On several classes of contra continuity

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Abstract

Further results and characterizations of several classes of contra continuous functions with their connections are discussed. Among the results: we show that every function is either somewhat nearly continuous or contra somewhat nearly continuous, and every function whose range is a regular space is somewhat nearly continuous.

Keywords

Contra continuous, contra β-continuous, contra semicontinuous, contra precontinuous, contra α-continuous, contra b-continuous, contra γ-continuous, contra β-continuous, contra somewhat continuous, contra somewhat nearly continuous.

AMS Subject Classification

54C08, 54C10, 54C05.

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1. Introduction and Preliminaries

In General Topology, covering spaces with closed sets was used by famous mathematicians. In 1918, Sierpinski [30] showed that whenever there is a family of pairwise disjoint closed sets that covers a connected compact Hausdorff space, then no more than one of those sets is nonempty. In 1992, Cater and Daily [13] proved that if a complete connected locally connected metric space is covered by countably many proper closed sets, then some two members of these sets must meet, which is an improvement of Sierpinski's result. In 1996, Dontchev [14] introduced the concept of strongly S-closed spaces in terms of covering by closed sets. This concept is a generalization of S-closed spaces given by Thompson [19]. Now, here a question arise: which class of continuous functions can preserve covering properties (open covering onto closed covering or vice versa). In [14] Dontchev defined contra continuity that transforms strongly S-closed spaces onto compact spaces. Then contra continuity has been generalized by many topologists for various purposes, see [15, 17, 18, 24]. The present paper studies further properties and characteriza-

tions of some types of generalized contra continuity.

Henceforward, the set of natural, rational and real numbers symbolized, respectively, by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$. The word space is referred to arbitrary topological space. The closure and interior of $A \subseteq X$ are named by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$, respectively, (when no confusion arise, we will use $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$).

Definition 1.1. *Let X be a space and $A \subseteq X$, A is said to be*

(1) regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$,

(2) α-open [25] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(3) preopen [22] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$,

(4) semiopen [20] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$,

(5) sc-open [6] if A is semiopen and a union of closed sets,

(6) β-open [1] or semipreopen [8] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

(7) γ-open [9] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,

(8) somewhat open (in short, sw-open) [28] if $\mathrm{Int}(A) \neq \emptyset$ or $A = \emptyset$,

(9) somewhat nearly open (in short, swn-open) [28] (for more details, see [5]) if $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$ or $A = \emptyset$. A nonempty swn-open is equivalent to SD-open in [2].

The class of entire preopen (resp. semiopen, α-open, γ-open, β-open) sets in X is symbolized by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $\gamma O(X)$, $\beta O(X)$).

A regular closed (resp. preclosed, semi-closed, sc-closed, α-closed, β-closed, γ-closed, sw-closed, swn-closed) set is the complement of regular open (resp. preopen, semiopen, sc-open, α-open, β-open, γ-open, sw-open, swn-open)

The intersection of entire preclosed (resp. semiclosed, α-closed, β-closed, γ-closed) sets in X including A is defined to be the preclosure (resp. semiclosure, α-closure, β-closure, γ-closure) of A, and is symbolized by $\text{Cl}_p(A)$ (resp. $\text{Cl}_s(A)$, $\text{Cl}_\alpha(A)$, $\text{Cl}_\beta(A)$, $\text{Cl}_\gamma(A)$).

The union of entire preopen (resp. semiopen, α-open, β-open, γ-open) sets in X included in A is called the preinterior (resp. semi-interior, α-interior, β-interior, γ-interior) of A, and is symbolized by $\text{Int}_p(A)$ (resp. $\text{Int}_s(A)$, $\text{Int}_\alpha(A)$, $\text{Int}_\beta(A)$, $\text{Int}_\gamma(A)$).

Fact 1.2. *The connection between the classes defined above is* $\tau \subseteq \alpha O(X) \subseteq PO(X) \cup SO(X) \subseteq \gamma O(X) \subseteq \beta O(X)$.

Definition 1.3. *Given $A \subseteq X$. A point $x \in X$ is in preclosure (resp. semi-closure, β-closure) of A if $U \cap A \neq \phi$ for every preopen (resp. semiopen, β-open) set U that contains x.*

Lemma 1.4. *[8, Theorem 2.4][5, Lemma 2.6] Let $A \subseteq X$.*

(1) A is semiopen iff $\text{Cl}(A) = \text{Cl}(\text{Int}(A))$.

(2) A is β-open iff $\text{Cl}(A) = \text{Cl}(\text{Int}(\text{Cl}(A)))$.

Lemma 1.5. *[5, Lemma 2.7] Let B be a nonempty subset of a space X.*

(i) If B is semiopen, then $\text{Int}(B) \neq \emptyset$.

(ii) If B is β-open, then $\text{Int}(\text{Cl}(B)) \neq \emptyset$.

Next we put Definition 1.1, Fact 1.2 and Lemma 1.5 into the following diagram:

open $\longleftarrow\!\!\!\!\diagdown$ sc-open $\longleftarrow\!$ regular closed

$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$

α-open $\longrightarrow$ semiopen $\longrightarrow\!$ sw-open

$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$

preopen $\longrightarrow$ γ-open $\longrightarrow$ β-open $\longrightarrow$ swn-open
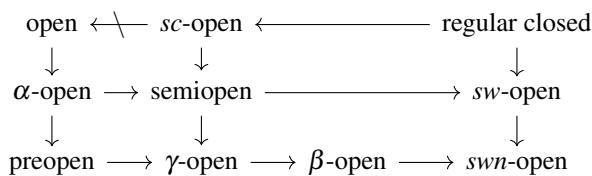
Diagram 1: Relations between some generalized open sets

The above implications cannot be inverted, generally, as shown below:

Example 1.6. *Consider $\mathbb{R}$ with the usual topology. Let $A = \mathbb{R} \setminus \{\frac{1}{n}\}_{n \in \mathbb{N}}$. Obviously A is α-open but not open. If $B = [0,1]$, B is semiopen but not α-open. If $C = \mathbb{Q}$, C is preopen but not α-open. Let $D = [0,1] \cup ((1,2) \cap \mathbb{Q})$. Then D is both*

γ-open and sw-open but neither preopen nor semiopen (*[9, Example 1]*). *Whenever $E = [0,1) \cap \mathbb{Q}$. One can check that E is β-open (also swn-open) but neither γ-open nor sw-open. Let $F = \mathscr{C} \cup [2,3]$, where $\mathscr{C}$ is the Cantor set. Then F is swn-open, while it is not β-open. Let $G = (0,1] = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. G is sc-open but neither open nor regular closed.*

Since $\mathbb{R}$ with the usual topology is T_1, we cannot find a semiopen set which is not sc-open (see Lemma 1.15). We recall the following example:

Example 1.7. *[6, Example 2.2.3] Consider $X = \{1,2,3\}$ with the topology $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. The set $\{1\}$ is semiopen, but it cannot be sc-open.*

Lemma 1.8. *[8, Theorems 3.13, 3.14 & 3.22][9, Proposition 2.6] For a subset A of X, we have*

(i) $\text{Cl}(\text{Int}(A)) = \text{Cl}(\text{Int}_s(A)) = \text{Cl}_\alpha(\text{Int}_\alpha(A)) = \text{Cl}(\text{Int}_\alpha(A)) = \text{Cl}_\alpha(\text{Int}(A))$,

(ii) $\text{Cl}(\text{Int}(\text{Cl}(A))) = \text{Cl}(\text{Int}_p(A)) = \text{Cl}(\text{Int}_\gamma(A)) = \text{Cl}(\text{Int}_\beta(A))$,

(iii) $\text{Int}(\text{Cl}(\text{Int}(A))) = \text{Cl}_s(\text{Int}(A))$.

Lemma 1.9. *The intersection of open with α-open (resp. pre-open, semiopen, β-open) is α-open (resp. preopen, semiopen, β-open).*

Proof. [25, Proposition 2] (resp. [29, Lemma 4.1], [26, Lemma 1], [1, Theorem 2.7]). $\square$

Lemma 1.10. *Let $A, B \subseteq X$. Then*

(1) $A \cap B$ is α-open in B, whenever A is α-open and B is dense, [4, Lemma 2.15].

(2) $A \cap B$ is semiopen in B, whenever A is semiopen and B is dense, [28, Lemma 1.1].

(3) $A \cap B$ is sw-open in B, whenever A is semiopen and B is dense, [28, Remark 1.2].

(4) $A \cap B$ is swn-open in B, whenever A is swn-open and B is open dense, [5, Proposition 3.8].

Lemma 1.11. *Let $A, B \subseteq X$. Then*

(1) $A \cap B$ is semiopen in B, whenever A is semiopen and B is preopen, [23, Lemma 1.1].

(2) $A \cap B$ is preopen in A, whenever A is semiopen and B is preopen, [23, Lemma 2.1].

(3) $A \cap B$ is α-open in A, whenever A is semiopen and B is α-open, [4, Lemma 2.23].

(4) $A \cap B$ is sc-open in B, whenever A is sc-open and B is preopen, [4, Lemma 2.18].

(5) $A \cap B$ is γ-open in B, whenever A is γ-open and B is α-open, [16, Theorem 1.1.6].

(6) $A \cap B$ is β-open in B, whenever A is β-open and B is α-open, [1, Lemma 2.7].

Lemma 1.12. *Let $A \subseteq Y \subseteq X$ where Y is a subspace.*

(1) If Y is semiopen, then A is semiopen in Y iff A is semiopen in X.

(2) If Y is β-open, then A is β-open in Y iff A is β-open in X.

(3) If Y is semiopen, then A is swn-open in Y iff A is swn-open in X.

(4) If Y is semiopen, then A is sw-open in Y iff A is sw-open in X.

Proof. (1) [21, Theorem 2.4].
(2) The fist direction is proved in [1, Theorem 2.7]. The converse can be followed from [20, Theorem 6] and from the fact that A is β-open iff there exist a preopen open U such that $U \subseteq A \subseteq \mathrm{Cl}(U)$.
(3) [5, Theorem 3.14].
(4) [5, Lemma 2.8] and by (1). $\square$

Remark 1.13. *Since every open (resp. dense) set is preopen, so $A \cap B$ is sc-open in B (by Lemma 1.11 (4)) whenever A is sc-open and B is either open or dense.*

Lemma 1.14. *[4, Lemma 2.22] Let $A \subseteq X$. If A is γ-open which is not sw-open, then A is preopen.*

Lemma 1.15. *[6, Proposition 2.2.10] A set A in a T_1-space is semiopen iff A is sc-open.*

Lemma 1.16. *[27, Lemma 3.1] A set $A \subseteq X$ is α-open iff A is both semiopen and preopen.*

Lemma 1.17. *[5, Proposition 3.13] A set $A \subseteq X$ is swn-open iff it is sw-open, whenever A is semiclosed.*

Lemma 1.18. *Let $A \subseteq X$. Then*

(1) A is a semiopen set iff $A \cap U$ is an sw-open set for all open $U \subseteq X$, [4, Lemma 2.29].

(2) A is an α-open set iff $A \cap U$ is an sw-open set for all α-open $U \subseteq X$, [4, Lemma 2.30].

(3) A is a preopen set iff $A \cap U$ is an swn-open set for all α-open $U \subseteq X$, [5, Proposition 3.17].

(4) A is a β-open set iff $A \cap U$ is an swn-open set for all open $U \subseteq X$, [5, Proposition 3.16].

(5) A is an α-open set iff $A \cap B$ is a semiopen set for all semiopen $U \subseteq X$, [25, Proposition 1].

(6) A is a preopen set iff $A \cap B$ is a β-open set for all semiopen $U \subseteq X$, [4, Lemma 2.26].

Lemma 1.19. *[4, Lemma 2.32] Let $A \subseteq X$. The following are equivalent:*

(i) A is regular closed;

(ii) A is sc-open and closed;

(iii) A is semiopen and closed;

(iv) A is γ-open and closed; and

(v) A is β-open and closed

Lemma 1.20. *[4, Lemma 2.33] For $A \subseteq X$, the following are equivalent:*

(i) If A is preopen, then it is α-open;

(ii) If A is β-open, then it is semiopen;

(iii) If A is preopen, then it is semiopen;

(iv) If A is dense, then it is semiopen;

(v) If A is dense, then it has an interior dense;

(vi) If A is co-dense, then it is nowhere dense;

(vii) If A is swn-open, then it is sw-open;

(viii) If A has nowhere dense boundary.

2. Relationships and Properties

This section is devoted to some properties of the following classes of continuous functions and their relationships.

Definition 2.1. *A function f from a space X to a space Y is called*

(1) completely continuous [10] if the preimage of all closed sets in Y are regular closed in X,

(2) somewhat nearly continuous [5, 28] (in short, swn-continuous) if the preimage of open set in Y is swn-open in X. An SD-continuous surjection in [3] is similar to swn-continuous,

(3) contra sc-continuous [7] if the preimage of all closed sets in Y are sc-open in X,

(4) contra semicontinuous [15] if the preimage of all closed sets in Y are semiopen in X,

(5) contra precontinuous [18] if the preimage of all closed sets in Y are preopen in X,

(6) contra α-continuous [17] if the preimage of all closed sets in Y are α-open in X,

(7) contra β-continuous [12] if the preimage of all closed sets in Y are β-open in X,

(8) contra γ-continuous [24] if the preimage of all closed sets in Y are γ-open in X,

(9) contra somewhat continuous [11] (in short, contra sw-continuous) if the preimage of all closed sets in Y are sw-open in X,

(10) contra somewhat nearly continuous (in short, contra swn-continuous) if the preimage of all closed sets in Y are swn-open in X.

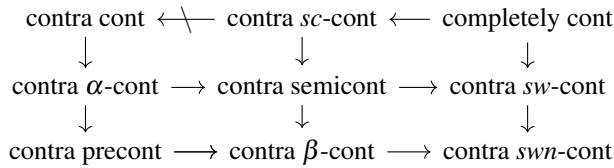The diagram given below shows the connections between functions mentioned above, which is the result of the Diagram 1:

$$\begin{array}{ccccc}
\text{contra cont} & \longleftarrow\!\!\!/\!\!\!\longleftarrow & \text{contra } sc\text{-cont} & \longleftarrow & \text{completely cont} \\
\downarrow & & \downarrow & & \downarrow \\
\text{contra } \alpha\text{-cont} & \longrightarrow & \text{contra semicont} & \longrightarrow & \text{contra } sw\text{-cont} \\
\downarrow & & \downarrow & & \downarrow \\
\text{contra precont} & \longrightarrow & \text{contra } \beta\text{-cont} & \longrightarrow & \text{contra } swn\text{-cont}
\end{array}$$

Diagram 2: Connections between some generalizations of contra continuity

Where "cont" means "continuous function" in the diagram, and no arrows can be reversed generally. As shown below.

Example 2.2. *Let $X = \{0,1\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{0\}, X\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is contra swn-continuous but not contra sw-continuous. The set $\{1\}$ is σ-closed has $\text{Int}(f^{-1}(\{1\})) = \emptyset$. This shows that $f^{-1}(\{1\})$ is never sw-open.*

Example 2.3. *Let τ be the usual topology on $X = \mathbb{R}$ and let d be the discrete topology on $Y = \{0,1\}$. Define the function $f : (X, \tau) \to (X, d)$ by*

$$f(x) = \begin{cases} 0, & x \notin (0,1) \cup \{2\}; \\ 1, & \text{otherwise}. \end{cases}$$

Then f is contra swn-continuous but not contra β-continuous. The preimage of any closed in Y contains an open interval, so it is always swn-open. On the other hand, the preimage of the closed set $\{0\}$ in Y is $(0,1) \cup \{2\}$. But $(0,1) \cup \{2\}$ is not β-open in X.

The examples showing other none implications are available in the literature or can be concluded from Examples 1.6-1.7.

Remark 2.4. *It is worth noting that every class of contra continuous functions, mentioned in Diagram 2, is independent to its original one.*

Since contra *swn*-continuity is new here, we provide some more details.

Definition 2.5. *For a subset A of a space X, we introduce the following:*

(i) $\text{Cl}_{swn}(A) = \bigcap\{F : F \text{ is swn-closed in } X \text{ and } A \subseteq F\}$.

(ii) $\text{Int}_{swn}(A) = \bigcup\{O : O \text{ is swn-open in } X \text{ and } O \subseteq A\}$.

(iii) $\ker(A) = \bigcap\{G \text{ is open in } X \text{ and } A \subseteq G\}$.

Remark 2.6. *Let X, Y be spaces. A function $f : X \to Y$ is contra swn-continuous iff $\forall x \in X$ and all closed sets F in Y including $f(x)$, one can find an swn-open set U in X including x such that $f(U) \subseteq F$.*

Proposition 2.7. *Let X, Y be spaces. For a function $f : X \to Y$, the following statements are equivalent:*

(1) f is contra swn-continuous;

(2) $f^{-1}(F)$ are swn-closed sets in X, for all open subsets F of Y;

(3) $f(\text{Cl}_{swn}(A)) \subset \ker(f(A))$, for all subsets A of X;

(4) $Cl_{swn}(f^{-1}(B)) \subset f^{-1}(\ker(B))$, for all subsets B of Y.

Proof. Standard. □

Theorem 2.8. *Given spaces X, Y. The following statements are equivalent for $f : X \to Y$*

(1) f is a contra swn-continuous function;

(2) for all closed sets F in Y with $f^{-1}(F) \neq \emptyset$, one can find a nonempty open set U in X such that $U \subseteq \text{Cl}(f^{-1}(F))$;

(3) for all open sets G in Y with $f^{-1}(G) \neq X$, one can find a closed E in X such that $\text{Int}(f^{-1}(G)) \subseteq E \neq X$.

Proof. From Definition 2.1 (10). □

Theorem 2.9. *Let $f : X \to Y$ be a function, where X, Y are arbitrary spaces, and let $U \subseteq X$.*

(1) If f is contra semicontinuous and U is preopen, then $f|_U$ is contra semicontinuous.

(2) If f is contra sc-continuous and U is preopen, then $f|_U$ is contra sc-continuous.

(3) If f is contra α-continuous and U is semiopen, then $f|_U$ is contra α-continuous.

(4) If f is contra precontinuous and U is semiopen, then $f|_U$ is contra precontinuous.

(5) If f is contra γ-continuous and U is α-open, then $f|_U$ is contra γ-continuous.

(6) If f is contra β-continuous and U is α-open, then $f|_U$ is contra β-continuous.

Proof. (1) Given a closed set F in Y. By assumption, $f^{-1}(F)$ is a semiopen set in X. Since U is preopen, by Lemma 1.11 (1), $f^{-1}(F) \cap U = f^{-1}|_U(F)$ is semiopen set in U. Hence $f|_U$ is contra semicontinuous.
(2) Same arguments as in (1) and then applying Lemma 1.11 (4).
(3) Same arguments as in (1) and then applying Lemma 1.11 (3).

(4) Same arguments as in (1) and then applying Lemma 1.11 (2).

(5) [24, Theorem 3.10].

(6) Same arguments as in (1) and then applying Lemma 1.11 (6). □

Remark 2.10. *The restriction of a contra sw-continuous (resp. contra swn-continuous) function to even an open subspace need not be contra sw-continuous (resp. contra swn-continuous), see [5, Remark 3.6].*

Theorem 2.11. *Let $f : X \to Y$ be a function, where X,Y are arbitrary spaces. We obtain the following provided that U is a dense subspace of X.*

(1) *If f is contra α-continuous, then $f|_U$ is contra α-continuous.*

(2) *If f is contra semicontinuous, then $f|_U$ is contra semicontinuous.*

(3) *If f is contra sw-continuous, then $f|_U$ is contra sw-continuous.*

(4) *If f is contra sc-continuous, then $f|_U$ is contra sc-continuous.*

(5) *If f is contra sw-continuous, then $f|_U$ is contra sw-continuous provided that U is also open.*

Proof. (1) Following the same steps given in the proof of Theorem 2.9 (1) and using Lemma 1.10 (1).

(2) Following the same steps given in the proof of Theorem 2.9 (1) and using Lemma 1.10 (2).

(3) Following the same steps given in the proof of Theorem 2.9 (1) and using Lemma 1.10 (3).

(4) Following the same steps given in the proof of Theorem 2.9 (1) and using Remark 1.13.

(5) Following the same steps given in the proof of Theorem 2.9 (1) and using Lemma 1.10 (4). □

From Proposition 2.28 in [28] and Remark 3.6 in [5], one can conclude that

Remark 2.12. *The restriction of contra precontinuous (resp. contra γ-continuous, contra β-continuous, contra swn-continuous) functions to dense subspaces may not be contra precontinuous (resp. contra γ-continuous, contra β-continuous, contra swn-continuous).*

Theorem 2.13. *Let X,Y be spaces. A contra γ-continuous function $f : X \to Y$ that is not contra sw-continuous is contra precontinuous.*

Proof. Follows from Lemma 1.14. □

Theorem 2.14. *[7, Proposition 4.5] Let X,Y be spaces such that X is T_1. A function $f : X \to Y$ is contra sc-continuous iff it is contra semicontinuous.*

Theorem 2.15. *A function f from a space X into another space Y is contra α-continuous iff it is contra semicontinuous and contra precontinuous.*

Proof. Follows from Lemma 1.16. □

Theorem 2.16. *A function f from a space X into another space Y is contra semicontinuous iff $f|_W$ is contra sw-continuous for every open set W in X.*

Proof. The first part follows from Theorem 2.9 (1), which implies that every contra semicontinuous restricted to an open set is again contra semicontinuous and hence contra *sw*-continuous.

Conversely, assume that $f|_W$ is contra *sw*-continuous for every open set $W \subseteq X$. Let F be a closed subset of Y. Then $f^{-1}|_W(F) = f^{-1}(F) \cap W$ is an *sw*-open set in W. Since W is open in X, by Lemma 1.12 (4), $f^{-1}(F) \cap W$ is an *sw*-open set in X and thus, by Lemma 1.18 (1), $f^{-1}(F)$ is a semiopen set in X. Hence f is contra semicontinuous. □

Theorem 2.17. *A function f from a space X into another space Y is contra α-continuous iff $f|_G$ is contra sw-continuous for every α-open set W in X.*

Proof. Following the same steps given in the proof of Theorem 2.16 and using Lemma 1.18 (2). □

Theorem 2.18. *A function f from a space X into another space Y is contra β-continuous iff $f|_W$ is contra swn-continuous for every open set W in X.*

Proof. Suppose f is contra β-continuous. Let F be a closed set in Y and let W be an open set in X. By assumption $f^{-1}(F)$ is β-open in X. By Lemma 1.9, $f^{-1}(F) \cap W$ is a β-open set in W and thus, by Lemma 1.5 (2), $f^{-1}(F) \cap W$ is an *swn*-open set in W. Hence, $f|_W$ is *swn*-continuous.

Conversely, assume that $f|_W$ is contra *swn*-continuous for every open set W in X. Let F be a closed set in Y. Then $f^{-1}|_W(F) = f^{-1}(F) \cap W$ is an *swn*-open set in W. Since W is an open (implies semiopen) subset of X, by Lemma 1.12 (3), $f^{-1}(F) \cap W$ is *swn*-open in X for every open W and so, by Lemma 1.18 (4), $f^{-1}(F)$ is a β-open set in X. Hence f is contra β-continuous. □

Theorem 2.19. *A function f from a space X into another space Y is contra precontinuous if $f|_W$ is contra swn-continuous for every α-open set W in X.*

Proof. Following the same steps given in the proof of Theorem 2.18 and Lemma 1.18 (3), one can obtain the proof. □

Theorem 2.20. *A function f from a space X into another space Y is contra α-continuous iff $f|_W$ is contra semicontinuous for every semiopen set W in X.*

Proof. Suppose f is contra α-continuous. Let F be a closed set in Y and let W be a semiopen set in X. By assumption $f^{-1}(F)$ is α-open in X. By Lemma 1.11 (3), $f^{-1}(F) \cap W$

is α-open in W and thus, by Diagram 1, $f^{-1}(F) \cap W$ is a semiopen set in W. Hence, $f|_W$ is contra semicontinuous.

Conversely, suppose that $f|_W$ is contra semicontinuous for every semiopen set W in X. Let F be a closed set in Y. Then $f^{-1}|_W(F) = f^{-1}(F) \cap W$ is semiopen in W. Since W is semiopen in X, by Lemma 1.12 (1), $f^{-1}(F) \cap W$ is semiopen in X for every semiopen set W and thus, by Lemma 1.18 (5), $f^{-1}(F)$ is α-open in X. Thus f is contra α-continuous. $\square$

Theorem 2.21. *A function f from a space X into another space Y is contra precontinuous iff $f|_W$ is contra β-continuous for every semiopen set W in X.*

Proof. Follows Theorem 2.20 and Lemma 1.18 (6). $\square$

Theorem 2.22. *Let f be a function from a space X into a regular space Y. If f is contra swn-continuous, then it is swn-continuous.*

Proof. Let $x \in X$ and V be an open set in Y including $f(x)$. By regularity of Y, there is an open set H in Y including $f(x)$ such that $f(x) \in \text{Cl}(H) \subseteq V$. Since f is contra *swn*-continuous, by Remark 2.6, there is an *swn*-open set G including x such that $f(G) \subseteq \text{Cl}(H) \subseteq V$ and so $f(G) \subseteq V$. Thus f is *swn*-continuous. $\square$

Theorem 2.23. *Every function f from a space X into another space Y is either swn-continuous or contra swn-continuous.*

Proof. Let V be an open set in Y. If $f^{-1}(V) = \emptyset$, then $f^{-1}(V)$ is *swn*-open and we are done.

If $f^{-1}(V) \neq \emptyset$, then suppose $\text{Int}(f^{-1}(V)) = \emptyset$, so

$$\text{Cl}(\text{Int}(f^{-1}(V))) \neq X$$

and thus $f^{-1}(V)$ is *swn*-closed.

On the other hand, suppose $f^{-1}(V)$ is not an *swn*-closed set, then $\text{Cl}(\text{Int}(f^{-1}(V))) = X$. Surely

$$\emptyset \neq \text{Int}(f^{-1}(V)) \subseteq \text{Int}(\text{Cl}(f^{-1}(V))).$$

Thus $f^{-1}(V)$ is *swn*-open. $\square$

From the above two theorem, we have

Theorem 2.24. *Every function f from a space X into a regular space Y is swn-continuous.*

Theorem 2.25. *For a function f from a space X into another space Y, the following are equivalent:*

(1) f is contra sw-continuous and contra semicontinuous;

(2) f is contra swn-continuous and contra semicontinuous.

Proof. Apply Lemma 1.17 $\square$

Theorem 2.26. *For a function f from a space X into another space Y, the following are equivalent:*

(1) f is completely continuous;

(2) f is contra sc-continuous and continuous;

(3) f is contra semicontinuous and continuous;

(4) f is contra γ-continuous and continuous;

(5) f is contra β-continuous and continuous.

Proof. Apply Lemma 1.19 $\square$

Theorem 2.27. *For a function f from a space X into another space Y, the following are equivalent:*

(i) every contra precontinuous function is contra α-continuous;

(ii) every contra β-continuous function is contra semicontinuous;

(iii) every contra precontinuous function is contra semicontinuous;

(iv) every contra swn-continuous function is contra sw-continuous.

Proof. Apply Lemma 1.20 $\square$

3. Characterizations

Theorem 3.1. *For any function f from a space X into another space Y, the following are equivalent:*

(1) f is contra semicontinuous;

(2) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exists a nonempty open set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(3) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, $f^{-1}(F) \cap U$ is sw-open;

(4) for every $x \in X$, every closed set F including $f(x)$ and every α-open U including x, there exists a nonempty open set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(5) for every $x \in X$, every closed set F including $f(x)$ and every α-open U including x, there exists a nonempty α-open set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(6) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exists a nonempty α-open set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(7) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exists a nonempty semiopen set V such that $V \subseteq U$ and $f(V) \subseteq F$.

Proof. (1) $\Longrightarrow$ (2) Let F be a closed set including $f(x)$ and let U be an open including x. By (1), $x \in f^{-1}(F) \subseteq \text{Cl}(\text{Int}(f^{-1}(F)))$. This implies that $\text{Int}(f^{-1}(F)) \cap U \neq \emptyset$. Set $V = \text{Int}(f^{-1}(F)) \cap U$. Therefore V is nonempty open and

$$f(V) \subseteq f(\text{Int}(f^{-1}(F)) \cap U) \subseteq f(f^{-1}(F)) \subseteq G.$$

This completes the proof of (2).

(2) $\implies$ (3) Let F be a closed set including $f(x)$ and let U be any open including x. By (2), there is a nonempty open set V such that $V \subseteq U$ and $f(V) \subseteq F$. Therefore $V \subseteq f^{-1}(F)$ and so $V \subseteq \text{Int}(f^{-1}(F))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}(f^{-1}(F)) \cap U = \text{Int}(f^{-1}(F) \cap U)$, which implies that $f^{-1}(F) \cap U$ is *sw*-open.

(3) $\implies$ (4) Let F be a closed set in Y including $f(x)$ and let U be an α-open set in X including x. Since every α-open set is semiopen, by Lemma 1.5 (1), $\text{Int}(U)$ is a nonempty open set. By (3) $\text{Int}(f^{-1}(F) \cap \text{Int}(U)) = \text{Int}(f^{-1}(F)) \cap \text{Int}(U) \neq \emptyset$. Set $V = \text{Int}(f^{-1}(F)) \cap \text{Int}(U)$. Clearly, V is a nonempty open set U and

$$f(V) \subseteq f(\text{Int}(f^{-1}(F)) \cap \text{Int}(U)) \subseteq f(f^{-1}(F)) \subseteq G.$$

This proves (4).

(4) $\implies$ (5), (5) $\implies$ (6) and (6) $\implies$ (7) are clear from the Diagram 1.

(7) $\implies$ (1) Let F be a closed set including $f(x)$. By (7), for every open set U including x, there is a nonempty semiopen set V such that $V \subseteq U$ and $f(V) \subseteq F$. Therefore $V \subseteq f^{-1}(F)$ and so $V \subseteq \text{Int}_s(f^{-1}(F))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}_s(f^{-1}(F)) \cap U$, which implies that $\text{Int}_s(f^{-1}(F)) \cap U \neq \emptyset$ for every open U including x. Hence $x \in \text{Cl}(\text{Int}_s(f^{-1}(F)))$. By Lemma 1.8 (i), $x \in f^{-1}(F) \subseteq \text{Cl}(\text{Int}(f^{-1}(F)))$. As x was taken arbitrarily, so f is contra semicontinuous. $\square$

The proofs of the following theorems are quite similar to the proof of Theorem 3.1. But for the sake of completeness, we provide them.

Theorem 3.2. *For any function f from a space X into another space Y, the following are equivalent:*

(1) f is contra α-continuous;

(2) for every $x \in X$, every closed set F including $f(x)$ and every semiopen U including x, there exist a nonempty open set V such that $V \subseteq U$ and $f(V) \subseteq F$.

Proof. (1) $\implies$ (2) Let F be a closed set including $f(x)$ and let U be a semiopen including x. By (1) and Lemma 1.8 (iii), $x \in f^{-1}(F) \subseteq \text{Int}(\text{Cl}(\text{Int}(f^{-1}(F)))) = \text{Cl}_s(\text{Int}(f^{-1}(F)))$. This implies that $\text{Int}(f^{-1}(F)) \cap B \neq \emptyset$ for every semiopen B including x and so $\text{Int}(f^{-1}(F)) \cap U \neq \emptyset$. If $W = \text{Int}(f^{-1}(F)) \cap U$, by Lemma 1.9, W is a nonempty semiopen set. Set $V = \text{Int}(W)$. By Lemma 1.5 (1), V is nonempty open and

$$f(V) \subseteq f(\text{Int}(f^{-1}(F)) \cap U) \subseteq f(f^{-1}(F)) \subseteq G.$$

This completes the proof of (2).

(2) $\implies$ (1) Let $x \in X$ and let F be a closed set including $f(x)$. By (2), for every semiopen set U including x, there is a nonempty open set V such that $V \subseteq U$ and $f(V) \subseteq F$. Then $V \subseteq f^{-1}(F)$ and so $V \subseteq \text{Int}(f^{-1}(F))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}(f^{-1}(F)) \cap U$, which implies that $\text{Int}(f^{-1}(F)) \cap$

$U \neq \emptyset$ for every semiopen U including x. Therefore $x \in \text{Cl}_s(\text{Int}(f^{-1}(F)))$. By Lemma 1.8 (iii),

$$x \in f^{-1}(F) \subseteq \text{Int}(\text{Cl}(\text{Int}(f^{-1}(F)))).$$

Hence f is contra α-continuous. $\square$

Theorem 3.3. *For any function f from a space X into another space Y, the following are equivalent:*

(1) f is contra β-continuous;

(2) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exist a nonempty preopen set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(3) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exist a nonempty γ-open set V such that $V \subseteq U$ and $f(V) \subseteq F$;

(4) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exist a nonempty β-open set V such that $V \subseteq U$ and $f(V) \subseteq F$.

Proof. (1) $\implies$ (2) Let F be a closed set including $f(x)$ and let U be an open set including x. By (1) and Lemma 1.8 (ii), $x \in f^{-1}(F) \subseteq \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(F)))) = \text{Cl}(\text{Int}_p(f^{-1}(F)))$. This implies that $\text{Int}_p(f^{-1}(F)) \cap O \neq \emptyset$ for every open O including x and so $\text{Int}_p(f^{-1}(F)) \cap U \neq \emptyset$. Set $V = \text{Int}_p(f^{-1}(F)) \cap U$. By Lemma 1.9, V is a nonempty preopen set with $V \subseteq U$ and

$$f(V) \subseteq f(\text{Int}_p(f^{-1}(F)) \cap U) \subseteq f(f^{-1}(F)) \subseteq G.$$

This proves (2).

(2) $\implies$ (3) and (3) $\implies$ (4) are follow from Diagram 1.

(4) $\implies$ (1) Let x be any point in X and let F be a closed set including $f(x)$. By (4), for every open set U including x, there is a nonempty β-open set V such that $V \subseteq U$ and $f(V) \subseteq F$. Then $V \subseteq f^{-1}(F)$ and so $V \subseteq \text{Int}_\beta(f^{-1}(F))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}_\beta(f^{-1}(F)) \cap U$, which implies that $\text{Int}_\beta(f^{-1}(F)) \cap U \neq \emptyset$ for every open U including x. Therefore $x \in \text{Cl}(\text{Int}_\beta(f^{-1}(F)))$. By Lemma 1.8 (ii), $x \in f^{-1}(F) \subseteq \text{Cl}(\text{Int}_\beta(f^{-1}(F))) = \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(F))))$. This prove that f is contra β-continuous. $\square$

Corollary 3.4. *For any function f from a space X into another space Y, the following are equivalent:*

(1) f is contra β-continuous;

(2) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, there exists a nonempty open set V such that $V \subseteq U$ and $V \subseteq \text{Cl}(f^{-1}(F))$;

(3) for every $x \in X$, every closed set F including $f(x)$ and every open U including x, $f^{-1}(F) \cap U$ is swn-open;

(4) for every x ∈ X, every closed set F including f(x) and every α-open U including x, there exists a nonempty open set V such that V ⊆ U and V ⊆ Cl(f⁻¹(F));

(5) for every x ∈ X, every closed set F including f(x) and every α-open U including x, there exists a nonempty α-open set V such that V ⊆ U and V ⊆ Cl(f⁻¹(F));

(6) for every x ∈ X, every closed set F including f(x) and every open U including x, there exists a nonempty α-open set V such that V ⊆ U and V ⊆ Cl(f⁻¹(F));

(7) for every x ∈ X, every closed set F including f(x) and every open U including x, there exists a nonempty semiopen set V such that V ⊆ U and V ⊆ Cl(f⁻¹(F)).

Proof. Follows from Theorem 3.1 and Lemma 1.18.

□

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