



Weak Roman domination in Chess graphs

P. Roushini Leely Pushpam^{1*} and M. Kamalam²

Abstract

Let $G = (V, E)$ be a graph and $f : V \rightarrow \{0, 1, 2\}$ be a function. A vertex u with weight $f(u)$ is said to be *undefended* with respect to f , if it is not adjacent to any vertex with positive weight. The function f is a *weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' : V \rightarrow \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight of f is $w(f) = \sum_{v \in V} f(v)$. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a weak Roman dominating function on G . In this paper, we present a constant time algorithm to obtain $\gamma_r(KN_{m,n})$, where, $KN_{m,n}$ is the Knight's graph.

Keywords

Weak Roman Domination Number, constant time algorithm.

AMS Subject Classification

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¹Department of Mathematics, D.B. Jain College, Chennai-600 097, Tamil Nadu, India.²Department of Mathematics, S.S. Shasun Jain College, Chennai-600 017, Tamil Nadu, India.*Corresponding author: ¹roushinip@yahoo.com; ²divinegrace27@gmail.com

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1. Introduction

The Roman Empire was the greatest in the History of Empires, designated so, owing to the Military Power it possessed, the

influence it had on the rest of the world at the time it existed, and the duration of time it existed. One of its emperors, Constantine, the Great, who faced continuous threat of attacks by his neighbouring enemies, devised a plan to protect all his territories. He created two types of armies for the protection of various territories that made up the Roman empire under his reign, armies that travelled from a territory at which they are stationed to any of its neighbouring territories to protect them and armies that remained stationery and protected only those territories at which they are stationed. Graph theoretically, the scenario could be depicted as a graph, in which the territories are represented as vertices and the neighbourhood relationship between pairs of territories are represented as edges. A formal discussion on the strategy of the Emperor, Constantine, the Great was first published by Ian Stewart [8]. Motivated by this, Cocayne et al. [5] defined the *Roman dominating function* (RDF) on a graph $G = (V, E)$ as a function $f : V \rightarrow \{0, 1, 2\}$ under which, if u is any node with $f(u) = 0$, then u is adjacent to a vertex v for which, $f(v) = 2$. The *weight of a Roman dominating function* f defined on a graph $G = (V, E)$ is denoted by $w(f)$ and is defined as $w(f) = \sum f(u)$, where the summation runs over all the vertices u in V . Denoting by V_i , the set of all vertices u in V with $f(u) = i$, there is a one-to-one correspondence between the functions $f : V \rightarrow \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) . We thus identify any function

$f : V \rightarrow \{0, 1, 2\}$, with an ordered partition (V_0, V_1, V_2) . The *Roman domination number* $\gamma_R(G)$, of a graph G , is the minimum weight of any RDF defined on G . A $\gamma_R(G)$ -function is an RDF with minimum weight. The Roman domination has been extensively studied in the literature [5, 14, 15, 17]. A similar parameter, the *weak Roman domination number* was defined by Henning and Hedetniemi [7] as follows. For a function $f : V \rightarrow \{0, 1, 2\}$, a vertex u with $f(u) = 0$ is said to be *undefended*, if u is not adjacent to any node v with $f(v) > 0$. The function $f : V \rightarrow \{0, 1, 2\}$, is called a *weak Roman dominating function* (WRDF), if u is any vertex with $f(u) = 0$, then u is adjacent to some vertex v with $f(v) > 0$, such that, under the function $f' : V \rightarrow \{0, 1, 2\}$, defined by $f'(u) = 1$, $f'(v) = f(v) - 1$, $f'(x) = f(x)$, for all $x \in V \setminus \{u, v\}$, no vertex is undefended. The *weight of a weak Roman dominating function* f defined on a graph $G = (V, E)$ is defined as $\sum f(u)$, where the summation runs over all the vertices in V . The minimum weight of a WRDF is called the *weak Roman domination number*, and is denoted by $\gamma_r(G)$. A $\gamma_r(G)$ -function is a WRDF with minimum weight. Many researchers have studied the parameter $\gamma_r(G)$ [3, 9, 11–13, 16].

Consider a $m \times n$ chess board with m rows and n columns. A graph in which the vertices are the mn squares of the chess board and edges are lines joining any two vertices from each of which a chess piece can move to the other using a legal move of that chess piece, specified by the rules of the game of chess, is called the *chess graph* associated with that chess piece. The *knight's graph* denoted by $KN_{m,n}$, is defined as the graph which has for its vertices the squares that make up the chessboard, and edges, the lines which join two vertices from each of which a knight chess piece can move to the other, using a legal knight's move. Thus vertex (i, j) is adjacent to the vertex (k, l) if and only if, $1 \leq i \neq k \leq m$, $1 \leq j \neq l \leq n$ and $|i - k| + |j - l| = 3$.

The Queen's graph denoted by $Q_{m,n}$ is defined as the graph which has its vertices, the squares that make up the $m \times n$ chessboard, and edges, the lines which join two vertices from each of which the queen chess piece can move to the other, using a legal queen's move. Thus, the vertex (i, j) is adjacent to the vertex (k, l) if and only if $i = j$ or $k = l$ or $|i - k| = |j - l|$.

The Rook's graph denoted by $R_{m,n}$ is defined as the graph which has its vertices, the squares that make up the $m \times n$ chessboard, and edges, the lines which join two vertices from each of which the rook chess piece can move to the other, using a legal rook's move. Thus, the vertex (i, j) is adjacent to the vertex (k, l) if and only if $i = k$ or $j = l$.

The Bishop's graph denoted by $B_{m,n}$ is defined as the graph which has its vertices, the squares that make up the $m \times n$ chessboard, and edges, the lines which join two vertices from each of which the bishop chess piece can move to the other, using a legal bishop's move. Thus, the vertex (i, j) is adjacent to the vertex (k, l) if and only if $|i - k| = |j - l|$.

The bishop's graph is a disconnected graph with two components, which are subgraphs induced by vertex sets consist-

ing of the white squares only and the black squares only. Thus the queen's graph is the union of the rook's graph and the bishop's graph.

The King's graph denoted by $K_{m,n}$ is defined as the graph which has its vertices, the squares that make up the $m \times n$ chessboard, and edges, the lines which join two vertices from each of which the king chess piece can move to the other, using a legal king's move. Thus, the vertex (i, j) is adjacent to the vertex (k, l) if and only if $i \neq k$ or $j \neq l$, $|i - k| \leq 1$ and $|j - l| \leq 1$.

Much has been studied on the chess graphs in [1, 2, 4, 6, 10, 18].

2. Definitions and Notation

Two vertices in the Knight's graph are said to be *harmonious*, if they have at least one vertex adjacent to them, in common. Given a pair of harmonious vertices, they can at the most be adjacent to two other vertices, in common, and no more. If two vertices are not harmonious, then we call them *inharmonious*. A set of three or more vertices are said to be mutually harmonious, if any two of them are harmonious. We make the following observations.

1. In a Knight's graph, $\Delta \leq 8$. For optimal labelling, we will label with 2 a vertex with degree at least 3. However, if the degree of a vertex is two, then the vertices adjacent to this vertex are harmonious and hence they can each be labelled with 1 for protecting at least three vertices.
2. In any Knight's graph, the maximum cardinality of a set of mutually harmonious vertices is four. For instance, in $KN_{4,4}$, $a_{1,2}, a_{2,1}, a_{2,3}$ and $a_{3,2}$ are mutually harmonious. Further the vertex $a_{2,2}$ is inharmonious with each of the vertices $a_{2,1}, a_{1,2}, a_{2,3}$ and $a_{3,2}$. Hence any vertex protected by $a_{2,2}$ will not be protected by $a_{2,1}, a_{1,2}, a_{2,3}$ and $a_{3,2}$.
3. Two vertices in the same row of the chess board and in successive columns are inharmonious as are two vertices in the same column and in successive rows.

The outer rim is the set $\{a_{i,j}/i = 1, m, 1 \leq j \leq n\} \cup \{a_{r,s}/s = 1, n, 1 \leq r \leq m\}$. If the outer rim is not the whole of the chess board, we define the inner rim as the set $\{a_{i,j}/i = 2, m - 1, 2 \leq j \leq n - 1\} \cup \{a_{r,s}/s = 2, n - 1, 2 \leq r \leq m - 1\}$. The set of all vertices that do not lie on the inner rim or on the outer rim is called the *central board*. A vertex on the central board is said to be a *central vertex*. Any central vertex is adjacent to eight vertices in the Knight's graph. It is to be noted that the central board or both the inner rim and the central board can be empty. If either $m = 2$ or $n = 2$, then the chess board consists of only the outer rim. For $m > 2$ and $n > 2$, the chess board will necessarily contain both the outer rim and the inner rim. If either $m < 5$ or $n < 5$, then the central board is empty. Based on how the vertices are labelled in a column, we name such columns



as follows. In a $m \times n$ chess board, a column in which at least one legion is placed at each vertex is called a *pad*. A pad in which all the vertices are placed with a single legion is denoted by $P^{(m)}$. We denote by $P^{(m)}(\{i, j, \dots, l\}, \{a, b, \dots, d\})$ a pad in which a single legion is placed at rows i, j, \dots, l and two legions are placed at rows a, b, \dots, d . A column in which all the vertices are placed with no legion is called *totally dependent* and is denoted by $TD^{(m)}$. A column in which at least one vertex has no legion in it is called a *dependent column*. We denote by $D^{(m)}(i, j, \dots, l)$ a dependent column in which a single legion is placed at rows i, j, \dots, l and no legion elsewhere and by $D^{(m)}(\{i, j, \dots, l\}, \{a, b, \dots, d\})$ a dependent column in which a single legion is placed at rows i, j, \dots, l , two legions are placed at each vertex in rows a, b, \dots, d and no legion elsewhere.

3. The WRDN of the Knight's graphs

$KN_{m,n}$

In this section, we find the γ_r -value of the Knight's graphs $KN_{m,n}$, for certain specific values of m and propose a linear time algorithm for finding the γ_r -value of the Knight's graphs $KN_{m,n}$ for arbitrary values of m and n .

3.1 Solution Methodology

For determining the γ_r -value of the Knight's graph, we adopt a strategy that is two pronged. In the first, we follow the integral board approach, in which we find the γ_r -value of the knight's graph $KN_{m,n}$, $m \leq 6$, $n \leq 6$, for each m and n , considering them as independent graphs. In the second, we consider the split board approach, in which we split the board into boards of smaller orders and use the optimal WRDF of the smaller boards to produce the optimal WRDF of the entire board. The split board method is suitable only in the case when the number of rows exceeds three (or six) and the number of columns exceeds six (or three) or explicitly for the knight's graph $KN_{m,n}$, $m > 3$ and $n > 6$ (or $m > 6$ and $n > 3$).

3.2 The Knight's graph $KN_{2,n}$

We shall now find the γ_r -value of the Knight's graphs $KN_{2,n}$, for arbitrary n . We have the following results.

Theorem 3.1. For the Knight's graph $KN_{2,n}$,

$$\gamma_r(KN_{2,n}) = \begin{cases} 2 \left\{ \left\lceil \frac{3(n+1)}{14} \right\rceil + \left\lceil \frac{3(n-1)}{14} \right\rceil \right\}, & \text{if } n \text{ is odd} \\ 4 \left\lceil \frac{3n}{14} \right\rceil, & \text{if } n \text{ is even} \end{cases}$$

Proof. The graph $KN_{2,n}$ is the disjoint union of four paths, two of which are of order $\lfloor \frac{n}{2} \rfloor$ and the other two, $\lceil \frac{n}{2} \rceil$. Hence by [7] it follows that,

$$\begin{aligned} \gamma_r(KN_{2,n}) &= 2 \left\lceil \frac{3 \lfloor \frac{n}{2} \rfloor}{7} \right\rceil + 2 \left\lceil \frac{3 \lceil \frac{n}{2} \rceil}{7} \right\rceil \\ &= \begin{cases} 2 \left\{ \left\lceil \frac{3(n+1)}{14} \right\rceil + \left\lceil \frac{3(n-1)}{14} \right\rceil \right\}, & \text{if } n \text{ is odd} \\ 4 \left\lceil \frac{3n}{14} \right\rceil, & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

□

3.3 The Knight's graph $KN_{3,n}$

We now consider the Knight's graphs $KN_{3,n}$ and find their γ_r -value for arbitrary values of n . Due to the triviality we do not consider the case $n = 1$. The case that $n = 2$ has already been handled in Theorem 3.1. So let $n > 2$.

Lemma 3.2. For the Knight's graph, $KN_{3,3}$, $\gamma_r(KN_{3,3}) = 5$.

Proof. There are nine vertices in the graph $KN_{3,3}$. No vertex in the graph has degree > 2 . Hence we do not label any vertex with 2. The graph is the disjoint union of a cycle on eight vertices and an isolated vertex (a_{22}). Hence $\gamma_r(KN_{3,3}) = \gamma_r(C_8) + 1$, where C_8 is a cycle on eight vertices. Hence $\gamma_r(KN_{3,3}) = 4 + 1 = 5$ [7]. To achieve this, we place the legions according to the placement pattern of legions given by $(P^{(3)}, D^{(3)}(2), D^{(3)}(3))$. □

Lemma 3.3. For the Knight's graph, $KN_{3,4}$, $\gamma_r(KN_{3,4}) = 6$.

Proof. It is a trivial verification that $\gamma_r(KN_{3,4}) = 6$ and place the legions given as $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)})$. □

Lemma 3.4. For the Knight's graph $KN_{3,5}$, $\gamma_r(KN_{3,5}) = 6$.

Proof. The vertex $a_{2,3}$ is adjacent to four vertices in the graph and no other vertex has this property. Placing two legions at $a_{2,3}$ would account for the protection of five vertices in the graph. If the remaining ten vertices are to be taken care of by inharmonic vertices, we would require five legions in all. However the mutually harmonious vertices $a_{1,3}, a_{2,2}, a_{2,4}$ and $a_{3,3}$ are sufficient for the protection of all the other vertices, if we place a single legion at each. Further there exists no other set containing fewer number of mutually harmonious and inharmonic vertices for the protection of all the other vertices. Hence the four mutually harmonious vertices $a_{1,3}, a_{2,2}, a_{2,4}$ and $a_{3,3}$ with a single legion placed at each of them together with two legions placed at $a_{2,3}$ would optimally protect all the vertices of the graph. Hence the optimal WRDF is given by the placement pattern of legions given by $(TD^{(3)}, D^{(3)}(2), D^{(3)}(\{1, 3\}, \{2\}), D^{(3)}(2), TD^{(3)})$ and hence $\gamma_r(KN_{3,5}) = 6$. □

Lemma 3.5. For the Knight's graph $KN_{3,6}$, $\gamma_r(KN_{3,6}) = 8$.

Proof. If the first five columns are labelled as in the case of the five columns of $KN_{3,5}$, in the same order, as given in Lemma 3.4, the vertices $a_{2,6}$ and one of $a_{1,6}$ or $a_{3,6}$ would have no protection. However they will get protection if a legion is placed at $a_{2,6}$ and one of $a_{1,6}$ or $a_{3,6}$. Thus $\gamma_r(KN_{3,6}) \leq 8$. However with fewer than eight legions, we cannot protect the graph $KN_{3,6}$. For, if we have to protect all the vertices of the fourth, fifth and the sixth columns alone, we would require five legions in all as shown in Lemma 3.2 and they have to be placed using the pattern, say, $(P^{(3)}, D^{(3)}(2), D^{(3)}(3))$ in the respective columns. In this case, no vertex in the first column is protected. If we place these five legions in columns three, four and five, then again the vertices $a_{1,1}, a_{1,3}$ and $a_{2,6}$ will have no protection. In a similar manner wherever we place five legions



in any of three consecutive columns, at least three vertices will lose protection, which cannot be protected by just two legions. Hence at least three additional legions are required. Therefore, $\gamma_r(KN_{3,6}) \geq 8$. It follows that, $\gamma_r(KN_{3,6}) = 8$. This can be achieved by the placement of legions $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, D^{(3)}(2), D^{(3)}(3))$ or $(TD^{(3)}, D^{(3)}(2), D^{(3)}(\{1,2\}, \{2\}), D^{(3)}(2), TD^{(3)}, D^{(3)}(2,3))$. \square

Lemma 3.6. For the Knight's graph $KN_{3,7}$, $\gamma_r(KN_{3,7}) = 9$.

Proof. Arguing similar to Lemma 3.5, we get the result. To achieve the γ_r -value, we use the pattern $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)})$. \square

Theorem 3.7. For $n \geq 7$,

$$\gamma_n(KN_{3,n}) = \begin{cases} n+2, & n \equiv 1 \pmod{3}, \\ n+3, & \text{otherwise} \end{cases}$$

Proof. The case $n = 7$ has been proved in Lemma 3.6.

Let $n > 7$.

Case 1: $n \equiv 1 \pmod{3}$

In this case, the placement of legions using the pattern given by $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, \dots, TD^{(3)}, TD^{(3)}, P^{(3)})$ defends all the vertices of the graph. Let the columns in the graph be $C_1, C_2, \dots, C_n = C_1, C_2, \dots, C_{3k+1}$, $k \geq 2$. The vertices in C_1 are labeled using $P^{(3)}$. The remaining columns can be partitioned as $(C_2, C_3, C_4), \dots, (C_{n-2}, C_{n-1}, C_n)$. Each of these partitions are labeled using $(TD^{(3)}, TD^{(3)}, P^{(3)})$. The number of legions in such a partition is 3 and the number of such partitions is $\frac{n-1}{3}$. Added with the three legions in C_1 , the total number of legions used is $(\frac{n-1}{3}) + 3 = n + 2$.

Case 2: $n \equiv 0 \pmod{3}$

So, $n - 2 \equiv 1 \pmod{3}$

By Case 1, all the vertices of the first $n - 2$ columns can be protected using $(n - 2) + 2 = n$ legions. To protect the last two columns, we will require at the least 3 legions, which can be accomplished by placing the legions in these columns using $D^{(3)}(2,3), D^{(3)}(2)$. Hence the total number of legions required for protection is $n + 3$. The placement pattern required is $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, \dots, TD^{(3)}, TD^{(3)}, P^{(3)}, D^{(3)}(2,3), D^{(3)}(2))$ and $\gamma_r(KN_{3,n}) = n + 3$, in this case.

Case 3: $n \equiv 2 \pmod{3}$

So, $n - 1 \equiv 1 \pmod{3}$

Arguing as in Case 2, the first $n - 1$ columns can be protected using $(n - 1) + 2 = n + 1$ legions. The last column can only be protected by placing a minimum of 2 legions, placed either using $D^{(3)}(1,2)$ or $D^{(3)}(2,3)$. Thus $\gamma_r(KN_{3,4}) = n + 1 + 2 = n + 3$. One placement pattern required to achieve this is $(P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, TD^{(3)}, TD^{(3)}, P^{(3)}, \dots, TD^{(3)}, TD^{(3)}, P^{(3)}, D^{(3)}(1,2))$. \square

4. The Knight's graph $KN_{4,n}$

Since the case $n = 1$ is trivial and the cases $n = 2$ and $n = 3$ have been discussed in the previous section, we start with the

case $n = 4$.

Lemma 4.1. For the Knight's graph $KN_{4,4}$, $\gamma_r(KN_{4,4}) = 6$.

Proof. It is a trivial verification that $\gamma_r(KN_{4,4}) = 6$. This can be achieved by placing legions according to the placement pattern given by $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3))$. \square

Lemma 4.2. For the Knight's graph $KN_{4,5}$, $\gamma_r(KN_{4,5}) = 6$.

Proof. An argument similar to the one given in Lemma 3.4 proves the result. The placement pattern of legions to achieve this γ_r -value is given as, $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)})$. \square

Lemma 4.3. For the Knight's graph $KN_{4,6}$, $\gamma_r(KN_{4,6}) = 9$.

Proof. The first five columns of $KN_{4,6}$ can be protected with six legions if we place legions in these columns following Lemma 4.2 using $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)})$. In that case, only one of the vertices $a_{2,6}$ or $a_{4,6}$ will have protection from $a_{3,4}$. If we allow the single legion placed at $a_{3,4}$ to protect $a_{2,6}$, the other vertices $a_{1,6}, a_{3,6}$ and $a_{4,6}$ will have no protection unless we place a single legion in each of these vertices. Thus $\gamma_r(KN_{4,6}) = 9$. This can be achieved by following the placement pattern of legions $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)}, D^{(4)}(1,3,4))$. \square

Lemma 4.4. For the Knight's graph $KN_{4,7}$, $\gamma_r(KN_{4,7}) = 10$.

Proof. The placement pattern of legions $(TD^{(4)}, D^{(4)}(2), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(2), TD^{(4)}, D^{(4)}(2,3,4))$ will protect all the vertices in the first six columns of the graph $KN_{4,6}$. However, in this case, $a_{4,7}$ will have no protection unless we place a single legion at the same. So, $\gamma_r(KN_{4,7}) = 10$. This is achieved by the placement pattern of legions $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)}, D^{(4)}(1,3,4), D^{(4)}(4))$. \square

Lemma 4.5. For the Knight's graph $KN_{4,8}$, $\gamma_r(KN_{4,8}) = 12$.

Proof. There are two inharmonious vertices $a_{3,3}$ and $a_{3,6}$, each adjacent to six vertices in the board. Hence we place two legions each at these vertices, which will account for the protection of fourteen vertices, in all. For the protection of the remaining fourteen vertices, if we place a legion each in each of these vertices, we will require fourteen additional legions. Further these fourteen vertices cannot be partitioned into seven distinct pairs of vertices. But, there are two disjoint sets of four mutually harmonious legions $\{a_{3,2}, a_{2,3}, a_{4,3}, a_{3,4}\}$ and $\{a_{3,5}, a_{2,6}, a_{4,6}, a_{3,7}\}$, each of which can account for the protection of sixteen vertices, if a single legion is placed at each of these eight vertices. However since the number of rows is only 4, each of these sets of vertices account for the protection of seven vertices, accounting together for the protection of the remaining fourteen vertices. Hence, $\gamma_r(KN_{4,8}) = 12$. To achieve this, we follow the placement pattern of legions at the vertices as given in $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)})$. \square



Note 4.6. The above solution for the 4×8 can be obtained using the split board approach, where we split the 4×8 chessboard into two 4×4 chessboards and solve the γ_r -value determination problem for the two 4×4 boards and use the solutions to obtain the solution of the original 4×8 chessboard. For 4×4 chessboard, we have $\gamma_r(KN_{4,4}) = 6$, achieved by placement of legions $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3))$. We use this pattern for the first four columns as well as for the last four columns of the 4×8 chessboard. Hence the vertices in all the eight columns of the 4×8 chessboard will be protected with twelve legions. Hence $\gamma_r(KN_{4,7}) = 12$. We get the alternate placement pattern $(TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3), TD^{(4)}, D^{(4)}(3), D^{(4)}(\{2,4\}, \{3\}), D^{(4)}(3))$ to achieve this.

Lemma 4.7. For the Knight's graph $KN_{4,9}$, $\gamma_r(KN_{4,9}) = 12$.

Proof. We use the split board approach and obtain the solution by splitting the 4×9 into a 4×5 chessboard and a 4×4 chessboard. The γ_r -values of both the boards are the same, both equal to six, which yields $\gamma_r(KN_{4,9}) = 12$. The placement pattern of legions can be obtained from Lemma 4.1 and 4.2. \square

We now state Lemma 4.8 without proof, since we can get the solution using the split board approach.

Lemma 4.8. For the Knight's graph $KN_{4,10}$, $\gamma_r(KN_{4,10}) = 12$.

Once the split board approach has come into play for the $4 \times n$ Knight's graph, we can apply the approach to obtain the γ_r -values of all the $4 \times n$ Knight's graphs for $n \geq 10$. For this we tabulate the γ_r -values of the $4 \times n$ Knight's graphs for $n < 10$ and use them for the purpose. The justification for using the split board approach has been detailed at the beginning of Section 7.

5. The Knight's graph $KN_{5,n}$

The 5×5 Knight's graph $KN_{5,5}$ is the most important of all the Knight's graphs considered so far, for the development of the further theory. It is the placement pattern of this Knight's graph in which every legion participate in the least redundant protection of all the legions. Further, the size of the 5×5 chessboard yields itself for the maximum number of vertices to be protected by just 6 legions placed in the chessboard. The following measure of efficiency will make the further discussion clear and smooth.

For a $m \times n$ Knight's graph $KN_{m,n}$, we define the *utilization factor* of the graph $KN_{m,n}$ as the ratio of $m \times n$ to $\gamma_r(KN_{m,n})$. We call the *wastage* of a vertex v the maximum number of vertices that v can protect, but it does not do so as a result of non-existence of the vertices or that some other vertex u is protecting a vertex w that is adjacent to v , independent of the legion placed at v , meaning that even after the legion placed at v is withdrawn, u can still protect w and all the other vertices that u was protecting prior to the withdrawal of the legion at v .

Lemma 5.1. For the Knight's graph $KN_{5,5}$, $\gamma_r(KN_{5,5}) = 6$.

Proof. The graph $KN_{5,5}$ has a single central vertex $a_{3,3}$ and placing two legions at $a_{3,3}$ will account for the protection of nine vertices including $a_{3,3}$. The remaining sixteen vertices will be protected by placing a legion each at the four mutually harmonious vertices $\{a_{2,3}, a_{3,2}, a_{3,4}, a_{4,3}\}$. The required placement pattern is $(TD^{(5)}, D^{(5)}(3), D^{(5)}(\{2,4\}, \{3\}), D^{(5)}(3), TD^{(5)})$ and $\gamma_r(KN_{5,n}) = 6$. \square

Note 5.2. We note that only the central vertex is capable of protecting eight other vertices and the four mutually harmonious vertices can protect a maximum of twelve other vertices. Hence the single central vertex and the four harmonious vertices can protect twenty vertices and account for the protection of twenty five vertices, in all, including themselves, with six legions. Hence there is no wastage for any vertex. The utilization factor for $KN_{5,5}$ is $\frac{25}{6} = 4.16$.

Lemma 5.3. For the Knight's graph $KN_{5,6}$, $\gamma_r(KN_{5,6}) = 10$.

Proof. The first five columns of $KN_{5,6}$ can be protected by six legions if these columns are placed with legions using the placement pattern of legions as given in Lemma 5.1 as $(TD^{(5)}, D^{(5)}(3), D^{(5)}(\{2,4\}, \{3\}), D^{(5)}(3), TD^{(5)})$. The single legion at $a_{4,3}$ can protect only one of the vertices $a_{2,6}$ or $a_{4,6}$. If we allow $a_{2,6}$ to be protected, the remaining four vertices in column six will have no protection unless we place a legion at each of the same. Hence $\gamma_r(KN_{5,6}) = 10$. One way to achieve this is by using the placement pattern of legions at the six columns of $KN_{5,6}$ as $(TD^{(5)}, D^{(5)}(3), D^{(5)}(\{2,4\}, \{3\}), D^{(5)}(3), TD^{(5)}, D^{(5)}(1,3,4,5))$. \square

We state without proof three lemmas, Lemma 5.4, Lemma 5.5 and Lemma 5.6. The proof of Lemma 5.4 is similar to the proof of Lemma 5.3, whereas the proofs of Lemma 5.5 and Lemma 5.6 are similar to that of Lemma 4.5 and that of Lemma 4.7.

Lemma 5.4. For the Knight's graph $KN_{5,7}$, $\gamma_r(KN_{5,7}) = 12$.

Lemma 5.5. For the Knight's graph $KN_{5,8}$, $\gamma_r(KN_{5,8}) = 12$.

Lemma 5.6. For the Knight's graph $KN_{5,9}$, $\gamma_r(KN_{5,9}) = 12$.

Note 5.7. In obtaining the γ_r -values of the graphs considered in Lemma 5.5 and Lemma 5.6, we use the split board approach.

6. The Knight's graph $KN_{6,n}$

We shall now consider the Knight's graphs $KN_{6,n}$ and find their γ_r -values. We start with the Knight's graph $KN_{6,6}$, as the other cases have already been discussed.

Lemma 6.1. For the Knight's graph $KN_{6,6}$, $\gamma_r(KN_{6,6}) = 12$.

Proof. There are four central vertices $\{a_{3,3}, a_{3,4}, a_{4,3}, a_{4,4}\}$ in the 5×6 chessboard, of which there are two pairs of harmonious vertices namely, $\{a_{3,3}, a_{4,4}\}$ and $\{a_{4,3}, a_{3,4}\}$. Hence



they can protect in all $32 - 4 = 28$ vertices of the chessboard if we place two legions at each of them. Thus these four vertices account for the protection of 32 vertices, including themselves. The remaining four vertices namely $a_{1,1}, a_{1,6}, a_{6,1}$ and $a_{6,6}$ need a legion placed at each of them for their protection. Hence $\gamma_r(KN_{5,6}) = 12$. This can be achieved by the placement pattern $(D^{(6)}(1, 6), TD^{(6)}, D^{(6)}(\{ \}, \{3, 4\}), D^{(6)}(\{ \}, \{3, 4\}), TD^{(6)}, D^{(6)}(1, 6))$. \square

Lemma 6.2. For the Knight's graph $KN_{6,7}$, $\gamma_r(KN_{6,7}) = 16$.

Proof. By the split board approach, if we split the 6×7 chessboard into a 6×4 and a 6×3 board, these split boards can be protected using nine and seven legions respectively, which sum up to sixteen legions. Hence, $\gamma_r(KN_{6,7}) = 16$, since by the integral board approach, the number of legions required is also the same. The placement pattern of legions using the split board approach can be obtained from the respective split boards that make up the given chessboard. The placement pattern using the integral board approach will be $(D^{(6)}(1, 6), TD^{(6)}, D^{(6)}(\{ \}, \{3, 4\}), D^{(6)}(\{ \}, \{3, 4\}), TD^{(6)}, D^{(6)}(1, 6), D^{(6)}(1, 2, 5, 6))$. \square

Lemma 6.3. For the Knight's graph $KN_{6,8}$, $\gamma_r(KN_{6,8}) = 18$.

Proof. The γ_r -value is obtained using the split board approach as using the integral board approach, the minimum number of legions required for protection is 20. Thus $\gamma_r(KN_{6,8}) = 18$. The placement pattern of legions can be obtained from the respective split boards that make up the given chessboard. \square

We now state the following two lemmas without proof.

Lemma 6.4. For the Knight's graph $KN_{6,9}$, $\gamma_r(KN_{6,9}) = 18$.

Lemma 6.5. For the Knight's graph $KN_{6,10}$, $\gamma_r(KN_{6,10}) = 20$.

7. The Knight's graph $KN_{7,n}$, $KN_{8,n}$ and $KN_{9,n}$

We start the discussion with $KN_{7,7}$

Lemma 7.1. For the Knight's graph $KN_{7,7}$, $\gamma_r(KN_{7,7}) = 20$.

Proof. If consider the integral board approach, we identify that there are eight central vertices $a_{3,3}, a_{3,4}, a_{3,5}, a_{4,3}, a_{4,5}, a_{5,3}, a_{5,4}$ and $a_{5,5}$, which when placed with two legions each will protect all the vertices of the graph $KN_{5,7}$ except the five vertices, $a_{1,1}, a_{1,7}, a_{4,4}, a_{7,1}$ and $a_{7,7}$, which can be protected only by placing a single legion at each one of these vertices. Hence the total number of legions required in this approach is 21. However using the split board approach, we get, $\gamma_r(KN_{7,7}) = 19$. The placement pattern of legions can be obtained from the respective split boards that make up the given chessboard. \square

The proof of the following lemmas are trivial using the split board approach as the integral board approach would not require lesser number of legions for the protection of all the vertices of the chessboards considered.

Lemma 7.2. For the Knight's graph $KN_{7,8}$, $\gamma_r(KN_{7,8}) = 20$.

Lemma 7.3. For the Knight's graph $KN_{7,9}$, $\gamma_r(KN_{7,9}) = 22$.

Simple verifications reveal that the γ_r -values of the Knight's graphs $KN_{8,n}$ and $KN_{9,n}$ can be obtained using the split board approach and the γ_r -values are tabulated in Table 1.

8. The general Knight's graph $KN_{m,n}$

We now tabulate the values of $\gamma_r(KN_{m,n})$ for $1 \leq m \leq 9$, $1 \leq n \leq 9$ in a lookup table (Refer Table 1), from which the γ_r -values of the Knight's graphs $KN_{m,n}$ can be obtained for $m \geq 10, n \geq 10$. From Table 1, it can be seen that whenever we can split m or n into a sum of two numbers such that their product is maximum, and at least one of the parts is greater than equal to 4, the split board approach applies. Given a $m \times n$ chessboard, split the chessboard in to four parts namely first, a $p \times q$ chessboard where p is the largest multiple of 5 that is less than $m - 5$ and q is the largest multiple of 5 less than $n - 5$, the second, a $s \times q$ chessboard, where s denotes the remaining number of rows of the given board that exists beyond row p , the third, a $p \times t$ chessboard where t is the remaining number of columns that exist beyond column q and the fourth, a $s \times t$ chessboard. Then the γ_r -value of the graph $KN_{m,n}$ is the sum of the values of the γ_r -values of the above given chessboards. This is because, the γ_r -value of the 5×5 chessboard has the maximum utilization factor and we split the given board to have as many 5×5 chessboards as possible and the remaining portion of the chessboard contains at least five rows and five columns. This is to accommodate the fact split board approach would not give optimal values for the $m \times n$ chessboards where $m \leq 6, n \leq 6$ and only the integral board approach gives the optimal values. The above splitting of the given chessboard gives the γ_r -values to the Knight's graphs considered. In respect of the chessboard with the number of columns and the number of rows, each less than or equal to 9, the γ_r -values are obtained directly from the lookup table.

Lookup table giving the γ_r -value of the Knight's graph - $KN_{m,n}$

Table 1

$n \backslash m$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	4	4	6	8	8	8	10
3	3	4	5	6	6	8	9	11	12
4	4	4	6	6	6	9	10	12	12
5	5	6	6	6	6	10	12	12	12
6	6	8	7	9	10	12	16	18	18
7	7	8	10	10	12	16	20	20	22
8	8	8	11	12	12	18	20	24	24
9	9	10	12	12	12	19	22	24	24



9. The algorithm for finding $\gamma_r(KN_{m,n})$

We now present the optimal algorithm for finding the γ_r -value of a given Knights graph $\gamma_r(KN_{m,n})$.

Optimal WRDN algorithm

Input : m, n - the order of the chessboard, the lookup table T
Algorithm :

Step 1 Find $x = \left\lfloor \frac{(m-5)}{5} \right\rfloor, y = \left\lfloor \frac{(n-5)}{5} \right\rfloor,$
 $p = 5 * x, q = 5 * y, s = m - p, t = n - q$

Step 2 Compute

$$g = 6 * x * y + T(s, t) + y * T(s, 5) + x * T(5, t)$$

Output : g - the WRDN of the Knight's graph $KN_{m,n}$

Theorem 9.1. *The Optimal WRDN algorithm is correct.*

Proof. The proof of correctness follows from the discussion given at the beginning of section 8. \square

Theorem 9.2. *Optimal WRDN algorithm is a constant time algorithm.*

Proof. The lookup table is a 9×9 table and requires constant time to populate and retrieve values from. Step 1 computes constant number of values in constant time. Step 2 computes the γ_r -value of the given $m \times n$ Knights graph in constant time. Hence the Optimal WRDN algorithm is a constant time algorithm. \square

10. The γ_r -value of some other Chess Graphs

We observe the following

1. $\gamma_r(Q_{m,n}) = \min(m, n),$
2. $\gamma_r(R_{m,n}) = \max(m, n),$
3. $\gamma_r(B_{m,n}) = m + n - 1,$
4. $\gamma_r(K_{m,n}) = \{2(m \bmod 3 + n \bmod 3) + x + y\},$

$$\text{where } x = \begin{cases} \left\lceil \frac{3m}{7} \right\rceil, & \text{if } m \equiv 1 \pmod{3} \\ 2m, & \text{if } m \equiv 2 \pmod{3} \end{cases}$$

$$\text{and } y = \begin{cases} \frac{3(n-1)}{7}, & \text{if } n \equiv 1 \pmod{3} \text{ and } m \equiv 1 \pmod{3} \\ \frac{3(n-2)}{7}, & \text{if } n \equiv 1 \pmod{3} \text{ and } m \equiv 2 \pmod{3} \\ 2(n-1), & \text{if } n \equiv 2 \pmod{3} \text{ and } m \equiv 1 \pmod{3} \\ 2(n-2), & \text{if } n \equiv 2 \pmod{3} \text{ and } m \equiv 2 \pmod{3} \end{cases}$$

11. Conclusion

There are not many constant time algorithms for graphs of arbitrary order. In this paper, we have designed a constant time algorithm, for finding the weak Roman domination number of the Knight's graphs, which are graphs of arbitrary order. This is possible, since the algorithm designed requires the order of the chess board considered as input and not the vertices

themselves. The findings of the paper reiterates the fact that constant time algorithms can be designed for processing arbitrarily large graphs, though such algorithms are rare, given the nonlinearity that is inherent in graphs.

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