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Zumkeller labeling of some path related graphs

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Abstract
A positive integer n is said to be Zumkeller number if its positive factors can be partitioned into 2 disjoint parts with the equal sum, that is each part with sum $\sigma(n)/2$. Let $G=(V(G),E(G))$ be a graph. An one to one function f defined on $V(G)$ to a subset of natural numbers is termed as Zumkeller labeling of G if the induced function $f^*:E(G)\to\mathbb{N}$ defined as $f^*(xy)=f(x)f(y)$ assigns a Zumkeller number for all $xy\in E(G)$, $x,y\in V(G)$. A graph $G=(V(G),E(G))$ admits Zumkeller labeling is called a Zumkeller graph. In this manuscript, we investigate Zumkeller labeling for several classes of path graph.

Keywords
Graph labeling, Zumkeller number.

AMS Subject Classification
05C78.

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Contents

1. Introduction

The assignments of values typically represented by integers using several appropriate mathematical rules to the vertices and / or edges of the given graph is termed as graph labeling. Formally, in 1967 Alex Rosa [1] introduced the concept of graph labeling. There is an enormous literature regarding labeling of many familiar classes of graphs [8].

A natural number is said to be perfect if $\sigma(n)=2n$, where $\sigma(n)$ represents the sum of positive factors. A generalization of concept of perfect number, termed as Zumkeller number. It has been explored by in 2003. Zumkeller observed a sequence of natural numbers in which the positive factors of all number has been partitioned into two disjoint parts whose sums are equal, *i.e.*, each part with sum $\sigma(n)/2$. Formally Zumkeller numbers was proposed by Clark et. al. [7]. Later, Peng and Bhaskara Rao were studied in detail about Zumkeller number and half Zumkeller number [13]. B. J. Balamurugan et. al. introduced the concept of Zumkeller labeling in in the year 2013 [2]. The idea of Zumkeller labeling of some simple graphs has been previously reported [3, 4, 5, 6, 11].

In this article, we investigated the existence of Zumkeller labeling in path related graphs such as splitting graph of path $S'(P_n)$, total graph of path $T(P_n)$, shadow graph of path $D_2(P_n)$, middle graph of path $M(P_n)$ etc.

2. Preliminaries

Definition 2.1. *The graph which is gained from G by the addition of new node v' for every node v in G and join v' to all nodes of G are neighbours to v are the splitting graph $S'(G)$.*

Definition 2.2. *For a graph G, the total graph $T(G)$ has the vertex set $V(G)\cup E(G)$ and two vertices are adjacent in $T(G)$ whenever their corresponding elements are either incident or adjacent in G.*

Definition 2.3. *The shadow graph $D_2(G)$ the connected graph G is built by taking two copies of G namely G' and G'' join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.*

Definition 2.4. *Middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G)\cup E(G)$ and in which two vertices are adjacent whenever either one is a vertex of G and other is an edge incident with it or they are adjacent edges of G.*

Definition 2.5. *If a positive integer n can be subdivided into two disjoint subset of equal sum by the positive divisors of n, then n is called Zumkeller number.*

Theorem 2.6. [7] *If n is a Zumkeller number and p is a prime with $(n, p) = 1$, then np^l Zumkeller for any positive integer l.*

Theorem 2.7. [7] *For any prime $p \neq 2$and positive integer k with $p \leq 2^{k+1} - 1$ the number $2^k p$ is a Zumkeller number.*

Theorem 2.8. [10] *Let $n = 2^k p^l$ be a positive integer. Then n is a Zumkeller number if and only if $p \leq 2^{k+1} - 1$ and l is an odd number.*

Definition 2.9. [12] *Let $G = (V(G), E(G))$ be a graph. An injective map $f : V(G) \to \mathbb{N}$ is said to be Zumkeller labeling of the graph G, if the induced function $f^* : E(G) \to \mathbb{N}$ defined as $f^*(xy) = f(x)f(y)$ Zumkeller number for all $xy \in E(G)$, $x, y \in V(G)$.*

3. Main Results

Theorem 3.1. *Splitting graph of path $S'(P_n)$ a Zumkeller graph.*

Proof. The vertex set of Splitting graph of path $S'(P_n)$ is $V = \{v_i, v_i^{'} : 1 \leq i \leq n\}$ and the edge set of $S'(P_n)$ is $E = \{e_i = v_i v_{i+1}, e_i^{'} = v_i v_{i+1}^{'}, e_i{''} = v_i^{'} v_{i+1} : 1 \leq i \leq n-1\}$.
There are following cases:
Case 1 : n is odd
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(v_i) = \begin{cases} 2^{\frac{i+1}{2}} p & i \equiv 1(mod2) \\ 2^{\frac{i}{2}} q & i \equiv 0(mod2) \end{cases} \qquad 1 \leq i \leq n$$

$$f(v_i^{'}) = \begin{cases} 2^{\frac{n+i+2}{2}} p & i \equiv 1(mod2) \\ 2^{\frac{n+i+1}{2}} q & i \equiv 0(mod2) \end{cases} \qquad 1 \leq i \leq n$$

where p is an odd prime less than 10 and q is any prime other than p. Now we define an induced function f^* to f as follows
$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1}) \qquad 1 \leq i \leq n-1$
$f^*(e_i^{'}) = f^*(v_i v_{i+1}^{'}) = f(v_i) f(v_{i+1}^{'}) \qquad 1 \leq i \leq n-1$
$f^*(e_i^{''}) = f^*(v_i^{'} v_{i+1}) = f(v_i^{'}) f(v_{i+1}) \qquad 1 \leq i \leq n-1$
Then by definition of f we obtain

$$f^*(v_i v_{i+1}) = 2^{\frac{i+1}{2}} p.2^{\frac{i+1}{2}} q = 2^{i+1} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i v_{i+1}) = 2^{\frac{i}{2}} q.2^{\frac{i+2}{2}} p = 2^{i+1} pq \qquad i \equiv 0(mod2)$$
$$f^*(v_i v_{i+1}^{'}) = 2^{\frac{i+1}{2}} p.2^{\frac{n+i+2}{2}} q = 2^{\frac{n+2i+3}{2}} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i v_{i+1}^{'}) = 2^{\frac{i}{2}} q.2^{\frac{n+i+3}{2}} p = 2^{\frac{n+2i+3}{2}} pq \qquad i \equiv 0(mod2)$$
$$f^*(v_i^{'} v_{i+1}) = 2^{\frac{n+i+2}{2}} p.2^{\frac{i+1}{2}} q = 2^{\frac{n+2i+3}{2}} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i^{'} v_{i+1}) = 2^{\frac{n+i+1}{2}} q.2^{\frac{i+2}{2}} p = 2^{\frac{n+2i+3}{2}} pq \qquad i \equiv 0(mod2)$$

i.e.,
$$f^*(e_i) = 2^{i+1} pq \qquad 1 \leq i \leq n-1$$
$$f^*(e_i^{'}) = 2^{\frac{n+2i+3}{2}} pq \qquad 1 \leq i \leq n-1$$
$$f^*(e_i^{''}) = 2^{\frac{n+2i+3}{2}} pq \qquad 1 \leq i \leq n-1$$

Using Theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers.

Case 2 : n is even
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(v_i) = \begin{cases} 2^{\frac{i+1}{2}} p & i \equiv 1(mod2) \\ 2^{\frac{i}{2}} q & i \equiv 0(mod2) \end{cases} \qquad 1 \leq i \leq n$$

$$f(v_i^{'}) = \begin{cases} 2^{\frac{n+i+1}{2}} p & i \equiv 1(mod2) \\ 2^{\frac{n+i}{2}} q & i \equiv 0(mod2) \end{cases} \qquad 1 \leq i \leq n$$

where p is an odd prime less than 10 and q is any prime other than p. Now we define an induced function f^* to f as follows
$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1}) \qquad 1 \leq i \leq n-1$
$f^*(e_i^{'}) = f^*(v_i v_{i+1}^{'}) = f(v_i) f(v_{i+1}^{'}) \qquad 1 \leq i \leq n-1$
$f^*(e_i^{''}) = f^*(v_i^{'} v_{i+1}) = f(v_i^{'}) f(v_{i+1}) \qquad 1 \leq i \leq n-1$
Then by definition of f we obtain

$$f^*(v_i v_{i+1}) = 2^{\frac{i+1}{2}} p.2^{\frac{i+1}{2}} q = 2^{i+1} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i v_{i+1}) = 2^{\frac{i}{2}} q.2^{\frac{i+2}{2}} p = 2^{i+1} pq \qquad i \equiv 0(mod2)$$
$$f^*(v_i v_{i+1}^{'}) = 2^{\frac{i+1}{2}} p.2^{\frac{n+i+1}{2}} q = 2^{\frac{n+2i+2}{2}} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i v_{i+1}^{'}) = 2^{\frac{i}{2}} q.2^{\frac{n+i+2}{2}} p = 2^{\frac{n+2i+2}{2}} pq \qquad i \equiv 0(mod2)$$
$$f^*(v_i^{'} v_{i+1}) = 2^{\frac{n+i+1}{2}} p.2^{\frac{i+1}{2}} q = 2^{\frac{n+2i+2}{2}} pq \qquad i \equiv 1(mod2)$$
$$f^*(v_i^{'} v_{i+1}) = 2^{\frac{n+i}{2}} q.2^{\frac{i+2}{2}} p = 2^{\frac{n+2i+2}{2}} pq \qquad i \equiv 0(mod2)$$

i.e.,
$$f^*(e_i) = 2^{i+1} pq \qquad 1 \leq i \leq n-1$$
$$f^*(e_i^{'}) = 2^{\frac{n+2i+2}{2}} pq \qquad 1 \leq i \leq n-1$$
$$f^*(e_i^{''}) = 2^{\frac{n+2i+2}{2}} pq \qquad 1 \leq i \leq n-1$$

Using Theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers. By the above cases we get Spliting graph of path is a Zumkeller graph. □

Example 3.2.

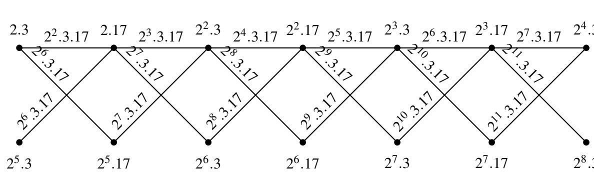

Figure 1. *$S'(P_7)$ is an Zumkeller graph with $p = 3$ and $q = 17$*

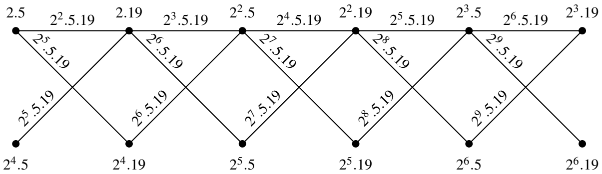

Figure 2. *$S'(P_6)$ is an Zumkeller graph with $p = 5$ and $q = 19$*

Theorem 3.3. *Total graph of path $T(P_n)$ is a Zumkeller graph.*

Proof. Let $V(T(P_n)) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n-1\}$ be the vertex set of the total graph of path $T(P_n)$. Let $E(T(P_n)) = \{v_iv_{i+1} : 1 \le i \le n-1\} \cup \{v_iu_i : 1 \le i \le n-1\} \cup \{v_{i+1}u_i : 1 \le i \le n-1\} \cup \{u_iu_{i+1} : 1 \le i \le n-2\}$.
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(v_i) = \begin{cases} 2^{\frac{i+1}{2}} & i \equiv 1(mod2) \\ 2^{\frac{i}{2}}p & i \equiv 0(mod2) \end{cases} \qquad 1 \le i \le n$$

$$f(v_i^{'}) = \begin{cases} 2^{\frac{i+1}{2}}q & i \equiv 1(mod2) \\ 2^{\frac{i}{2}}r & i \equiv 0(mod2) \end{cases} \qquad 1 \le i \le n-1$$

where p, q, r are distinct odd primes less than 10.
An induced function $f^* : E \to \mathbb{N}$ such that
$f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) \qquad 1 \le i \le n-1$
$f^*(v_iu_i) \quad = f(v_i)f(u_i) \qquad 1 \le i \le n-1$
$f^*(v_{i+1}u_i) = f(v_{i+1})f(u_i) \qquad 1 \le i \le n-1$
$f^*(u_iu_{i+1}) = f(u_i)f(u_{i+1}) \qquad 1 \le i \le n-2$
Now we can show that the labels on the edges of Total graph of path $T(P_n)$ Zumkeller numbers through the following cases.

$$f^*(v_iv_{i+1}) = 2^{\frac{i+1}{2}}.2^{\frac{i+1}{2}}p = 2^{i+1}p \qquad i \equiv 1(mod2)$$
$$f^*(v_iv_{i+1}) = 2^{\frac{i}{2}}p.2^{\frac{i+2}{2}} = 2^{i+1}p \qquad i \equiv 0(mod2)$$
$$f^*(v_iu_i) \quad = 2^{\frac{i+1}{2}}.2^{\frac{i+1}{2}}q = 2^{i+1}q \qquad i \equiv 1(mod2)$$
$$f^*(v_iu_i) \quad = 2^{\frac{i}{2}}p.2^{\frac{i}{2}}r = 2^{i}pr \qquad i \equiv 0(mod2)$$
$$f^*(v_{i+1}u_i) = 2^{\frac{i+1}{2}}p.2^{\frac{i+1}{2}}q = 2^{i+1}pq \qquad i \equiv 1(mod2)$$
$$f^*(v_{i+1}u_i) = 2^{\frac{i+2}{2}}.2^{\frac{i}{2}}r = 2^{i+1}r \qquad i \equiv 0(mod2)$$
$$f^*(u_iu_{i+1}) = 2^{\frac{i+1}{2}}q.2^{\frac{i+1}{2}}r = 2^{i+1}qr \qquad i \equiv 1(mod2)$$
$$f^*(u_iu_{i+1}) = 2^{\frac{i}{2}}r.2^{\frac{i+2}{2}}q = 2^{i+1}qr \qquad i \equiv 0(mod2)$$

Consuming theorem 2.6 and Theorem 2.7 we get all edge labels are Zumkeller numbers.
Therefore Total graph of path $T(P_n)$ is a Zumkeller graph. □

Example 3.4.

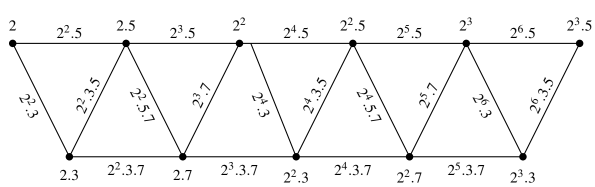

Figure 3. *$T(P_6)$ is an Zumkeller graph with $p = 5, q = 3$ and $r = 7$*

Theorem 3.5. *Shadow graph of path $D_2(P_n)$ admits Zumkeller labeling.*

Proof. Let $V(D_2(P_n)) = \{u_i, v_i : 1 \le i \le n\}$ be the vertex set of the shadow graph of path $D_2(P_n)$.
Let $E(D_2(P_n)) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1}, v_iu_{i+1} : 1 \le i \le n-1\}$ be the edge set of the shadow graph of path $D_2(P_n)$.
We have two cases:
case 1 : $n \equiv 1(mod2)$
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(u_i) = 2^{\frac{k}{2}}.p^{i} \qquad 1 \le i \le n$$
$$f(v_i) = 2^{\frac{k}{2}}.p^{n+i+1} \qquad 1 \le i \le n$$

where, p is an odd prime number and $p \le 2^{k+1} - 1$ and k is an even number and an induced function $f^* : E \to \mathbb{N}$ such that
$f^*(u_iu_{i+1}) = f(u_i)f(u_{i+1}) \qquad 1 \le i \le n-1$
$f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) \qquad 1 \le i \le n-1$
$f^*(u_iv_{i+1}) = f(u_i)f(v_{i+1}) \qquad 1 \le i \le n-1$
$f^*(v_iu_{i+1}) = f(v_i)f(u_{i+1}) \qquad 1 \le i \le n-1$
Now, we have to prove that the numbers on the edges are Zumkeller numbers.

$$f^*(u_iu_{i+1}) = 2^{\frac{k}{2}}p^{i}.2^{\frac{k}{2}}p^{i+1} = 2^{k}p^{2i+1}$$
$$f^*(v_iv_{i+1}) = 2^{\frac{k}{2}}p^{n+i+1}.2^{\frac{k}{2}}p^{n+i+2} = 2^{k}p^{2n+2i+3}$$
$$f^*(u_iv_{i+1}) = 2^{\frac{k}{2}}p^{i}.2^{\frac{k}{2}}p^{n+i+2} = 2^{k}p^{n+2i+2}$$
$$f^*(v_iu_{i+1}) = 2^{\frac{k}{2}}p^{n+i+1}.2^{\frac{k}{2}}p^{i+1} = 2^{k}p^{n+2i+2}$$

Using Theorem 2.8 all edge labels are Zumkeller numbers.

case 2 : $n \equiv 0(mod2)$
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(u_i) = 2^{\frac{k}{2}}.p^{i} \qquad 1 \le i \le n$$
$$f(v_i) = 2^{\frac{k}{2}}.p^{n+i} \qquad 1 \le i \le n$$

where, p is an odd prime number and $p \le 2^{k+1} - 1$ and k is an even number and an induced function $f^* : E \to \mathbb{N}$ such that
$f^*(u_iu_{i+1}) = f(u_i)f(u_{i+1}) \qquad 1 \le i \le n-1$
$f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) \qquad 1 \le i \le n-1$
$f^*(u_iv_{i+1}) = f(u_i)f(v_{i+1}) \qquad 1 \le i \le n-1$
$f^*(v_iu_{i+1}) = f(v_i)f(u_{i+1}) \qquad 1 \le i \le n-1$
Now, we have to prove that the numbers on the edges are Zumkeller numbers.

$$f^*(u_iu_{i+1}) = 2^{\frac{k}{2}}p^{i}.2^{\frac{k}{2}}p^{i+1} = 2^{k}p^{2i+1}$$
$$f^*(v_iv_{i+1}) = 2^{\frac{k}{2}}p^{n+i}.2^{\frac{k}{2}}p^{n+i+1} = 2^{k}p^{2n+2i+1}$$
$$f^*(u_iv_{i+1}) = 2^{\frac{k}{2}}p^{i}.2^{\frac{k}{2}}p^{n+i+1} = 2^{k}p^{n+2i+1}$$
$$f^*(v_iu_{i+1}) = 2^{\frac{k}{2}}p^{n+i}.2^{\frac{k}{2}}p^{i+1} = 2^{k}p^{n+2i+1}$$

Using Theorem 2.8 all edge labels are Zumkeller numbers. Thus Shadow graph of path $D_2(P_n)$ admits Zumkeller labeling. □

Example 3.6.

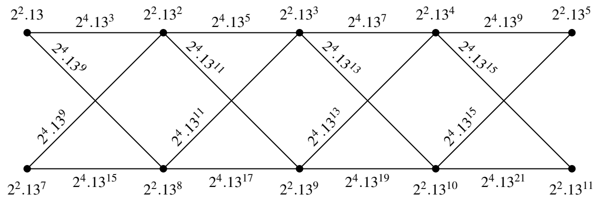

Figure 4. *$D_2(P_5)$ is an Zumkeller graph with $p = 13$ and $k = 4$.*

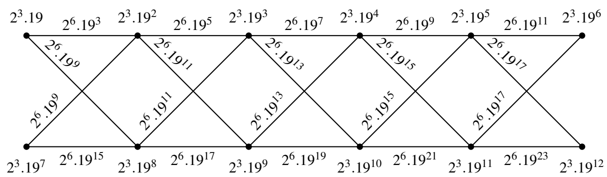

Figure 5. *$D_2(P_6)$ is an Zumkeller graph with $p = 19$ and $k = 6$.*

Theorem 3.7. *Middle graph of path $M(P_n)$ admits Zumkeller labeling.*

Proof. Let $V(M(P_n)) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n-1\}$ and $E(M(P_n)) = \{v_iu_i : 1 \le i \le n-1\} \cup \{v_iu_{i-1} : 2 \le i \le n\} \cup \{u_iu_{i+1} : 1 \le i \le n-2\}$ be the vertex set, edge set respectively of middle graph of path $M(P_n)$.
Define a one to one map $f : V \to \mathbb{N}$ such that

$$f(u_i) = r^i \qquad 1 \le i \le n-1$$
$$f(v_i) = \begin{cases} 2^{k+\frac{i-1}{2}}p & i \equiv 1(mod2) \\ 2^{k+\frac{i-2}{2}}q & i \equiv 0(mod2) \end{cases} \qquad 1 \le i \le n$$

.
where, p and q are distinct odd prime numbers and $p,q \le 2^{k+1}-1$ $(k \ge 2, k \in \mathbb{N})$ and r is any prime number number other than p and q.
An induced function $f^* : E \to \mathbb{N}$ such that
$f^*(v_iu_i) \quad = f(v_i)f(u_i) \qquad 1 \le i \le n-1$
$f^*(v_iu_{i-1}) = f(v_i)f(u_{i-1}) \qquad 2 \le i \le n$
$f^*(u_iu_{i+1}) = f(u_i)f(u_{i+1}) \qquad 1 \le i \le n-2$
Now we calculate the edge labels

$$f^*(v_iu_i) = \begin{cases} 2^{k+\frac{i-1}{2}}pr^i & i \equiv 1(mod2) \\ 2^{k+\frac{i-2}{2}}qr^i & i \equiv 0(mod2) \end{cases}$$
$$f^*(v_iu_{i-1}) = \begin{cases} 2^{k+\frac{i-2}{2}}pr^i & i \equiv 1(mod2) \\ 2^{k+\frac{i-3}{2}}qr^i & i \equiv 0(mod2) \end{cases}$$
$$f^*(u_iu_{i+1}) = 2^{2k+i-1}pq$$

Using Theorem 2.6 and Theorem 2.7 we get all the edge labels are Zumkeller numbers.
Thus the graph Middle graph of path $M(P_n)$ admits Zumkeller labeling. □

Example 3.8.

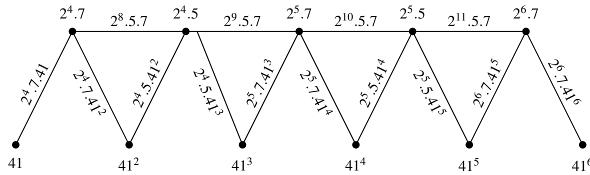

Figure 6. *$M(P_6)$ is an Zumkeller graph with $p = 7, q = 5, r = 41$ and $k = 4$*

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