mathscr{G}-$ Lindeloff subset of X. □

Theorem 4.6. *Let $(X,\mathscr{G})$ be a GTS and X is $\mathscr{G}-$ Lindeloff . Let $A\subseteq X$ is $\mathscr{G}-$ closed in X. Then $(A,\mathscr{G}_A)$ is $\mathscr{G}-$ Lindeloff*

Proof. Assume X is $\mathscr{G}-$ Lindeloff f and $A\subseteq X$ is $\mathscr{G}-$ closed in X. Let $\mathscr{F}$ be a $\mathscr{G}$-open cover of A by members in $\mathscr{G}_A$. For each $U\in\mathscr{F}$, fix a $\mathscr{G}$-open set $V(U)$ in X such that $A\cap V(U)=U$. Then the family $\mathscr{L}=\{V(U):U\in\mathscr{F}\}\cup\{X-A\}$ is a $\mathscr{G}$-open cover of X and hence admits a countable $\mathscr{G}$-open sub cover consisting of $V(U_1),V(U_2),\ldots,V(U_n),\ldots$ and possibly $X-A$. But then $\{U_1,U_2,\ldots,U_n,\ldots\}$ covers A and is a countable $\mathscr{G}$-open sub cover of $\mathscr{F}$. Hence $(A,\mathscr{G}_A)$ is $\mathscr{G}-$ Lindeloff. □

Theorem 4.7. *Every uncountable sub set of a $\mathscr{G}-$ Lindeloff space X has at least one $\mathscr{G}$-cluster point in X.*

Proof. Assume X is a $\mathscr{G}$-Lindeloff space and A be an uncountable sub set of X. Assume that A has no $\mathscr{G}$-cluster point in X. Then for each $x\in X$, there exist $U_x\in\mathscr{G}$ with $x\in U_x$ and such that $U_x\cap A=\{x\}$ or ϕ. Now the collection $\{U_x:x\in X\}$ is a $\mathscr{G}$-open cover of X. since X is $\mathscr{G}-$ Lindeloff this collection has a countable sub cover, say $U_{x_1},U_{x_2},\ldots,U_{x_n},\ldots$ But $(U_{x_1}\cap A)\cup(U_{x_2}\cap A)\cup\ldots\cup(U_{x_n}\cap A)\cup\ldots=\{x_1\}\cup\{x_2\}\cup\ldots\cup\{x_n\}\cup\ldots$ or $\phi\Longrightarrow(U_{x_1}\cup U_{x_2}\cup\ldots\cup U_{x_n}\ldots)\cap A=\{x_1,x_2,\ldots,x_n,\ldots\}or$ ϕ, contradicts that A is uncountable. □

Theorem 4.8. *The $\mathscr{G}$-Lindeloff property is preserved under $\mathscr{G}$-continuous functions.*

Proof. Let $(X,\mathscr{G}_1)$ and $(X,\mathscr{G}_2)$ are two GTS's. Let $f:(X,\mathscr{G}_1)\to(X,\mathscr{G}_2)$ be a $(\mathscr{G}_1,\mathscr{G}_2)$ continuous function from X onto Y. Let X be $\mathscr{G}_1-$ Lindeloff. Let $\mathscr{F}$ be any $\mathscr{G}_2$-open cover of Y, then the collection $\left\{f^{-1}(\mathscr{G}):\mathscr{G}\in\mathscr{F}\right\}$ is a $\mathscr{G}_1$-open cover of X. since X is $\mathscr{G}_1-$ Lindeloff , there exist a countable $\mathscr{G}_1$-open sub cover of X, say $\left\{f^{-1}(\mathscr{G}_1),f^{-1}(\mathscr{G}_2),\ldots,f^{-1}(\mathscr{G}_n),\ldots.\right\},\mathscr{G}_1,\mathscr{G}_2,\ldots\mathscr{G}_n,\ldots\in\mathscr{F}$. Since the mapping is onto, the collection $\{\mathscr{G}_1,\mathscr{G}_2,\ldots,\mathscr{G}_n,\ldots\}$ is a $\mathscr{G}_2$-open sub cover of Y. Hence Y is $\mathscr{G}_2-$ Lindeloff. □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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