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Minimum hub color energy of a graph

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Abstract

This paper deals with the concept of Minimum hub color energy $E_{H_\chi}(\Gamma)$ of a connected graph and then the minimum hub color energy $E_{H_\chi}(\Gamma)$ of few familiar standard family of graphs has been computed. Also few basic properties of the minimum hub color energy of a connected graph has also been discussed.

Keywords

minimum hub color eigenvalues, minimum hub color energy, minimum hub color matrix, minimum hub color set.

AMS Subject Classification

05C15, 05C31, 05C50, 05C69, 05C92.

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Contents

1. Introduction

Let Γ be a graph which doesn't consist of loops, multiple or any directed edges.

In the year 2006, the idea of hub numbers was innovated by M. Walsh [5]. A set $H \subseteq \Gamma$ is called as Hub set if for any vertices x,y ε V(Γ)-H ,there exists a H-path (intermediate vertices should be one of the members from the set H) in Γ between x and y (ignoring the existing trivial paths). The least cardinality of a hub set H in Γ is called the hub number of Γ and it is denoted by h(Γ).

I. Gutman [2] conceptualised the concept of Energy of a graph Γ in the year 1978. Let Γ be a simple connected graph with order n and size m. Let us consider A = (a_{ij}) as the adjacency matrix of the graph Γ. The eigenvalues $\rho_1, \rho_2, \rho_3, ..., \rho_n$ of Γ are the eigenvalues obtained from $A - I\Gamma$. As the matrix of A is real symmetric, the eigenvalues of Γ are real with its total sum equal to zero. The energy E(Γ) of Γ is found to be the sum of the absolute values of all the eigenvalues of Γ i.e., E(Γ) $=\sum_{i=1}^{n} |\rho_i|$.

The process of coloring a graph Γ is assigning color to its vertices in such a way that no two adjacent vertices obtain the same color. The minimum number of such colors needed to color a graph Γ is known as chromatic number and it is denoted by $\chi(\Gamma)$.

The color matrix of Γ is the square adjacency matrix $A_\chi(\Gamma)$ of order n whose each entry is as follows:

$$(a_{ij}) = \begin{cases} 1\, if\, v_i\, and\, v_j\, are\, adjacent\, and\; c(v_i) \neq c(v_j) \\ -1\; if\, v_i\, and\, v_j\, are\, not\, adjacent\, and\; c(v_i) = c(v_j) \\ 0\; otherwise \end{cases}$$

Let $\omega_1, \omega_2, \omega_3, ..., \omega_n$ be the eigenvalues of the color matrix $A_\chi(\Gamma)$ of Γ. The color energy $E_\chi(\Gamma)$ of a graph Γ given by C.Adiga et al [1] is defined as $E_\chi(\Gamma) = \sum_{i=1}^{n} |\omega_i|$.

The above concepts motivated us to introduce the concept of minimum hub color energy $E_{H\chi}(\Gamma)$ of a graph Γ and compute minimum hub color energy of few standard graphs.

Similar studies are found in [3] and [4].

2. The Minimum Hub Color Energy of a Graph

Let Γ be a graph consisting of n vertices and V = $\{v_1, v_2, ..., v_n\}$ be its vertex set. Let H be a minimum hub set of the graph Γ. The minimum hub color matrix of Γ is the n $\times$ n square matrix $A_{H\chi}(\Gamma) = (a_{ij})$, where

$$(a_{ij}) = \begin{cases} 1\ if\ i = j\ if\ v_i \varepsilon H\ or \\ 1\ if\ v_i\ and\ v_j\ are\ adjacent\ and \\ c(v_i) \neq c(v_j) \\ -1\ if\ v_i\ and\ v_j\ are\ not\ adjacent\ and \\ c(v_i) = c(v_j) \\ 0\ otherwise \end{cases}$$

The characteristic polynomial of $A_{H\chi}(\Gamma)$ denoted by $f_n(\Gamma, \lambda_\chi)$ is defined as $f_n(\Gamma, \lambda_\chi) = det(\lambda_\chi I - A_{H\chi}(\Gamma))$.

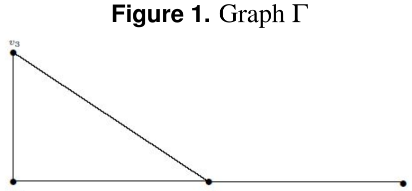
Since the adjacency matrix $A_{H\chi}(\Gamma)$ contains real symmetric entries, its hub color eigenvalues are real and we denote them as $\lambda_{\chi 1}, \lambda_{\chi 2}, \lambda_{\chi 3}, ..., \lambda_{\chi n}$. The minimum hub color energy of Γ is defined to be : $E_{H\chi}(\Gamma) = \sum_{i=1}^{n} |\lambda_{\chi i}|$.

3. Illustration

Let Γ be a graph and V = $\{v_1, v_2, v_3, v_4\}$ be its vertex set.Its minimum Hub set H = $\{v_1\}$ and χ= 3 with $v_1 = c_1, v_2 = c_2$, $v_3 = c_3, v_4 = c_2$ as in below Figure.1 . Then its adjacency matrix is found to be $A_{H\chi}(\Gamma) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

The characteristic polynomial of $A_{H\chi}(\Gamma)$ is $f_n(\Gamma, \lambda_\chi) = \lambda_\chi^4 - \lambda_\chi^3 - 5\lambda_\chi^2 + 2\lambda_\chi + 4$ and the minimum hub color eigen values are 2.4393 , -0.8193 , 1.1386 and -1.7565. Thus the minimum hub color energy of G is $E_{H\chi}(\Gamma)$ = 6.1537 .

Figure 1. Graph Γ



4. Basic Theorems on Minimum Hub Color Energy

Theorem 4.1. *Let Γ be a graph of order n, size m and hub number h(Γ). If $f_{H\chi}(\Gamma, \lambda_\chi) = a_0\lambda_\chi^n + a_1\lambda_\chi^{n-1} + ... + a_n$ is the characteristic polynomial of the minimum hub colored matrix of the Graph Γ, then*
1) a_0=1
2) $a_1 = -h(\Gamma)$
3) $a_2 = \binom{h(G)}{2}$ - $[m + number\ of\ pairs\ of$ $non-adjacent\ vertices\ receiving\ same\ color\ in\ \Gamma$]

Proof. (1) It follows from the definition of $f_{H\chi}(\Gamma, \lambda_\chi) := det(\lambda_\chi I - A_{H\chi}(\Gamma))$ that a_0=1.

(2) Since the sum of the diagonal elements of adjacency matrix $A_{H\chi}(\Gamma)$ is equal to hub number $h(\Gamma)$ of the corresponding graph Γ, we get $a_1 = -h(\Gamma)$.

(3) The sum of determinants of all principal 2×2 submatrices of $A_{H\chi}(G)$ is equal to $(-1)^2 a_2$, which leads to

$$\begin{aligned} a_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{ji} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i < j \le n} (a_{ii}a_{jj}) - \sum_{1 \le i < j \le n} (a_{ij})^2 \\ &= \binom{h(\Gamma)}{2} - [m + number\ of\ pairs\ of \\ & non-adjacent\ vertices\ receiving\ same\ color\ in\ \Gamma] \end{aligned}$$

□

Theorem 4.2. *Let Γ be a graph with a minimum hub set H. Then $E_{H\chi}(\Gamma) \equiv |H| (mod2)$ if the numerical value of minimum hub color energy is found to be a rational number where $|H|$ denotes the cardinality of minimum hub color H of Γ.*

Proof. Let $\lambda_{\chi 1}, \lambda_{\chi 2}, \lambda_{\chi 3}, ..., \lambda_{\chi n}$ be the minimum hub color eigenvalues of a graph Γ of which $\lambda_{\chi 1}, \lambda_{\chi 2}, \lambda_{\chi 3}, ..., \lambda_{\chi p}$ (for $p < n$) are positive values and the remaining being non-positive values, then

$$\begin{aligned} \sum_{i=1}^{n} |\lambda_{\chi i}| &= (\lambda_{\chi 1} + \lambda_{\chi 2} + \lambda_{\chi 3} + ... + \lambda_{\chi p}) - \\ & (\lambda_{\chi p+1} + \lambda_{\chi p+2} + \lambda_{p+3} + .. + \lambda_{\chi n}) \\ &= 2(\lambda_{\chi 1} + \lambda_{\chi 2} + \lambda_{\chi 3} + ... + \lambda_{\chi p}) - \\ & (\lambda_{\chi 1} + \lambda_{\chi 2} + \lambda_{\chi 3} + ... + \lambda_{\chi n}) \\ &= 2(\lambda_{\chi 1} + \lambda_{\chi 2} + \lambda_{\chi 3} + ... + \lambda_{\chi p}) - h(\Gamma) \end{aligned} \tag{4.1}$$

Therefore, $E_{H\chi}(\Gamma) = 2m - H\ where\ m = (\lambda_{\chi 1} + \lambda_{\chi 2} + \lambda_{\chi 3} + ... + \lambda_{\chi p})$

Since the eigen values are algebraic integers,then their sum will also be algebraic integer. Thus, the value of 'm' will be an integer if the energy $E_{H\chi}(\Gamma)$ is a rational value. Hence the proof. □

Theorem 4.3. *Let Γ be a simple connected graph with order n . Let $\lambda_{\chi 1}, \lambda_{\chi 2}, \lambda_{\chi 3}, ..., \lambda_{\chi n}$ be the eigenvalues of minimum hub color adjacency matrix $A_{H\chi}(\Gamma)$. Then*
(1). $\sum_{i=1}^{n} \lambda_{\chi i} = h(\Gamma)$, Hub number of Γ
(2). $\sum_{i=1}^{n} \lambda_{\chi i}^2 = h(\Gamma) + 2(m + m_c')$ where m_c' denotes the number of pairs of non-adjacent vertices receiving same color in Γ

Proof. (1) Since we know that the sum of the eigenvalues of $A_{H\chi}(\Gamma)$ is equal to the trace of the $A_{H\chi}(\Gamma)$, then $\sum_{i=1}^{n} \lambda_{\chi i} = \sum_{i=1}^{n} a_{ii} = |H| = h(\Gamma)$ where $|H|$ denotes the cardinality of minimum hub set H of Γ.

(2) Similarly we know that the sum of the squares of eigenvalues of $A_{H\chi}(\Gamma)$ is equal to the trace of $(A_{H\chi}(\Gamma))^2$.

Then

$$\sum_{i=1}^{n}\lambda_i^2=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}$$
$$=\sum_{i=1}^{n}(a_{ii})^2+2\sum_{i\neq j}^{n}a_{ij}a_{ji}$$
$$=\sum_{i=1}^{n}(a_{ii})^2+2\sum_{i<j}^{n}(a_{ij})^2$$

Therefore, $\sum_{i=1}^{n}\lambda_{\chi i}^2=h(\Gamma)+2\left(m+m_c^{'}\right)$. □

5. Minimum Hub Color Energy of Few Familiar Standard Graphs

Theorem 5.1. *For $n\geq 2$, the minimum hub color energy of the Star graph $K_{1,n-1}$ of order n is $(n-2)+\sqrt{n^2+2n-3}$*

Proof. For Star graph $K_{1,n-1}$ with n vertices $V=\{v_1,v_2,\ldots,v_n\}$ and the minimum hub set $H=\{v_0\}$.

Since hub number $h(K_{1,n-1})$ =1 and chromatic number χ =2, we get $A_{H\chi}(K_{1,n-1})=$

$$\begin{bmatrix}1&1&\ldots&\ldots&1&1\\1&0&\ldots&\ldots&-1&-1\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\1&-1&\ldots&\ldots&0&-1\\1&-1&\ldots&\ldots&-1&0\end{bmatrix}_{n\times n}$$

Then the Characteristic polynomial is $(-1)^n(\lambda-1)^{n-2}\left(\lambda^2+(n-3)\lambda-(2n-3)\right)$

Spectrum, $Spec_{H\chi}(K_{1,n-1})=$

$$\begin{pmatrix}1&\frac{-(n-3)\pm\sqrt{n^2+2n-3}}{2}\\n-2&1\end{pmatrix}$$

Therefore, minimum hub color energy is

$$E_{H\chi}(K_{1,n-1})=\sum_{i=1}^{n}|\lambda_i|$$
$$=|1|(n-2)+$$
$$\left|\frac{-(n-3)\pm\sqrt{n^2+2n-3}}{2}\right|1$$
$$=(n-2)+\sqrt{n^2+2n-3}$$

The minimum hub color energy of the Star graph is $(n-2)+\sqrt{n^2+2n-3}$. □

Theorem 5.2. *For $n\geq 2$, the minimum hub color energy of the Complete Bipartite graph $K_{n,n}$ of order 2n is $(2n-1)+\sqrt{4n^2+4n-7}$*

Proof. For Complete Bipartite graph $K_{n,n}$ with vertex set $V=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$, the minimum hub set $H=\{u_1,v_1\}$. Since the hub number $h(K_{n,n})=2$ and the chromatic number $\chi=2$, we get $A_{H\chi}(K_{n,n})=$

$$\begin{bmatrix}1&-1&\ldots&\ldots&1&1\\-1&0&\ldots&\ldots&1&1\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\1&1&\ldots&\ldots&1&-1\\1&1&\ldots&\ldots&-1&0\end{bmatrix}_{2n\times 2n}$$

Then the Characteristic polynomial is $(\lambda-2)(\lambda-1)^{2n-3}\left(\lambda^2+(2n-3)\lambda-4(n-1)\right)$

Spectrum, $Spec_{H\chi}(K_{n,n})=$

$$\begin{pmatrix}2&1&\frac{-(2n-3)\pm\sqrt{4n^2+4n-7}}{2}\\1&2n-3&1\end{pmatrix}$$

The minimum hub color energy is,

$$E_{H\chi}(K_{1,n-1})=\sum_{i=1}^{n}|\lambda_i|$$
$$=|2|(1)+|1|(2n-3)+$$
$$\left|\frac{-(2n-3)\pm\sqrt{4n^2+4n-7}}{2}\right|1$$
$$=(2n-1)+\sqrt{4n^2+4n-7}$$

The minimum hub color energy of the Complete Bipartite graph is $(2n-1)+\sqrt{4n^2+4n-7}$. □

Theorem 5.3. *For $n\geq 2$, the minimum hub color energy of the Friendship graph F_n of order 2n+1 is $(3n-2)+\sqrt{n^2+6n+1}$*

Proof. For Friendship graph F_n with Vertex set V and the minimum hub set $H=\{v_0\}$.

Since the hub number $h(F_n)=1$ and the chromatic number χ=3, we get $A_{H\chi}(F_n)=$

$$\begin{bmatrix}1&1&1&\ldots&\ldots&1&1\\1&0&1&\ldots&\ldots&-1&0\\1&1&0&\ldots&\ldots&0&-1\\\vdots&\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\\vdots&\vdots&\vdots&\ddots&\ddots&\vdots&\vdots\\1&-1&0&\ldots&\ldots&0&1\\1&0&-1&\ldots&\ldots&1&0\end{bmatrix}_{(2n+1)\times(2n+1)}$$

Then the Characteristic polynomial is $(-1)(\lambda-2)^{n-1}(\lambda)^{n-1}(\lambda+n)\left(\lambda^2-(n+1)\lambda-n\right)$

Spectrum, $Spec_{H\chi}(F_n)=$

$$\begin{pmatrix}2&0&-n&\frac{-(n-3)\pm\sqrt{n^2+6n+1}}{2}\\n-1&n-1&1&1\end{pmatrix}$$

The minimum hub color energy is,

$$
\begin{aligned}
E_{H\chi}(F_n) &= \sum_{i=1}^{n} |\lambda_i| \\
&= |2|\,(n-1) + |0|\,(n-1) + |-n|\,1 + \\
&\left|\frac{-(n-3) \pm \sqrt{n^2+6n+1}}{2}\right| 1 \\
&= (3n-2) + \sqrt{n^2+6n+1}
\end{aligned}
$$

The minimum hub color energy of the Friendship graph is $(3n-2)+\sqrt{n^2+6n+1}$. □

6. Conclusion

The Minimum Hub Color energy of Friendship Graph, Star graph and Complete Bipartite graph are obtained in this paper. From the results, it is observed that the choice of the minimum hub set plays an important role in determining the minimum hub Color energy of any graph.

References

[1] Chandrashekar Adiga, Sampathkumar E , Sriraj.M.A, Shrikanth A. S, Color Energy of a Graph ,*Proceedings of the Jangjeon Mathematical Society*, 16 (2013), 335- - 351.

[2] Gutman I, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz* , 103 (1978), 1 — 22.

[3] S.Sreeja, U.Mary, *Minimum Hub Harary Energy of a Graph*, Advances in Mathematics, Scientific Journal, 9(12)(2020), 10745–10753.

[4] Veena Mathad, Sulthan Senan Mahde, The Minimum Hub Energy of a Graph, *Palestine Journal of Mathematics,* 6(1) (2016), 247–256.

[5] Walsh M, The hub number of Graphs, *International Journal of Mathematics and Computer Science,* 1 (2006), 117– 124.

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