



A parameter uniform convergence for a system of two singularly perturbed initial value problems with different perturbation parameters and Robin initial conditions

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Abstract

In this paper an initial value problem for a system of two singularly perturbed first order differential equations with different perturbation parameters and Robin initial conditions is considered on the interval $(0, 1]$. A numerical method composed of a classical finite difference scheme on a piecewise uniform Shishkin mesh is suggested. This method is proved to be first - order convergent in the maximum norm uniformly in the perturbation parameters. A numerical illustration is provided to support the theory.

Keywords

Singular perturbation problems, different perturbation parameters, Robin initial conditions, Finite difference schemes, Shishkin mesh, Parameter uniform convergence.

AMS Subject Classification

34K10, 34K20, 34K26, 34K28.

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1. Introduction

A coupled system of two Singularly Perturbed Ordinary Differential Equations of first order with the prescribed Robin initial conditions is considered. The leading term of each equation is multiplied by a small positive parameter and the parameters may differ. The solution exhibits overlapping lay-

ers. A Shishkin mesh is constructed in the domain of the IVP. A Finite Difference scheme applied on this mesh (which is piecewise uniform) is proved to be uniformly convergent almost first order accurate in both the parameters. Numerical results are presented in support of the theory. Consider the singularly perturbed linear system

$$\vec{L}\vec{u}(x) = \begin{cases} \varepsilon_1 u_1'(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x) \\ \varepsilon_2 u_2'(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega \end{cases} \quad (1.1)$$

$$\vec{u}(0) - \varepsilon_i \vec{u}'(0) = \vec{\phi} \quad (1.2)$$

where, $\vec{u}(x) = (u_1(x), u_2(x))^T$, $\vec{\phi} = (\phi_1, \phi_2)^T$. The parameters ε_i , $i = 1, 2$ are assumed to be distinct.

Assumption 1.1. The functions a_{ij} , $f_i \in \mathbb{C}^{(2)}(\bar{\Omega})$, $i, j = 1, 2$,

satisfy the following inequalities

$$\left. \begin{array}{l} (i) \ a_{11}(x) > |a_{12}(x)|, \ a_{22}(x) > |a_{21}(x)| \\ (ii) \ a_{12}(x), \ a_{21}(x) \leq 0 \end{array} \right\} \forall x \in \bar{\Omega}. \quad (1.3)$$

Assumption 1.2. The positive number α satisfy

$$0 < \alpha < \min_{x \in \bar{\Omega}} \{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}. \quad (1.4)$$

Assumption 1.3. The singular perturbation parameters $\varepsilon_1, \varepsilon_2$ satisfy $0 < \varepsilon_1 < \varepsilon_2 \leq 1$.

The problem (1.1) and (1.2) can also be written in the operator form

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega \quad (1.5)$$

with

$$\vec{\beta}\vec{u}(0) = \vec{\phi} \quad (1.6)$$

where the operators $\vec{L}, \vec{\beta}$ are defined by

$$\vec{L} = ED + A, \quad \vec{\beta} = I - ED$$

where $E = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$, I is the identity operator and $D = \frac{d}{dx}$ is the first order differential operator.

The above problem is singularly perturbed in the following sense. The reduced problem obtained by putting each $\varepsilon_i = 0$, $i = 1, 2$ in the system (1.1) is the linear algebraic system

$$A(x)\vec{u}_0(x) = \vec{f}(x) \quad (1.7)$$

where $A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$, $\vec{u}_0(x) = (u_{0,1}(x), u_{0,2}(x))^T$ and $\vec{f}(x) = (f_1(x), f_2(x))^T$.

Notice that the equation (1.7) has a unique solution for each value of x , and hence the arbitrary robin initial conditions (1.2) cannot be imposed. This shows that there are initial layers in the components of the solution in the neighborhood of $x = 0$.

For the case $\varepsilon_1 < \varepsilon_2$, the solution $\vec{u} = (u_1, u_2)^T$ has the following layer pattern. Both the components u_1 and u_2 exhibit an initial layer of width $O(\varepsilon_2)$, while the component u_1 has an additional layer of width $O(\varepsilon_1)$.

2. Analytical Results

The operator \vec{L} satisfies the following maximum principle.

Lemma 2.1. Let $A(x)$ satisfy (1.3) and (1.4). Let $\vec{\psi} = (\psi_1, \psi_2)^T$ be any function in the domain of \vec{L} such that $\vec{\beta}\vec{\psi}(0) \geq \vec{0}$. Then $\vec{L}\vec{\psi}(x) \geq \vec{0}$ on $x \in \Omega$ implies that $\vec{\psi}(x) \geq \vec{0}$ on $x \in \bar{\Omega}$.

Proof. Let i^*, x^* be such that $\psi_{i^*}(x^*) \leq \psi_i(x)$. Assume that the lemma is false. Then $\psi_{i^*}(x^*) < 0$. For $x^* = 0$, then

$$\begin{aligned} (\vec{\beta}\vec{\psi})_{i^*}(0) &= \psi_{i^*}(0) - \varepsilon_{i^*}\psi'_{i^*}(0) \\ &< 0, \text{ which is a contradiction.} \end{aligned}$$

Therefore, $x^* \neq 0$.

Suppose $x^* \in (0, 1]$, then

$$(\vec{L}\vec{\psi})_{i^*}(x^*) = \varepsilon_{i^*}\psi'_{i^*}(x^*) + \sum_{j=1}^2 a_{i^*j}(x^*)\psi_{i^*}(x^*)$$

< 0 , which contradicts the assumption.

Hence our assumption $\psi_{i^*}(x^*) < 0$ is wrong. It follows that $\psi_{i^*}(x^*) \geq 0$ and thus that $\vec{\psi}(x) \geq \vec{0}$, for all $x \in \bar{\Omega}$, which proves the lemma. \square

As an immediate consequence of the above lemma the stability result is established in the following.

Lemma 2.2. Let $A(x)$ satisfy (1.3) and (1.4). Let $\vec{\psi}$ be any vector-valued function in the domain of \vec{L} , then for each $x \in [0, 1]$, then

$$|\vec{\psi}(x)| \leq \max\{\|\vec{\beta}\vec{\psi}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|\}$$

Proof. Consider the two functions

$$\bar{\theta}^\pm(x) = \max\left\{\|\vec{\beta}\vec{\psi}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|\right\} \pm \vec{\psi}(x), \quad x \in \bar{\Omega}$$

$$\bar{\theta}^\pm(x) = M \pm \vec{\psi}(x)$$

where $M = \max\{\|\vec{\beta}\vec{\psi}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|\}$. Then, it is not hard to verify that $\vec{\beta}\bar{\theta}^\pm(0) \geq 0$ and $\vec{L}\bar{\theta}^\pm(x) \geq 0$ on Ω . It follows from Lemma 2.1 that $\bar{\theta}^\pm(x) \geq 0$ on $\bar{\Omega}$. Hence,

$$|\vec{\psi}(x)| \leq \max\{\|\vec{\beta}\vec{\psi}(0)\|, \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|\}$$

\square

3. Estimates of Derivatives

Lemma 3.1. Let $A(x)$ satisfy (1.3) and (1.4). Let \vec{u} be the solution of (1.1), (1.2). Then, for each i , $i = 1, 2$ and $x \in \bar{\Omega}$, there exists a constant C such that

$$|u_i(x)| \leq C \left\{ \|\vec{\phi}\| + \|\vec{f}\| \right\}$$

$$|u'_i(x)| \leq C\varepsilon_i^{-1} \left\{ \|\vec{\phi}\| + \|\vec{f}\| \right\}$$

$$|u''_i(x)| \leq C\varepsilon_i^{-2} \left\{ \|\vec{\phi}\| + \|\vec{f}\| + \|\vec{f}'\| \right\}.$$

Proof. From Lemma 2.2, it is evident that,

$$|\vec{u}(x)| \leq \|\vec{\beta}\vec{\psi}(0)\| + \frac{1}{\alpha}\|\vec{L}\vec{\psi}\|.$$

Thus,

$$|u_i(x)| \leq C \left\{ \|\vec{\phi}\| + \|\vec{f}\| \right\}.$$



Rewrite the differential equation (1.1), we get

$$\vec{u}'(x) = E^{-1}(\vec{f} - A\vec{u})$$

$$\text{Hence, } |u'_i(x)| \leq C\varepsilon_i^{-1}(\|\vec{\phi}\| + \|\vec{f}\|).$$

Differentiating (1.1) once, we get

$$E\vec{u}''(x) + A(x)\vec{u}'(x) = \vec{f}'(x) - A'(x)\vec{u}(x).$$

Using the bounds of \vec{u}' and \vec{u} , we get the desired result, that is

$$|u''_i(x)| \leq C\varepsilon_i^{-2}[\|\vec{f}'\| + \|\vec{\phi}\| + \|\vec{f}\|].$$

□

The Shishkin decomposition of the solution \vec{u} of (1.1) is given by

$$\vec{u} = \vec{v} + \vec{w} \quad (3.1)$$

where the smooth component $\vec{v} = (v_1, v_2)^T$ of the solution \vec{u} satisfies

$$\vec{L}\vec{v} = \vec{f} \text{ on } (0, 1] \quad (3.2)$$

with

$$\vec{\beta}\vec{v}(0) = \vec{u}_0(0) - E\vec{u}'_0(0) \quad (3.3)$$

and the singular component $\vec{w} = (w_1, w_2)^T$ is the solution of

$$\vec{L}\vec{w}(x) = \vec{0} \text{ for } x \in (0, 1] \quad (3.4)$$

with

$$\vec{\beta}\vec{w}(0) = \vec{\phi} - \vec{\beta}\vec{v}(0). \quad (3.5)$$

The smooth component \vec{v} is subjected to further decomposition. Since the component u_1 has ε_1 sublayer, the component v_1 is given a further decomposition, $v_1 = u_{0,1} + \varepsilon_2(v_{1,2} + \varepsilon_1 v_{1,1})$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \varepsilon_2 & \varepsilon_2 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} v_{1,1} & v_{1,2} \\ 0 & v_{2,2} \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.6)$$

$$\vec{v}(x) = \vec{u}_0(x) + \vec{\Upsilon}(x) \quad (3.7)$$

where

$$\vec{\Upsilon}(x) = (\Upsilon_1, \Upsilon_2)^T, \quad \Upsilon_j = \vec{\varepsilon}_j^j (\vec{v}_j^j)^T \quad (3.8)$$

$$\vec{\varepsilon}_1^1 = (\varepsilon_1 \varepsilon_2, \varepsilon_2), \quad \vec{\varepsilon}_2^2 = (0, \varepsilon_2)$$

From (3.2),

$$\vec{L}\vec{v} = \vec{f} \text{ on } (0, 1] \quad (3.9)$$

with

$$\vec{\beta}\vec{v}(0) = \vec{\beta}(\vec{u}_0 + \vec{\Upsilon})(0) \quad (3.10)$$

From(3.6), it is observed that the components $v_{i,j}, i = 1, 2, j = 1, 2$ satisfy the following system of equations

$$a_{11}v_{1,2} + a_{12}v_{2,2} = -\frac{\varepsilon_1}{\varepsilon_2}u'_{0,1} \quad (3.11)$$

$$\varepsilon_2 v'_{2,2} + a_{21}v_{1,2} + a_{22}v_{2,2} = -u'_{0,2} \text{ with } (v_{2,2} - \varepsilon_2 v'_{2,2})(0) = 0 \quad (3.12)$$

and

$$\varepsilon_1 v'_{1,1} + a_{11}v_{1,1} = -v'_{1,2} \text{ with } (v_{1,1} - \varepsilon_1 v'_{1,1})(0) = 0. \quad (3.13)$$

The singular component of the solution \vec{u} satisfies

$$\vec{L}\vec{w} = \vec{0} \text{ on } (0, 1] \quad (3.14)$$

with

$$\vec{\beta}\vec{w}(0) = \vec{\beta}(\vec{u} - \vec{v})(0). \quad (3.15)$$

From the expressions and using lemma for \vec{v} , it is found that for $k = 0, 1, 2$

$$\begin{aligned} |v_{1,2}^{(k)}| &\leq C(1 + \varepsilon_2^{-1}) \text{ for } k = 0, 1 \\ |v_{1,1}^{(k)}| &\leq C(1 + \varepsilon_1^{-1}\varepsilon_2^{-1}) \text{ for } k = 2. \end{aligned} \quad (3.16)$$

From (3.7), (3.8) and (3.16), the following bounds for $v_i, i = 1, 2$, hold

$$\begin{aligned} |v_i^{(k)}| &\leq C \text{ for } k = 0, 1 \\ |v_i^{(k)}| &\leq C\varepsilon_i^{-1} \text{ for } k = 2. \end{aligned} \quad (3.17)$$

To find bounds on the layer component \vec{w} of \vec{u} , consider the layer functions

$$B_1(x) = e^{-\frac{\alpha x}{\varepsilon_1}}, \quad B_2(x) = e^{-\frac{\alpha x}{\varepsilon_2}}. \quad (3.18)$$

Lemma 3.2. Let $A(x)$ satisfy (1.3) and (1.4). Then the solution $\vec{w}(x) = (w_1(x), w_2(x))^T$ of the problem (1.1) satisfies

$$\begin{aligned} |w_1(x)| &\leq CB_2(x) \\ |w_2(x)| &\leq CB_2(x) \\ |w'_1(x)| &\leq C(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)) \\ |w'_2(x)| &\leq C\varepsilon_2^{-1}B_2(x) \\ |w''_1(x)| &\leq C\varepsilon_1^{-1}(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)) \\ |w''_2(x)| &\leq C\varepsilon_1^{-1}(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)) \end{aligned}$$

Proof. To derive the bound on \vec{w} , define the two functions,

$$\theta_i^\pm(x) = CB_2(x) + w_i(x), \text{ for } i = 1, 2 \text{ and } x \in \bar{\Omega}.$$



For a proper choice of C ,

$$\vec{\beta}\vec{\theta}^\pm(0) \geq \vec{0}.$$

To derive the first order derivatives, consider the equation $(\vec{L}\vec{w})_1 = 0$

$$\varepsilon_1 w_1'(x) + a_{11}(x)w_1(x) + a_{12}(x)w_2(x) = 0$$

which implies that

$$|w_1'(x)| \leq C\varepsilon_1^{-1}B_2(x).$$

In particular,

$$|w_1'(0)| \leq C\varepsilon_1^{-1}.$$

Similarly,

$$|w_2'(x)| \leq C\varepsilon_2^{-1}B_2(x).$$

To derive the sharper bound for $w_1'(x)$, consider the equation satisfied by w_1 ,

$$\begin{aligned} \varepsilon_1 w_1'(x) + a_{11}(x)w_1(x) + a_{12}(x)w_2(x) &= 0 \\ (ie), \quad \varepsilon_1 w_1'(x) + a_{11}(x)w_1(x) &= -a_{12}(x)w_2(x) \end{aligned}$$

Differentiating (1.1) once, we get

$$L_1 w_1'(x) = (-a_{12}(x)w_2(x))' - a_{11}'(x)w_1(x)$$

which leads to,

$$|(\vec{L}\vec{w}')_1(x)| \leq C\varepsilon_2^{-1}B_2(x).$$

Consider the functions,

$$\theta^\pm(x) = C(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)).$$

Then,

$$\begin{aligned} (\vec{L}\theta^\pm)_1(x) &= C(-\alpha + a_{11}(x))(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)) \\ &\geq 0, \quad \text{for a proper choice of } C. \end{aligned}$$

Therefore, $(\vec{L}\theta^\pm)_1(x) \geq 0$ on Ω . Further it is not hard to see that $\vec{\beta}\theta^\pm(0) \geq \vec{0}$. Hence by using the maximum principle for the operator \vec{L} , the required bound on $w_1'(x)$ follows.

Differentiating $(\vec{L}\vec{w})_1 = 0$ and $(\vec{L}\vec{w})_2 = 0$ once and using the estimates of $w_1'(x)$ and $w_2'(x)$, one can obtain the desired result for $i = 1, 2$,

$$|w_i''(x)| \leq C\varepsilon_i^{-1}(\varepsilon_1^{-1}B_1(x) + \varepsilon_2^{-1}B_2(x)).$$

This completes the proof. \square

4. The Shishkin mesh

A piecewise uniform Shishkin mesh $\overline{\Omega}^N$ with N mesh-intervals is now constructed on $\overline{\Omega} = [0, 1]$ as follows for the case $\varepsilon_1 < \varepsilon_2$. In the case $\varepsilon_1 = \varepsilon_2$ a simpler construction requiring just one parameter τ suffices. The interval $[0, 1]$ is subdivided into 3 sub-intervals $[0, \tau_1] \cup (\tau_1, \tau_2] \cup (\tau_2, 1]$. The parameters τ_1, τ_2 which determine the points separating the uniform meshes, are defined by $\tau_0 = 0, \tau_3 = \frac{1}{2}$

$$\tau_2 = \min \left\{ \frac{1}{2}, \frac{\varepsilon_2}{\alpha} \ln N \right\} \quad (4.1)$$

and

$$\tau_1 = \min \left\{ \frac{\tau_2}{2}, \frac{\varepsilon_1}{\alpha} \ln N \right\} \quad (4.2)$$

Clearly,

$$0 < \tau_1 < \tau_2 \leq \frac{1}{2}.$$

Then, on the sub-interval $(\tau_2, 1]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $(0, \tau_1]$ and $(\tau_1, \tau_2]$, a uniform mesh of $\frac{N}{8}$ mesh points is placed. Note that, when both the parameters $\tau_r, r = 1, 2$, take on their left hand value, the Shishkin mesh becomes a classical uniform mesh on $[0, 1]$. This construction leads to a class of four possible Shishkin piecewise uniform meshes $M_{\vec{b}}$, where $\vec{b} = (b_1, b_2)$ with $b_i = 0$ if $\tau_i = \frac{\tau_i+1}{2}$ and $b_i = 1$ otherwise.

5. The Discrete Problem

The Initial Value Problem (1.1), (1.2) is discretised using the backward Euler scheme applied on the piecewise uniform fitted mesh $\overline{\Omega}^N$. The discrete problem is

$$\vec{L}^N \vec{U}(x_j) = ED^- \vec{U}(x_j) + A(x_j) \vec{U}(x_j) = \vec{f}(x_j), \quad j = 1(1)N \quad (5.1)$$

$$\vec{U}(x_0) - ED^+ \vec{U}(x_0) = \vec{\phi}. \quad (5.2)$$

The problem (5.1), (5.2) can also be written in the operator form

$$\vec{L}^N \vec{U} = \vec{f} \text{ on } \Omega^N \text{ with}$$

$$\vec{\beta}^N \vec{U}(0) = \vec{\phi}$$

$$\text{where } \vec{L}^N = ED^- + A \text{ with}$$

$$\vec{\beta}^N = I - ED^+$$

and D^+, D^- are the difference operators

$$D^- \vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}},$$

$$D^+ \vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}, \quad j = 1, 2, \dots, N.$$

The following discrete results are analogous to those for the continuous case.



Lemma 5.1. Let $A(x)$ satisfy (1.3) and (1.4). Let $\vec{\Psi} = (\Psi_1, \Psi_2)^T$ be any vector-valued mesh function, such that $\vec{\beta}^N \vec{\Psi}(0) \geq \vec{0}$. Then $\vec{L}^N \vec{\Psi} \geq \vec{0}$ on Ω^N implies that $\vec{\Psi} \geq \vec{0}$ on $\overline{\Omega}^N$.

Proof. Let i^*, j^* be such that $\Psi_{i^*} = \min_{i=1,2} \min_{0 \leq j \leq N} \Psi_i(x_j)$ and assume that the lemma is false. Then, $\Psi_{i^*}(x_{j^*}) < 0$. If $x_{j^*} = 0$, then

$$(\vec{\beta}^N \vec{\Psi})_{i^*}(0) = \Psi_{i^*}(0) - \varepsilon_{i^*} D^+ \Psi_{i^*}(0) < 0, \text{ a contradiction}$$

Therefore, $x_{j^*} \neq 0$.

Suppose $x_{j^*} \in \Omega^N$, then

$$(\vec{L}^N \vec{\Psi})_{i^*}(x_{j^*}) = \varepsilon_{i^*} D^- \Psi_{i^*}(x_{j^*}) + \sum_{k=1}^2 a_{i^*k}(x_{j^*}) \Psi_k(x_{j^*}) < 0$$

which is a contradiction. Hence proves the lemma. \square

Lemma 5.2. Let $A(x)$ satisfy (1.3) and (1.4). Let $\vec{\Psi}$ be any vector-valued mesh function on $\overline{\Omega}^N$, then for $i = 1, 2$,

$$\Psi_i(x_j) \leq \max \left\{ \|\vec{\beta}^N \vec{\Psi}(0)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \right\}, \quad 0 \leq j \leq N$$

Proof. Consider the two mesh functions

$$\vec{\Theta}^\pm(x_j) = \max \left\{ \|\vec{\beta}^N \vec{\Psi}(0)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \right\} \pm \vec{\Psi}(x_j)$$

Using the properties of $A(x)$, it is not hard to verify that $\vec{\beta}^N \vec{\Theta}^\pm(0) \geq \vec{0}$ and $\vec{L}^N \vec{\Theta}^\pm \geq \vec{0}$ on Ω^N . Applying the discrete maximum principle (Lemma 5.1) then gives $\vec{\Theta}^\pm \geq \vec{0}$, and so

$$\Psi_i(x_j) \leq \max \left\{ \|\vec{\beta}^N \vec{\Psi}(0)\|, \frac{1}{\alpha} \|\vec{L}^N \vec{\Psi}\| \right\}$$

as required. \square

6. The Local Truncation Error

From Lemma 5.2, it follows that in order to bound the error $\|\vec{U} - \vec{u}\|$, it suffices to bound $L^N(\vec{U} - \vec{u})$. But this expression satisfies

$$\begin{aligned} \vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j)) &= \vec{L}^N \vec{U}(x_j) - \vec{L}^N \vec{u}(x_j) \\ &= \vec{f}(x_j) - \vec{L}^N \vec{u}(x_j) \\ &= \vec{L} \vec{u}(x_j) - \vec{L}^N \vec{u}(x_j) \\ &= (\vec{L} - \vec{L}^N) \vec{u}(x_j) \end{aligned}$$

and

$$((\vec{L} - \vec{L}^N) \vec{u})_i(x_j) = (D^- - D)v_i(x_j) + (D^- - D)w_i(x_j)$$

which is the local truncation of the first derivative. Then, by the triangle inequality,

$$|(\vec{L}^N(\vec{U} - \vec{u}))_i(x_j)| \leq |(D^- - D)v_i(x_j)| + |(D^- - D)w_i(x_j)|.$$

Analogous to the continuous case, the discrete solution \vec{U} can be decomposed into \vec{V} and \vec{W} which are defined to be solutions of the following discrete problems

$$(\vec{L}^N \vec{V})(x_j) = \vec{f}(x_j) \text{ on } \Omega^N, \quad \vec{\beta}^N \vec{V}(0) = \vec{\beta} \vec{v}(0) \quad (6.1)$$

and

$$(\vec{L}^N \vec{W})(x_j) = \vec{0} \text{ on } \Omega^N, \quad \vec{\beta}^N \vec{W}(0) = \vec{\beta} \vec{w}(0) \quad (6.2)$$

where \vec{u} and \vec{w} are the solutions of (3.2), (3.3) and (3.4), (3.5) respectively.

Further, for $i = 1, 2$,

$$\begin{aligned} |(\vec{\beta}^N(\vec{V} - \vec{v}))_i(0)| &= |(D - D^+)v_i(0)| \\ |(\vec{\beta}^N(\vec{W} - \vec{w}))_i(0)| &= |(D - D^+)w_i(0)| \\ |(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| &= |\varepsilon_i(D^- - D)v_i(x_j)| \end{aligned} \quad (6.3)$$

$$|(\vec{L}^N(\vec{W} - \vec{w}))_i(x_j)| = |\varepsilon_i(D^- - D)w_i(x_j)| \quad (6.4)$$

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $\vec{U}(x_j) - \vec{u}(x_j)$. Then the local truncation error $\vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j))$ has the decomposition

$$\vec{L}^N(\vec{U} - \vec{u})(x_j) = \vec{L}^N(\vec{V} - \vec{v})(x_j) + \vec{L}^N(\vec{W} - \vec{w})(x_j)$$

Therefore, the local truncation error of the smooth and singular components can be treated separately. In view of this, it is to be noted that for any smooth function ψ , the following two distinct estimates of the local truncation of its first derivative hold.

$$|(D^- - D)\psi(x_j)| \leq 2 \max_{s \in I_j} |\psi'(s)| \quad (6.5)$$

and

$$|(D^- - D)\psi(x_j)| \leq \frac{h_j}{2} \max_{s \in I_j} |\psi''(s)| \quad (6.6)$$

where $I_j = x_j - x_{j-1}$.

The error in the smooth and singular components are bounded in the following section.

7. Error estimate

The proof of the theorem on the error estimate is split into two parts. First, a theorem concerning the error in the smooth component is established. Then the error in the singular component is established.

The following theorem gives the estimate of the error in the smooth component \vec{V} .



Theorem 7.1. Let $A(x)$ satisfy (1.3) and (1.4). Let \vec{v} denote the smooth component of the solution of (1.1), (1.2) and \vec{V} denote the smooth component of the solution of the problem (5.1), (5.2). Then

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq CN^{-1}.$$

Proof. From the expression (6.6),

$$\begin{aligned} |(\vec{\beta}^N(\vec{V} - \vec{v}))_i(0)| &\leq C\epsilon_i(x_1 - x_0) \max_{s \in [x_0, x_1]} |v_i''(s)| \\ &\leq CN^{-1}. \end{aligned} \quad (7.1)$$

By the local truncation error, we have

$$\left| (\vec{L}^N(\vec{V} - \vec{v}))_i(x_j) \right| \leq C\epsilon_i(x_j - x_{j-1})|\vec{v}|_2$$

It is to be noted that $x_j - x_{j-1} \leq 2N^{-1}$ holds for all choices of the piecewise uniform mesh the estimate for \vec{v} obtained above then yields

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| \leq CN^{-1} \quad (7.2)$$

as required. \square

In order to estimate the error in the singular component of the solution \vec{u} , the following lemmas are required.

Theorem 7.2. Let conditions (1.3) and (1.4) hold. If \vec{w} denotes the singular component of the solution of (1.1), (1.2) and \vec{W} be the singular component of the solution of the problem (5.1), (5.2) then,

$$|(\vec{L}^N(\vec{W} - \vec{w}))_i(x_j)| \leq CN^{-1} \ln N.$$

Proof. From the expression (6.6),

$$\begin{aligned} |(\vec{\beta}^N(\vec{W} - \vec{w}))_i(0)| &\leq C\epsilon_i(x_1 - x_0) \max_{s \in [x_0, x_1]} |w_i''(s)| \\ &\leq CN^{-1} \ln N. \end{aligned} \quad (7.3)$$

It is to be noted that for each $i = 1, 2$ and $j = 1, 2, \dots, N$, on each mesh $M_{\vec{b}}$, and from Lemma (3.2), it can be seen that for $j = 1, 2, \dots, \frac{N}{2}$,

$$\begin{aligned} |\epsilon_i(D^- - D)w_i(x_j)| &\leq Ch_j \max_{s \in I_j} |\epsilon_i w_i''(s)| \\ &\leq Ch_j \sum_{q=1}^2 \frac{B_q(x_{j-1})}{\epsilon_q} \\ &\leq C \frac{h_j}{\epsilon_1} \end{aligned} \quad (7.4)$$

Hence,

$$|\epsilon_i(D^- - D)w_i(x_j)| \leq C \frac{h_j}{\epsilon_1}$$

The theorem is proved for four cases.

Case (i): On mesh $M_{\vec{b}}$ with $\vec{b} = (0, 0)$

Here the mesh is uniform and hence $h_j = x_j - x_{j-1} = N^{-1}$.

Since $\tau_1 = \frac{1}{2}, \frac{\epsilon_1}{\alpha} \ln N \geq \frac{1}{2}$ (or) $\epsilon_1^{-1} \leq C \ln N$.

Therefore,

$$|\epsilon_i(D^- - D)w_i(x_j)| \leq C \frac{h_j}{\epsilon_1} \leq CN^{-1} \ln N \quad \text{using (7.4)}$$

Case (ii): On mesh $M_{\vec{b}}$ with $\vec{b} = (0, 1)$

In this case, the mesh is piecewise uniform and the following are true: $\tau_1 = \frac{\tau_2}{2}, \tau_2 = \frac{\epsilon_2}{\alpha} \ln N$. Hence, $\frac{\tau_2}{2} < \frac{\epsilon_1}{\alpha} \ln N$ (or) $\epsilon_1 > \frac{\epsilon_2}{2}$. Also, $\tau_2 - \tau_1 = \tau_1$.

On the interval $(0, \tau_1]$,

$$\begin{aligned} |\epsilon_i(D^- - D)w_i(x_j)| &\leq C \frac{h_j}{\epsilon_1} \\ &\leq C \frac{\tau_1 N^{-1}}{\epsilon_1} \\ &\leq CN^{-1} \ln N \quad \text{using (7.4)} \end{aligned}$$

On the interval $(\tau_1, \tau_2]$

$$\begin{aligned} |\epsilon_i(D^- - D)w_i(x_j)| &\leq C \frac{h_j}{\epsilon_1} \\ &\leq C \frac{(\tau_2 - \tau_1)N^{-1}}{\epsilon_1} \\ &\leq CN^{-1} \ln N \quad \text{as } \tau_2 - \tau_1 = \tau_1 \end{aligned}$$

On the interval $(\tau_2, 1]$

$$\begin{aligned} |\epsilon_i(D^- - D)w_i(x_j)| &\leq C\epsilon_1 \max_{[x_{j-1}, x_j]} |w_i''(s)| \\ &\leq CB_2(x_{j-1}) \quad \text{since } \epsilon_1 \leq \epsilon_2 \\ &\leq CN^{-1}. \end{aligned}$$

Case (iii): On mesh $M_{\vec{b}}$ with $\vec{b} = (1, 0)$

Here, the mesh is piecewise uniform. As $\tau_1 = \frac{1}{2}, \epsilon_2 \leq C \ln N$

and as $\tau_1 = \frac{\epsilon_1}{\alpha} \ln N, \epsilon_1 < \frac{\epsilon_2}{2}$.

On the interval $(0, \tau_1]$,

$$|\epsilon_i(D^- - D)w_i(x_j)| \leq C \frac{h_j}{\epsilon_1} \leq CN^{-1} \ln N.$$

On the interval $(\tau_1, \tau_2]$,

$$\begin{aligned} |\epsilon_i(D^- - D)w_i(x_j)| &\leq \epsilon_i(D^- - D)[w_{i,1}(x_j) + w_{i,2}(x_j)] \\ &\leq CB_1(x_{j-1}) + CN^{-1} \ln N \end{aligned}$$

Since, $x_j > \tau_1, x_{j-1} \geq \tau_1$ and hence $B_1(x_{j-1}) \leq N^{-1}$.

$$|\epsilon_i(D^- - D)w_i(x_j)| \leq CN^{-1} \ln N.$$

On the interval $(\tau_2, 1]$, proceeding as in the interval $(\tau_1, \tau_2]$,

$$|\epsilon_i(D^- - D)w_i(x_j)| \leq CB_i(x_{j-1}) + CN^{-1} \ln N.$$



Since, $x_j > \tau_2$, $x_{j-1} \geq \tau_2$ and hence $B_i(x_{j-1}) \leq N^{-1}$, as $\tau_2 > \frac{2\varepsilon_1}{\alpha} \ln N > \frac{\varepsilon_1}{\alpha} \ln N$.
Hence,

$$|\varepsilon_i(D^- - D)w_i(x_j)| \leq CN^{-1} \ln N.$$

Case (iv): On mesh $M_{\vec{b}}$ with $\vec{b} = (1, 1)$
On the interval $(0, \tau_1]$,

$$|\varepsilon_i(D^- - D)w_i(x_j)| \leq CN^{-1} \ln N.$$

On the interval $(\tau_1, \tau_2]$, the proof follows by using the same procedure used in case (iii) in the respective intervals.
On the interval $(\tau_2, 1]$,

$$|\varepsilon_i(D^- - D)w_i(x_j)| \leq CB_2(x_{j-1}) \leq CN^{-1} \ln N.$$

Hence the theorem. □

Theorem 7.3. Let \vec{u} be the solution of the continuous problem (1.1), (1.2) and \vec{U} be the solution of the discrete problem (5.1), (5.2). Then

$$\|(\vec{L}^N(\vec{U} - \vec{u}))\| \leq CN^{-1}$$

Proof. From Lemma 5.2, it is clear that, in order to prove the above theorem it suffices to prove that $\|(\vec{L}^N(\vec{U} - \vec{u}))\| \leq CN^{-1}$. But, $\|(\vec{L}^N(\vec{U} - \vec{u}))\| \leq \|(\vec{L}^N(\vec{V} - \vec{v}))\| + \|(\vec{L}^N(\vec{W} - \vec{w}))\|$. Hence using theorems 7.1 and 7.2, the above result is derived. □

8. Numerical Illustration

The numerical method proposed above is illustrated through an example presented in this section.

Example 8.1. Consider the initial value problem

$$\left. \begin{aligned} \varepsilon_1 u_1'(x) + 3u_1(x) - u_2(x) &= 3, \\ \varepsilon_2 u_2'(x) - u_1(x) + 5u_2(x) &= 1 \end{aligned} \right\} \forall x \in (0, 1]$$

with

$$\begin{aligned} u_1(0) - \varepsilon_1 u_1'(0) &= 2 \\ u_2(0) - \varepsilon_2 u_2'(0) &= 1 \end{aligned}$$

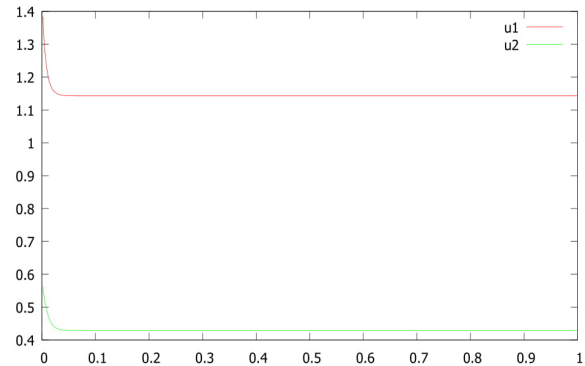
The numerical solution obtained by applying the fitted mesh method (5.1) and (5.2) to the Example 8.1 is shown in Figure 1. The order of convergence and the error constant are calculated and are presented in Table 1.

Table 1

Values of $D_\varepsilon^N, D^N, p^N, p^*$ and C_p^N generated for the example 8.1.

η	Number of mesh points N				
	64	128	256	512	1024
1.0000	0.0490	0.0282	0.0152	0.0078	0.0040
0.1250	0.0649	0.0481	0.0324	0.0203	0.0120
0.0156	0.0649	0.0481	0.0324	0.0203	0.0120
0.0019	0.0649	0.0481	0.0324	0.0203	0.0120
0.0002	0.0649	0.0481	0.0324	0.0203	0.0120
D^N	0.0649	0.0481	0.0324	0.0203	0.0120
p^N	0.431	0.571	0.677	0.752	
C_p^N	1.510	1.510	1.370	1.150	0.923
The order of $\vec{\varepsilon}$ -uniform convergence $p^* = 0.431$					
Computed $\vec{\varepsilon}$ -uniform error constant, $C_p^N = 1.510$					

Figure 1



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