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# Density of an $m$-Bipolar Fuzzy Graph

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**Abstract**

In this article, density, balanced, strictly balanced, complement of an $m$-bipolar fuzzy graph ($m$-BPFG), self-complementary between $m$-BPFG and its complement are defined and corresponding properties are studied.

**Keywords**

$m$-BPFG, Density of an m-BPFG, Balanced, Self complementary.

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## Contents



## 1. Introduction

Fuzzy sets are introduced for the parameters to solve problems related to vague and uncertain in real life situations are demonstrated by Zadeh [11] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy concept in 1994 by Zhang [12, 13]. This was further improved by Chen et al. [3] to m-polar fuzzy set theory.

Free body diagrams using set of nodes connected by lines representing pairs are good problem solving tools in non-deterministic real life situations. Thus, Kaufmann [7] was first to introduced the idea of fuzzy graph from Zadeh fuzzy relation. Rosenfeld [8] gave the idea of fuzzy vertex, fuzzy edges and fuzzy cycle etc. Akram [1] studied the some properties of bipolar fuzzy graphs. Later Rashmanlou et al. [10] studied the categorical properties of bipolar fuzzy graphs. Ghorai and Pal [4-6] introduced the concept of m-polar fuzzy graphs and studied some properties on it. Ramprasad et al. [9] introduced the product m-polar fuzzy line and intersection graphs. Bera and pal [2] introduced the concept of m-polar interval-valued fuzzy graph and studied the algebraic properties of density, regularity and irregularity etc. on m-PIVFG.

This paper attempts to develop theory to analyze parameters combining concepts from mpolar fuzzy graph and bipolar fuzzy graph as a unique effort. The resultant graph is turned mBPFG and studied properties on it.

## 2. Preliminaries

All the vertices and edges of an $m$-polar fuzzy graph have m components and those components are fixed. But these components may be bipolar. Using this idea, $m$-BPFG has been introduced.

Before defining m-bipolar fuzzy graph, we assume the following:

For a given set $V$, define an equivalence relation $\leftrightarrow$ on $V\times V-\{(k,k):k\in V\}$ as follows: $(k_1,l_1)\leftrightarrow(k_2,l_2)\Leftrightarrow$ either $(k_1,l_1)=(k_2,l_2)$ or $k_1=l_2, l_1=k_2$. The quotient set got in this way is denoted by $\overrightarrow{V^2}$.

**Definition 2.1.** *An m-bipolar fuzzy set (m-BPFS) S on V is defined by*

$$S(s)=\{\langle[p_1\circ\psi_S^p(s),p_1\circ\psi_S^n(s)],[p_2\circ\psi_S^p(s),p_2\circ\psi_S^n(s)],\cdots,[p_m\circ\psi_S^p(s),p_m\circ\psi_S^n(s)]\rangle\}$$

*for all $s\in V$ or shortly*

$$S(s)=\left\{\left\langle\left[p_j\circ\psi_S^p(s),p_j\circ\psi_S^n(s)\right]_{j=1}^m\right\rangle \mid s\in V\right\}$$

*where the functions $p_j\circ\psi_S^p:V\to[0,1]$ and $p_j\circ\psi_S^n:V\to[-1,0]$ denote the positive memberships and negative memberships of the element respectively.*

**Definition 2.2.** *Let S be an m-BPFS on a set V. An m-bipolar fuzzy relation on a set S is m-BPFS T of $V\times V, T(s,t)=$*

$\{\langle [p_1 \circ \psi_T^p(s,t), p_1 \circ \psi_T^n(s,t)], [p_2 \circ \psi_T^p(s,t), p_2 \circ \psi_T^n(s,t)], \cdots, [p_m \circ \psi_T^p(s,t), p_m \circ \psi_T^n(s,t)] \rangle\}$ *for all* $s,t \in V$ *or shortly*

$$T(s,t) = \left\{ \left\langle [p_j \circ \psi_T^p(s,t), p_j \circ \psi_T^n(s,t)]_{j=1}^m \right\rangle \mid s,t \in V \right\}$$

*such that* $p_j \circ \psi_T^p(s,t) \le \min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}$ $p_j \circ \psi_T^n(s,t) \ge \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$, *for every* $j = 1,2,\cdots,m$ *and* $s,t \in V$.

**Definition 2.3.** *An m-bipolar fuzzy graph (m-BPFG) of a graph* $G^* = (V,E)$ *is a pair* $G = (V,S,T)$ *where*

$$S = \left\langle [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \right\rangle,$$

$p_j \circ \psi_S^p : V \to [0,1]$ *and* $p_j \circ \psi_S^n : V \to [-1,0]$ *is an* $m-BPFS$ *on* $V$; *and* $T = \left\langle [p_j \circ \psi_T^p, p_j \circ \psi_T^n]_{j=1}^m \right\rangle$, $p_j \circ \psi_T^p : \vec{V}^2 \to [0,1]$ *and* $p_j \circ \psi_T^n : \vec{V}^2 \to [-1,0]$ *is an m- BPFS in* $\overrightarrow{V^2}$ *such that* $p_j \circ \psi_T^p(k,l) \le \min\{p_j \circ \psi_S^p(k), p_j \circ \psi_S^p(l)\}$, $p_j \circ \psi_T^n(k,l) \ge \max\{p_j \circ \psi_S^n(k), p_j \circ \psi_S^n(l)\}$ *for all* $(k,l) \in \vec{V}^2$, $j = 1,2,\cdots,m$ *and* $p_j \circ \psi_T^p(k,l) = p_j \circ \psi_T^n(k,l) = 0$ *for all* $(k,l) \in \overrightarrow{V^2} - E$.

**Definition 2.4.** *Let* $G = (V,S,T)$ *be an m-BPFG of a graph* $G^* = (V,E)$. *An m-BPFG* $N = (Q,C,D)$ *is said to be an m-bipolar fuzzy subgraph of G induced by Q if* $Q \subseteq V$ $C(s) = S(s)$ *for all* $s \in Q$ *and* $D(s,t) = T(s,t)$ *for all* $(s,t) \in \overrightarrow{Q^2}$.

**Definition 2.5.** *An m-BPFG* $G = (V,S,T)$ *of a graph* $G^* = (V,E)$ *is complete if for every* $s,t \in V$ *and* $j = 1,2,\cdots m$ *satisfying* $p_j \circ \psi_T^p(s,t) = \min\{p_j \circ \psi_s^p(s), p_j \circ \psi_s^p(t)\}$ $p_j \circ \psi_T^n(s,t) = \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$.

**Definition 2.6.** *An m-BPFG* $G = (V,S,T)$ *of a graph* $G^* = (V,E)$ *is strong if for every* $(s,t) \in E$ *and* $j = 1,2,\cdots,m$ *satisfying* $p_j \circ \psi_T^p(s,t) = \min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}$ $p_j \circ \psi_T^n(s,t) = \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$.

## 3. Density of an m-BPFG

In this section, complement, density, balanced, strictly balanced and self complementary of an m-BPFG are defined and studied some of its properties.

**Definition 3.1.** *Let* $G = (V,S,T)$ *be an m-BPFG of* $G^* = (V,E)$. *The complement of G is an m-BPFG* $\bar{G} = (V,\bar{S},\bar{T})$ *of* $\overline{G^*} = \left(V, \overline{V^2}\right)$ *such that* $\bar{S} = S$ *and* $\bar{T}$ *is defined by*

$$p_j \circ \psi_{\bar{T}}(s,t) = [p_j \circ \psi_{\bar{T}}^p(s,t), p_j \circ \psi_{\bar{T}}^n(s,t)]$$
$$p_j \circ \psi_{\bar{T}}^p(s,t) = \{p_j \circ \psi_S^p(s) \wedge p_j \circ \psi_S^p(t)\} - p_j \circ \psi_T^p(s,t)$$
$$p_j \circ \psi_{\bar{T}}^n(s,t) = \{p_j \circ \psi_S^n(s) \vee p_j \circ \psi_S^n(t)\} - p_j \circ \psi_T^n(s,t)$$

*for every* $(s,t) \in \vec{V}^2$ *and* $j = 1,2,\cdots m$.

**Definition 3.2.** *The density of an m-BPFG* $G = (V,S,T)$ *of* $G^* = (V,E)$ *is*

$$\chi(G) = \left\langle [p_j \circ \chi^p(G), p_j \circ \chi^n(G)]_{j=1}^m \right\rangle = \left\langle \left[ \frac{2\sum_{s,t \in V} p_j \circ \psi_T^p(s,t)}{\sum_{(s,t)\in E} \{p_j \circ \psi_S^p(s) \wedge p_j \circ \psi_S^p(t)\}}, \frac{2\sum_{s,t \in V} p_j \circ \psi_T^n(s,t)}{\sum_{(s,t)\in E} \{p_j \circ \psi_S^n(s) \vee p_j \circ \psi_S^n(t)\}} \right]_{j=1}^m \right\rangle.$$

**Definition 3.3.** *An m-BPFG* $G = (V,S,T)$ *of* $G^* = (V,E)$ *is balanced if* $\chi(Q) \le \chi(G)$ *for all non-empty subgraphs Q of G. i.e. for all* $j = 1,2,\cdots m$, *we have* $p_j \circ \chi^p(Q) \le p_j \circ \chi^p(G)$ *and* $p_j \circ \chi^n(Q) \le p_j \circ \chi^n(G)$.

**Definition 3.4.** *An m-BPFG* $G = (V,S,T)$ *of* $G^* = (V,E)$ *is strictly balanced if* $\chi(G) = \chi(Q)$ *for all non-empty subgraphs Q of G.*

**Example 3.5.** *Consider a 2-BPFG* $G = (V,S,T)$ *of* $G^* = (V,E)$ *as shown in the Figure 1.*

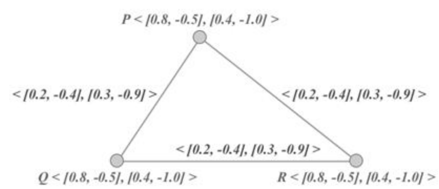


**Figure 1.** Strictly balanced 2 -BPFG $G$

For this 2 -BPFG, we have $\chi(G) = \langle [0.5, 1.6], [1.5, 1.8] \rangle$ and the densities of a non-empty subgraphs of $G$ are $\chi(H_1 = (P,Q)) = \chi(H_2 = (Q,R)) = \chi(H_3 = (R,P)) = \langle [0.5, 1.6], [1.5, 1.8] \rangle$. Therefore $G$ is a strictly balanced 2 -BPFG.

**Definition 3.6.** *Let* $G_1 = (V_1,S_1,T_1)$ *and* $G_2 = (V_2,S_2,T_2)$ *be two m-BPFGs on crisp graphs* $G_1^* = (V_1,E_1)$ *and* $G_2^* = (V_2,E_2)$ *respectively. An isomorphism between* $G_1$ *and* $G_2$ *is a bijective mapping* $\phi$ *from* $V_1$ *to* $V_2$ *such that for each* $j = 1,2,\cdots m$ $p_j \circ \psi_{S_1}^p(s) = p_j \circ \psi_{S_2}^p(\phi(s))$, $p_j \circ \psi_{S_1}^n(s) = p_j \circ \psi_{S_2}^n(\phi(s)) \forall s \in V_1$.

$$p_j \circ \psi_{T_1}^p(s,t) = p_j \circ \psi_{T_2}^p(\phi(s),\phi(t)), p_j \circ \psi_{T_1}^n(s,t) = p_j \circ \psi_{T_2}^n(\phi(s),\phi(t))$$

$\forall (s,t) \in \overrightarrow{V_1^2}$

**Definition 3.7.** *An m-BPFG is said to be self-complementary if* $G \cong \bar{G}$.

**Theorem 3.8.** *Every complete m-BPFG is balanced.*

*Proof.* Let $G=(V,S,T)$ be a complete $m$ -BPFG and $Q$ be a non empty subgraph of $G$. Then for every $j=1,2,\cdots m$

$$\begin{aligned}
&[p_j\circ\chi^p(G), p_j\circ\chi^n(G)] \\
&=\left[\frac{2\sum_{s,t\in V} p_j\circ\psi_T^p(s,t)}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}},\right. \\
&\qquad\left.\frac{2\sum_{s,t\in V} p_j\circ\psi_T^n(s,t)}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}\right] \\
&=\left[\frac{2\sum_{s,t\in V}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}},\right. \\
&\qquad\left.\frac{2\sum_{s,t\in V}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}\right]=[2,2]
\end{aligned}$$

$$\begin{aligned}
&[p_j\circ\chi^p(Q), p_j\circ\chi^n(Q)] \\
&=\left[\frac{2\sum_{s,t\in V(Q)} p_j\circ\psi_T^p(s,t)}{\sum_{(s,t)\in E(Q)}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}},\right. \\
&\qquad\left.\frac{2\sum_{s,t\in V(Q)} p_j\circ\psi_T^n(s,t)}{\sum_{(s,t)\in E(Q)}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}\right] \\
&\le\left[\frac{2\sum_{s,t\in V(Q)}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}{\sum_{(s,t)\in E(Q)}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}},\right. \\
&\qquad\left.\frac{2\sum_{s,t\in V(Q)}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}{\sum_{(s,t)\in E(Q)}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}\right]=[2,2]
\end{aligned}$$

(where $V(Q)$ and $E(Q)$ represents the vertex set and edge set of $Q$ ). Thus $G=(V,S,T)$ is a balanced m-BPFG. □

**Corollary 3.9.** *Every strong m-BPFG is balanced.*

**Theorem 3.10.** *The complement of a strictly balanced m-BPFG is strictly balanced.*

*Proof.* Follows from the definition. □

**Theorem 3.11.** *Let $G=(V,S,T)$ be a strictly balanced m-BPFG and let $\bar{G}=(V,\bar{S},\bar{T})$ be its complement. Then $\chi(G)+\chi(\bar{G})=\langle[2,2],[2,2],\cdots,[2,2]\rangle$*

*Proof.* Let $Q$ be any non-empty subgraph of $G$. Since $G$ is strictly balanced $m$ -BPFG then $\chi(Q)=\chi(G)$ for every $Q\subseteq G$. Now, for all $(s,t)\in\overline{V^2}$ and $j=1,2,\cdots,m$ we have

(i) $p_j\circ\psi_{\bar{T}}^p(s,t)=\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}-p_j\circ\psi_T^p(s,t)$

(ii) $p_j\circ\psi_{\bar{T}}^n(s,t)=\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}-p_j\circ\psi_T^n(s,t)$

Dividing (i) by $\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}$ on both sides, we get

$$\frac{p_j\circ\psi_{\bar{T}}^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}=1-\frac{p_j\circ\psi_T^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}$$

$$\sum_{s,t\in V}\frac{p_j\circ\psi_{\bar{T}}^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}=1-\sum_{s,t\in V}\frac{p_j\circ\psi_T^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}$$

$$2\sum_{s,t\in V}\frac{p_j\circ\psi_{\bar{T}}^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}=2-2\sum_{s,t\in V}\frac{p_j\circ\psi_T^p(s,t)}{\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}}$$

i.e. $p_j\circ\chi^p(\bar{G})=2-p_j\circ\chi^p(G)$. Similarly, $p_j\circ\chi^n(\bar{G})=2-p_j\circ\chi^n(G)$. Therefore $\chi(G)+\chi(\bar{G})=\langle[2,2],[2,2],\cdots,[2,2]\rangle$. □

**Corollary 3.12.** *Every balanced m-BPFG is need not be complete.*

**Theorem 3.13.** *Let $G=(V,S,T)$ be a strong m-BPFG graph, then $\bar{G}$ is balanced.*

*Proof.* Suppose that $G$ is a strong m-BPFG. Then for all $(s,t)\in E$ and $j=1,2,\cdots,m$. We have $p_j\circ\psi_T^p(s,t)=\min\left\{p_j\circ\psi_S^p(s),p_j\circ\psi_S^p(t)\right\}, p_j\circ\psi_T^n(s,t)=\max\left\{p_j\circ\psi_S^n(s),p_j\circ\psi_S^n(t)\right\}$.

From $\bar{G}$, for all $(s,t)\in E$ and $j=1,2,\cdots,m$ we have

$$\begin{aligned}
p_j\circ\psi_{\bar{T}}^p(s,t)&=\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}-p_j\circ\psi_T^p(s,t)\\
p_j\circ\psi_{\bar{T}}^n(s,t)&=\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}-p_j\circ\psi_T^n(s,t).
\end{aligned}$$

As $G$ is strong, we get

$$\left[p_j\circ\psi_{\bar{T}}^p(s,t),p_j\circ\psi_{\bar{T}}^n(s,t)\right]=[0,0],$$

$\forall(s,t)\in E$ and $p_j\circ\psi_{\bar{T}}^p(s,t)=\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}$ and $p_j\circ\psi_{\bar{T}}^n(s,t)=\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}$ for all $\forall(s,t)\in\bar{E}$. Hence, $\bar{G}$ is a strong $m-BPFG$ and it implies that $\bar{G}$ is balanced. □

**Theorem 3.14.** *Let $G=(V,S,T)$ be a self complementary m-BPFG of $G^*=(V,E)$. Then for all $(s,t)\in E$ and $j=1,2,\cdots,m$.*
*We have $\sum_{s\neq t}p_j\circ\psi_T^p(s,t)=\frac{1}{2}\sum_{s\neq 1}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}$ $\sum_{s\neq l}p_j\circ\psi_T^n(s,t)=\frac{1}{2}\sum_{s\neq t}\left(p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right)$.*

*Proof.* Suppose that $G=(V,S,T)$ is a self complementary m-BPFG of $G^*$. Then there exists an isomorphism $\phi$ from $G$ to $\bar{G}$ such that $p_j\circ\psi_S^p(s)=p_j\circ\psi_{\bar{s}}^p(\phi(s)), p_j\circ\psi_S^n(s)=p_j\circ\psi_{\bar{s}}^n(\phi(s))\ \forall s\in V$ and $p_j\circ\psi_T^p(s,t)=p_j\circ\psi_{\bar{T}}^p(\phi(s),\phi(t)), p_j\circ\psi_T^n(s,t)=p_j\circ\psi_{\bar{T}}^n(\phi(s),\phi(t))\forall(s,t)\in\vec{V}^2$. Let $(s,t)\in\vec{V}^2$. Then for $j=1,2,\cdots,m$, we have

$$\begin{aligned}
p_j\circ\psi_{\bar{T}}^p(\phi(s),\phi(t))&=\left\{p_j\circ\psi_S^p(\phi(s))\wedge p_j\circ\psi_S^p(\phi(t))\right\}\\
&\quad-p_j\circ\psi_T^p(\phi(s),\phi(t))\\
p_j\circ\psi_{\bar{T}}^n(\phi(s),\phi(t))&=\left\{p_j\circ\psi_S^n(\phi(s))\vee p_j\circ\psi_S^n(\phi(t))\right\}\\
&\quad-p_j\circ\psi_T^n(\phi(s),\phi(t))\\
\text{i.e. } p_j\circ\psi_T^p(s,t)&=\left\{p_j\circ\psi_S^p(\phi(s))\wedge p_j\circ\psi_S^p(\phi(t))\right\}\\
&\quad-p_j\circ\psi_T^p(\phi(s),\phi(t))\\
p_j\circ\psi_T^n(s,t)&=\left\{p_j\circ\psi_S^n(\phi(s))\vee p_j\circ\psi_S^n(\phi(t))\right\}\\
&\quad-p_j\circ\psi_T^n(\phi(s),\phi(t))
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{s\neq t}p_j\circ\psi_T^p(\phi(s),\phi(t))+\sum_{s\neq t}p_j\circ\psi_T^p(s,t)\\
&=\sum_{s\neq t}\left\{p_j\circ\psi_S^p(\phi(s))\wedge p_j\circ\psi_S^p(\phi(t))\right\}\\
&=\sum_{s\neq t}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}
\end{aligned}$$

That is,

$$2\sum_{s\neq t} p_j\circ\psi_T^p(s,t)=\sum_{s\neq t}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\},$$
$$\sum_{s\neq t} p_j\circ\psi_T^p(s,t)=\frac{1}{2}\sum_{s\neq t}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}$$

Similarly

$$\sum_{s\neq t} p_j\circ\psi_T^n(s,t)=\frac{1}{2}\sum_{s\neq l}\left(p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right).$$

□

**Theorem 3.15.** *Let $G_1=(V_1,S_1,T_1)$ and $G_2=(V_2,S_2,T_2)$ be two isomorphic m-BPFGs. Then if $G_2$ is balanced then $G_1$ is balanced.*

*Proof.* Since $G_1$ and $G_2$ are isomorphic, therefore there exists a bijective mapping $\phi: V_1\to V_2$ such that $p_j\circ\psi_{S_1}^p(s)=p_j\circ\psi_{S_2}^p(\phi(s)), p_j\circ\psi_{S_1}^n(s)=p_j\circ\psi_{S_2}^n(\phi(s))\forall s\in V_1$ and

$$p_j\circ\psi_{T_1}^p(s,t)=p_j\circ\psi_{T_2}^p(\phi(s),\phi(t)), p_j\circ\psi_{T_1}^n(s,t)=p_j\circ\psi_{T_2}^n(\phi(s),\phi(t))$$

$\forall(s,t)\in\overrightarrow{V_1^2},\forall j=1,2,\cdots,m$. Then,
$\sum_{s\in V_1}p_j\circ\psi_{S_1}^p(s)=\sum_{\phi(s)\in V_2}p_j\circ\psi_{S_2}^p(\phi(s)),\sum_{s\in V_1}p_j\circ\psi_{S_1}^n(s)=\sum_{\phi(s)\in V_2}p_j\circ\psi_{S_2}^n(\phi(s))$ and

$$\sum_{s,t\in V_1}p_j\circ\psi_{T_1}^p(s,t)=\sum_{\phi(s),\phi(t)\in V_2}p_j\circ\psi_{T_2}^p(\phi(s),\phi(t))$$
$$\sum_{s,t\in V_1}p_j\circ\psi_{T_1}^n(s,t)=\sum_{\phi(s),\phi(t)\in V_2}p_j\circ\psi_{T_2}^n(\phi(s),\phi(t))$$

Let $N_1$ and $N_2$ be two non-empty subgraphs of $G_1$ and $G_2$ respectively. Then, $p_j\circ\psi_{S_1}^p(s)=p_j\circ\psi_{S_2}^p(\phi(s)),p_j\circ\psi_{S_1}^n(s)=p_j\circ\psi_{S_2}^n(\phi(s))$ and $p_j\circ\psi_{T_1}^p(s,t)=p_j\circ\psi_{T_2}^p(\phi(s),\phi(t)),p_j\circ\psi_{T_1}^n(s,t)=p_j\circ\psi_{T_2}^n(\phi(s),\phi(t)),\forall s,t\in V_1(N_1),\forall j=1,2,\cdots,m$. Here, $V_1(N_1)$ are the vertices of $N_1$. Since $G_2$ is balanced, we have for $j=1,2,\cdots,m$ $p_j\circ\chi(N_2)\leq p_j\circ\chi(G_2)$.

$$2\sum_{(s,t)\in E_2(N_2)}\frac{p_j\circ\psi_{T_2}^p(s,t)}{p_j\circ\psi_{S_2}^p(s)\wedge p_j\circ\psi_{S_2}^p(t)}\leq 2\sum_{(s,t)\in E_2}\frac{p_j\circ\psi_{T_2}^p(s,t)}{p_j\circ\psi_{S_2}^p(s)\wedge p_j\circ\psi_{S_2}^p(t)}$$
$$2\sum_{(s,t)\in E_1(N_1)}\frac{p_j\circ\psi_{T_1}^p(s,t)}{p_j\circ\psi_{S_1}^p(s)\wedge p_j\circ\psi_{S_1}^p(t)}\leq 2\sum_{(s,t)\in E_1}\frac{p_j\circ\psi_{T_1}^p(s,t)}{p_j\circ\psi_{S_1}^p(s)\wedge p_j\circ\psi_{S_1}^p(t)}$$

Similarly,

$$2\sum_{(s,t)\in E_1(N_1)}\frac{p_j\circ\psi_{T_1}^n(s,t)}{p_j\circ\psi_{S_1}^n(s)\wedge p_j\circ\psi_{S_1}^n(t)}\leq 2\sum_{(s,t)\in E_1}\frac{p_j\circ\psi_{T_1}^n(s,t)}{p_j\circ\psi_{S_1}^n(s)\wedge p_j\circ\psi_{S_1}^n(t)}$$

i.e. $p_j\circ\chi(N_1)\leq p_j\circ\chi(G_1)$, i.e. $G_1$ is balanced. □

**Theorem 3.16.** *Let $G=(V,S,T)$ be a m-BPFG of $G^*=(V,E)$. If*

$$p_j\circ\psi_T^p(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\},$$
$$p_j\circ\psi_T^n(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\},$$

*for all $(s,t)\in\overrightarrow{V^2}$, $j=1,2,\cdots,m$, Then G is self- complementary.*

*Proof.* Suppose that $G=(V,S,T)$ is *a* m-BPFG of $G^*=(V,E)$ satisfying

$$p_j\circ\psi_T^p(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\},$$
$$p_j\circ\psi_T^n(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}$$

for all $(s,t)\in\overrightarrow{V^2}, j=1,2,\cdots,m$, then the identity mapping $I:V\to V$ is an isomorphism from $G$ to $\bar{G}$. Clearly, $p_j\circ\psi_S^p(s)=p_j\circ\psi_{\bar{S}}^p(I(s)),p_j\circ\psi_S^n(s)=p_j\circ\psi_{\bar{S}}^n(I(s))\forall s\in V$ and we have for all $j=1,2,\cdots,m$ and $(s,t)\in\overrightarrow{V^2}$.

$$\begin{aligned}p_j\circ\psi_{\bar{T}}^p(I(s),I(t))&=p_j\circ\psi_{\bar{T}}^p(s,t)\\&=\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}-p_j\circ\psi_T^p(s,t)\\&=\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}-\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}\\&=\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}=p_j\circ\psi_T^p(s,t)\end{aligned}$$

Similarly, $p_j\circ\psi_{\bar{T}}^n(I(s),I(t))=p_j\circ\psi_T^n(s,t)$
i.e. $p_j\circ\psi_{\bar{T}}^p(I(s),I(t))=p_j\circ\psi_T^p(s,t),p_j\circ\psi_{\bar{T}}^n(I(s),I(t))=p_j\circ\psi_T^n(s,t)$ for all $(s,t)\in\overrightarrow{V^2}, j=1,2,\cdots,m$.
Therefore $G\cong\bar{G}$. i.e. $G$ is self-complementary. □

**Theorem 3.17.** *Let $G=(V,S,T)$ be an m-BPFG such that for each $j=1,2,\cdots,m$ and $(s,t)\in\vec{V}^2$,*

$$p_j\circ\psi_T^p(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}$$
$$p_j\circ\psi_T^n(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}.$$

*Then $\chi(G)=\langle[1,1],[1,1],\cdots,[1,1]\rangle$.*

*Proof. Let* $G=(V,S,T)$ be an m-BPFG such that for each $j=1,2,\cdots,m$ and $(s,t)\in\overrightarrow{V^2}$,

$$p_j\circ\psi_T^p(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\},$$
$$p_j\circ\psi_T^n(s,t)=\frac{1}{2}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}$$

Therefore,

$$\left[\frac{2\sum_{s,t\in V} p_j\circ\psi_T^p(s,t)}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^p(s)\wedge p_j\circ\psi_S^p(t)\right\}},\right.$$
$$\left.\frac{2\sum_{s,t\in V} p_j\circ\psi_T^n(s,t)}{\sum_{(s,t)\in E}\left\{p_j\circ\psi_S^n(s)\vee p_j\circ\psi_S^n(t)\right\}}\right]=[1,1]$$

it follows that $\chi(G)=\langle[1,1],[1,1],\cdots,[1,1]\rangle$ □

## 4. Direct product of two m-bipolar fuzzy graphs

**Definition 4.1.** *Let $G_1=(V_1,S_1,T_1)$ and $G_2=(V_2,S_2,T_2)$ be two m-BPFGs of $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$ respectively such that $V_1\cap V_2=\phi$. Then the direct product of $G_1$ and $G_2$ is defined to be the m -BPFG $G_1\cap G_2=(S_1\cap S_2,T_1\cap T_2)$ of the graph $G^*=(V_1\times V_2,E)$ where,*

$$E=\{((s_1,t_1),(s_2,t_2))\mid(s_1,s_2)\in E_1,(t_1,t_2)\in E_2\}\subseteq\left(\overrightarrow{V_1\times V_2}\right)^2$$

*and for each $j=1,2,\cdots,m$*

*(i)* $p_j\circ\psi^p_{(s_1\cap s_2)}(s,t)=\min\left\{p_j\circ\psi^p_{S_1}(s),p_j\circ\psi^p_{S_2}(t)\right\}$
$p_j\circ\psi^n_{(S_1\cap s_2)}(s,t)=\max\left\{p_j\circ\psi^n_{S_1}(s),p_j\circ\psi^n_{S_2}(t)\right\}$
*for all* $(s,t)\in V_1\times V_2$

*(ii)* $p_j\circ\psi^p_{(T_1\cap T_2)}((s_1,t_1),(s_2,t_2))$
$=\min\left\{p_j\circ\psi^p_{T_1}(s_1,s_2),p_j\circ\psi^p_{T_2}(t_1,t_2)\right\}$
$p_j\circ\psi^n_{(T_,\cap T_2)}((s_1,t_1),(s_2,t_2))$
$=\max\left\{p_j\circ\psi^n_{T_1}(s_1,s_2),p_j\circ\psi^n_{T_2}(t_1,t_2)\right\}$
*for all* $(s_1,s_2)\in E_1,(t_1,t_2)\in E_2$

*(iii)* $p_j\circ\psi^p_{(T_1\cap T_2)}((p,q),(r,s))=0,$
$p_j\circ\psi^n_{(T_1\cap T_2)}((p,q),(r,s))=0$
*for all* $((p,q),(r,s))\in\left(\left(\overrightarrow{V_1\times V_2}\right)^2-E\right)$

**Theorem 4.2.** *The direct product $G_1\cap G_2$ of two m - BPFGs $G_1$ and $G_2$ is also an m - BPFG.*

*Proof.* Let $((s_1,t_1),(s_2,t_2))\in E$. Then $(s_1,s_2)\in E_1$ and $(t_1,t_2)\in E_2$. Hence for each $j=1,2,\cdots,m$; we have

$$\begin{aligned}&p_j\circ\psi^p_{(T_1\cap T_2)}((s_1,t_1),(s_2,t_2))\\&=p_j\circ\psi^p_{T_1}(s_1,s_2)\wedge p_j\circ\psi^p_{T_2}(t_1,t_2)\\&\le p_j\circ\psi^p_{S_1}(s_1)\wedge p_j\circ\psi^p_{S_1}(s_2)\wedge p_j\circ\psi^p_{S_2}(t_1)\wedge p_j\circ\psi^p_{S_2}(t_2)\\&=p_j\circ\psi^p_{(S_1\cap S_2)}(s_1,t_1)\wedge p_j\circ\psi^p_{(S_1\cap s_2)}(s_2,t_2)\end{aligned}$$

Also, for all $((p,q),(r,s))\in\left(\left(V_1\overrightarrow{\times}V_2\right)^2-E\right)$, $j=1,2,\cdots,m$.

$$\begin{aligned}&p_j\circ\psi^p_{(T_1\cap T_2)}((p,q),(r,s))\\&=0\le p_j\circ\psi^p_{(S_1\cap S_2)}(p,r)\wedge p_j\circ\psi^p_{(S_1\cap S_2)}(q,s).\end{aligned}$$

Similarly, $p_j\circ\psi^n_{(T_1\cap T_2)}((s_1,t_1),(s_2,t_2))\ge p_j\circ\psi^n_{(S_1\cap S_2)}(s_1,t_1)\vee p_j\circ\psi^n_{(S_1\cap S_2)}(s_2,t_2)$. This shows that, $G_1\cap G_2$ is an m-BPFG. □

**Theorem 4.3.** *Let $G_1=(V_1,S_1,T_1)$ and $G_2=(V_2,S_2,T_2)$ be two $m-$ BPFGs of $G_1^*=(V_1,E_1)$ and $G_2{}^*=(V_2,E_2)$ respectively. Then $\chi(G_1)=\chi(G_2)=\chi(G_1\cap G_2)$ if and only if $\chi(G_1)\le\chi(G_1\cap G_2)$ and $\chi(G_2)\le\chi(G_1\cap G_2)$*

*Proof.* Let $\chi(G_1)\le\chi(G_1\cap G_2)$ and $\chi(G_2)\le\chi(G_1\cap G_2)$. Then for $j=1,2,\cdots,m$

$$\begin{aligned}&p_j\circ\chi^p(G_1)\\&=\frac{2\sum_{s_1,s_2\in V_1}p_j\circ\psi^p_{T_1}(s_1,s_2)}{\sum_{s_1,s_2\in V_1}p_j\circ\psi^p_{S_1}(s_1)\wedge p_j\circ\psi^p_{S_1}(s_2)}\\&\ge\frac{2\sum_{\substack{s_1,s_2\in V_1\\t_1,t_2\in V_2}}p_j\circ\psi^p_{T_1}(s_1,s_2)\wedge p_j\circ\psi^p_{S_2}(t_1)\wedge p_j\circ\psi^p_{S_2}(t_2)}{\sum_{\substack{s_1,s_2\in V_1\\t_1,t_2\in V_2}}p_j\circ\psi^p_{S_1}(s_1)\wedge p_j\circ\psi^p_{S_1}(s_2)\wedge p_j\circ\psi^p_{S_2}(t_1)\wedge p_j\circ\psi^p_{S_2}(t_2)}\\&=\frac{2\sum_{\substack{s_1,s_2\in V_1\\t_1,t_1\in V_2}}p_j\circ\psi^p_{T_1}(s_1,s_2)\wedge p_j\circ\psi^p_{T_2}(t_1,t_2)}{\sum_{\substack{s_1,s_2\in V_1\\t_1,t_2\in V_2}}p_j\circ\psi^p_{S_1}(s_1)\wedge p_j\circ\psi^p_{S_1}(s_2)\wedge p_j\circ\psi^p_{S_2}(t_1)\wedge p_j\circ\psi^p_{S_2}(t_2)}\\&=\frac{2\sum_{\substack{s_1,s_2\in V_1\\t_1,t_2\in V_2}}p_j\circ\psi^p_{(T_1\cap T_2)}((s_1,t_1),(s_2,t_2))}{\sum_{\substack{s_1,s_2\in V_1\\t_1,t_2\in V_2}}p_j\circ\psi^p_{(S_1\cap S_2)}(s_1,t_1)\wedge p_j\circ\psi^p_{(S_1\cap S_2)}(s_2,t_2)}\\&=p_j\circ\chi^p(G_1\cap G_2).\end{aligned}$$

Similarly, $p_j\circ\chi^n(G_1)\ge p_j\circ\chi^n(G_1\cap G_2)$ i.e. $\chi(G_1)\ge\chi(G_1\cap G_2)$.
Similarly, $\chi(G_2)\ge\chi(G_1\cap G_2)$. Therefore, $\chi(G_1)=\chi(G_2)=\chi(G_1\cap G_2)$. □

**Theorem 4.4.** *Let $G_1=(V_1,S_1,T_1)$ and $G_2=(V_2,S_2,T_2)$ be two balanced $m-$ BPFGs. Then $G_1\cap G_2$ is balanced if and only if $\chi(G_1)=\chi(G_2)=\chi(G_1\cap G_2)$*

*Proof.* Let $G_1\cap G_2$ be balanced. Then $\chi(G_1)\le\chi(G_1\cap G_2)$ and $\chi(G_2)\le\chi(G_1\cap G_2)$. Hence, by Theorem 4.3, we have $\chi(G_1)=\chi(G_2)=\chi(G_1\cap G_2)$.

Conversely, assume that $\chi(G_1)=\chi(G_2)=\chi(G_1\cap G_2)$ and $N$ is a non-empty subgraph of $G_1\cap G_2$. Then there exist two subgraphs $N_1$ and $N_2$ of $G_1$ and $G_2$ respectively. Let

$$\chi(G_1)=\chi(G_2)=\left\langle\left[\frac{l_j^p}{k_j^p},\frac{l_j^n}{k_j^n}\right]_{j=1}^m\right\rangle,$$
$$\chi(N_1)=\left\langle\left[\frac{c_j^p}{d_j^p},\frac{c_j^n}{d_j^n}\right]_{j=1}^m\right\rangle$$
$$\chi(N_2)=\left\langle\left[\frac{e_j^p}{f_j^p},\frac{e_j^n}{f_j^n}\right]_{j=1}^m\right\rangle$$

for $j=1,2,\cdots,m,l_j^p,l_j^n,k_j^p,k_j^n,c_j^p,c_j^n,d_j^p,d_j^n,e_j^p,e_j^n,f_j^p,f_j^n\in R$.

Since $G_1$ and $G_2$ are balanced and

$$\chi(G_1)=\chi(G_2)=\left\langle\left[\frac{l_j^p}{k_j^p},\frac{l_j^n}{k_j^n}\right]_{j=1}^m\right\rangle,$$

where

$$0 \leq \left\langle \left[ \frac{l_j^p}{k_j^p}, \frac{l_j^n}{k_j^n} \right]_{j=1}^{m} \right\rangle \leq \langle [2,2],[2,2],\cdots,[2,2] \rangle,$$

$$\chi(N_1) = \left\langle \left[ \frac{c_j^p}{d_j^p}, \frac{c_j^n}{d_j^n} \right]_{j=1}^{m} \right\rangle \leq \left\langle \left[ \frac{l_j^p}{k_j^p}, \frac{l_j^n}{k_j^n} \right]_{j=1}^{m} \right\rangle$$

$$\chi(N_2) = \left\langle \left[ \frac{e_j^p}{f_j^p}, \frac{e_j^n}{f_j^n} \right]_{j=1}^{m} \right\rangle \leq \left\langle \left[ \frac{l_j^p}{k_j^p}, \frac{l_j^n}{k_j^n} \right]_{j=1}^{m} \right\rangle$$

Thus $c_j^p k_j^p + e_j^p k_j^p \leq d_j^p l_j^p + f_j^p l_j^p, c_j^n k_j^n + e_j^n k_j^n \leq d_j^n l_j^n + f_j^n l_j^n$ for all $j = 1,2,\cdots,m$. Hence,

$$\chi(N) \leq \left\langle \left[ \frac{c_j^p + e_j^p}{d_j^p + f_j^p}, \frac{c_j^n + e_j^n}{d_j^n + f_j^n} \right]_{j=1}^{m} \right\rangle \leq \left\langle \left[ \frac{l_j^p}{k_j^p}, \frac{l_j^n}{k_j^n} \right]_{j=1}^{m} \right\rangle = \chi(G_1 \cap G_2).$$

Thus, $\chi(N) \leq \chi(G_1 \cap G_2)$ for any subgraph $N$ of $G_1 \cap G_2$. Therefore, $G_1 \cap G_2$ is balanced. □

## 5. Conclusion

In this article, density and balanced m-BPFGs are defined. We studied the properties on selfcomplementary and density of an m-BPFG. We will extend our work to study the properties of morphism between two m-BPFGs and m- bipolar fuzzy line and intersection graphs.

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