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Graceful distance labeling for some particular graphs

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Abstract

In this paper, we define a new type of labeling for graphs which we call graceful distance labeling (GDL). An injective mapping f from the vertex set $V(G)$ into the set of non-negative integers such that the absolute difference of labels of vertices u and v is greater than or equal to distance between them i.e. $|f(u)-f(v)| \geq d(u,v)$ where $d(u,v)$ denotes the distance between the vertices u and v in G. The graceful distance labeling number (GDLN), $\lambda_d(G)$ of G is the minimum k where G has a graceful distance labeling f with k being the absolute difference between the largest and smallest image points of f i.e. $\lambda_d(G) = \min k$, where $k = \max|f(u)-f(v)|$. In this paper, we find the values of k for different graphs.

Keywords

GDLN, hairy cycle, corona of graphs, double graphs.

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Contents

1. Introduction

Most of the labeling techniques trace their origin to a paper of Rosa [9]. For a simple, connected and undirected graph $G(V,E)$ with n vertices, a graceful distance labeling (GDL) with span k is an injective function $f : V(G) \rightarrow \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq d(u,v)$ where $d(u,v)$ denotes the distance between the vertices u and v in G. The span k over f is the largest number in $f(V)$ i.e span $(k) = \max f(v)$. The minimum span k taken over all graceful distance labeling of G denoted as $\lambda_d(G)$ is called graceful distance labeling number (GDLN) [1],[3],[4],[6],[8].

2. Preliminaries

GDL Algorithm for a graph:

In this section, we develop an algorithm for giving the GDL to a simple, connected and undirected graph $G(V,E)$ from the set $Z_+ = \{0,1,2,\ldots\}$. The steps of the algorithm are given below:

Step 1: assign zero to one of the vertex which has maximum diameter.

Step 2: remove 0 from the set $\{0,1,2,3,\ldots\}$

Step 3: assign 1 to one of the vertex which is adjacent to the vertex with zero label.

Step 4: remove 1 from the set $\{1,2,3,\ldots\}$

Step 5: continue this process until all the vertices have the distinct labels.

For example: GDL of a tree is shown in figure 1.

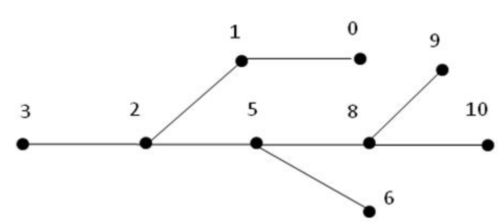

Figure 1. GDL of a tree with 9 vertices

In the following theorems, we will find out the GDLN for path, cycle, complete graph and star graph.

Theorem 2.1. *All paths* (P_n) *admit graceful distance labeling with* $\lambda_d(P_n) = n-1$.

Proof. Let, $v_1, v_2, \ldots, v_n$ be then vertices of a path P_n. The distance between initial and final vertex of path P_n is $n-1$.Let us consider the initial vertex of the path P_n has minimum label and one of the remaining vertices has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that $|\ f(u) - f(v)\ | \geq d(u,v)$ for $u, v \in V(P_n)$. Hence all paths admit graceful distance labeling. Consider a map $f : V(P_n) \to \{0,1,2,\ldots,k\}$ such that $\left|f(v_i) - f(v_j)\right| \geq d(v_i, v_j)$, where $d(v_i, v_j)$ denotes the distance between any two vertices v_i and v_j in P_n.

$$
\begin{aligned}
f(v_i) &= i-1; i = 1,2,\ldots,n \\
\because \lambda_d(G) &= k = \max \mid f(u) - f(v) \\
\therefore \lambda_d(P_n) &= k = \max \left|f(v_i) - f(v_j)\right| = n-1
\end{aligned}
$$

Hence, $\lambda_d(P_n) = n-1$. □

For example, Path graph P_5 and its GDL are shown in following figure:

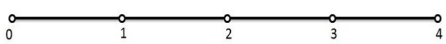

Figure 2. GDL of Path graph P_5

Theorem 2.2. *All cycle admits GDL and GDLN of cycle of n vertices are* $n-1$ *i.e.* $\lambda_d(C_n) = n-1$.

Proof. Let $(v_1, v_2, \ldots, v_n)$ be the vertices of cycle C_n with n edges.Let us consider the initial vertex of the path C_n has minimum label and one of the remaining vertex has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that $|f(u) - f(v)| \geq d(u,v)$ for $u, v \in V(C_n)$.Hence all cycles admit graceful distance labeling. Consider a map $f : V(C_n) \to \{0,1,2,\ldots,k\}$ such that $|\ f(v_i) - f(v_j)\ | \geq d(v_i, v_j)$, where $d(v_i, v_j)$ denotes the distance between any two vertices v_i and v_j in C_n.

$$
\begin{aligned}
f(v_i) &= i-1; i = 1,2,\ldots,n \\
\because \lambda_d(G) &= k = \max |f(u) - f(v)| \\
\therefore \lambda_d(C_n) &= k = \max \left|f(v_i) - f(v_j)\right| = n-1
\end{aligned}
$$

Hence, $\lambda_d(C_n) = n-1$. □

For example, Cycle graph c_6 and its GDL are shown in following figure:

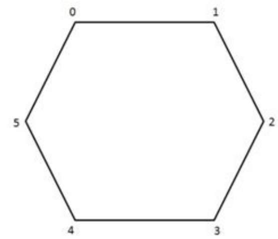

Figure 3. GDL of Cycle graph C_6

Theorem 2.3. *All complete graphs admit GDL and its GDLN is* $n-1$ *i.e.* $\lambda_d(K_n) = n-1$.

Proof. Let $V(K_n) = v_1, v_2, \ldots, v_n$ and consider the initial vertex of K_n has minimum label and one of the remaining vertices has maximum label. Then distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that $|\ f(u) - f(v)\ | \geq d(u,v)$ for $u, v \in V(K_n)$. Hence all complete graphs admit graceful distance labeling. Consider a mapf : $V(K_n) \to \{0,1,2,\ldots,k\}$ such that $\left|f(v_i) - f(v_j)\right| \geq d(v_i, v_j)$, where $d(v_i, v_j)$ denotes the distance between any two vertices v_i and v_j in K_n.

$$
\begin{aligned}
f(v_i) &= i-1; i = 1,2,\ldots,n \\
\because \lambda_d(G) &= k = \max \mid f(u) - f(v) \\
\therefore \lambda_d(K_n) &= k = \max \left|f(v_i) - f(v_j)\right| = n-1
\end{aligned}
$$

Hence, $\lambda_d(K_n) = n-1$. □

For example, complete graph K_4 and its GDL is shown in following figure:

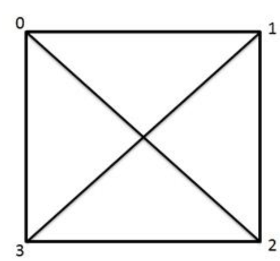

Figure 4. GDL of complete graph K_4

Theorem 2.4. *All star graphs admits GDN and GDLN of star graph of n vertices is* $2(n-2)$, *i.e.* $\lambda_d(K_{1,n-1}) = 2(n-2)$.

Proof. Let $K_{1,n-1}$ be a star graph with

$$
V(K_{1,n-1}) = \{v_1, v_2, \ldots, v_n\}.
$$

Let v_1 be the centre vertex of star and $v_2, v_3, \ldots, v_n$ be the pendant vertices that adjacent to centre vertexv $_1$. Let us consider the initial vertex of the path $K_{1,n-1}$ has minimum label and one of the remaining vertex has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that $|f(u) - f(v)| \geq d(u,v)$ for $u, v \in V(K_1, n-1)$. Hence all star graphs admit graceful distance labeling. Consider a map $f : V(K_{1,n-1}) \to \{0,1,2,\dots,k\}$ such that $\left|f(v_i) - f(v_j)\right| \geq d(v_i, v_j)$, where $d(v_i, v_j)$ denotes the distance between any two vertices v_i and v_j in $K_1, n-1$.

$$f(v_i) = \begin{cases} 1; & i = 1 \\ 2i-4; & i = 2,3,\dots,n \end{cases}$$

$$\because \lambda_d(G) = k = \max|f(u) - f(v)|$$

$$\therefore \lambda_d(K_{1,n-1}) = k = \max\left|f(v_i) - f(v_j)\right| = 2(n-2)$$

Hence, $\lambda_d(K_{1,n-1}) = 2(n-2)$. □

For example, Star graph $K_{1,10}$ and its GDL is shown in following figure:

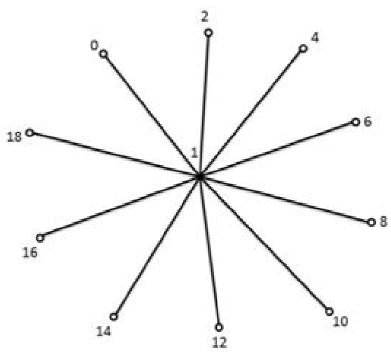

Figure 5. GDL of Star graph $K_{1,10}$

3. Corona of two graphs

A new operation, named corona of two graphs, was presented by Frucht and Harary [2], [5] in 1970. Let G be a graph of order p and H be another graph of finite order. The corona of the graph G and H, denoted by $G \odot H$ is a graph that is obtained by taking one copy of G and p copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H by an edge. i.e.

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i)$$

And

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G)$$

and $u_i \in V(H_i)$.

Theorem 3.1. *All comb graphs $P_n \odot K_1$ admit GDL and its GDLN is*

$$\lambda_d(P_n \odot K_1) = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

Proof. Let $V(P_n \odot K_1) = \{u_1, u_2 \dots, u_n, v_1, v_2 \dots, v_n\}$ be the vertex set of $P_n \odot K_1$ with each u_i belongs to P_n and each v_i belongs to K_i. Let us consider the initial vertex of the comb graph $P_n \odot K_1$ has minimum label and one of the remaining vertices has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that $|f(u) - f(v)| \geq d(u,v)$ for $u, v \in V(P_n \odot K_1)$. Hence all comb graphs admit graceful distance labeling. Consider a map $f : V(P_n \odot K_1) \to \{0,1,2,\dots,k\}$ such that $|f(u) - f(v)| \geq d(u,v)$, where $d(u,v)$ denotes the distance between any two vertices u and v in $P_n \odot K_1$.

$$f(u_i) = \begin{cases} 3i-4; & 2 \leq i \leq n \\ 1; & i = 1 \end{cases}$$

$$f(v_i) = 3(i-1); \quad i = 1,2,\dots,n$$

Here

$$\text{Max}\{f(u) : u \in V(P_n \odot K_1)\} = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

and $\min\{f(u) : u \in V(P_n \odot K_1)\} = 0$

$$\therefore \lambda_d(P_n \odot K_1) = k = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

Hence,

$$\lambda_d(P_n \odot K_1) = k = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

□

For example, Comb graph $P_9 \odot K_1$ and its GDL is shown in following figures;

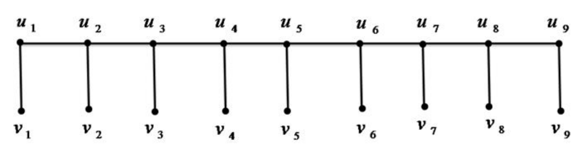

Figure 6. Comb graph $P_9 \odot K_1$

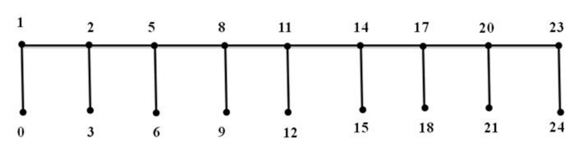

Figure 7. GDL of comb graph $P_9 \odot K_1$

Theorem 3.2. *All hairy cycles $C_n \odot K_1$ admit GDL andits GDLN is*

$$\lambda_d(C_n \odot K_1) = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

Proof. Let $V(C_n \odot K_1) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ such that the pendant vertices $v_1, v_2, \dots, v_n$ are adjacent to $u_1, u_2, \dots, u_n$ respectively.Let us consider the initial vertex of the hairy cycle graph $C_n \odot K_1$ has minimum label and one of the remaining vertex has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u) - f(v)|$ has maximum label. Also, we observe that

$|f(u)-f(v)| \geq d(u,v)$ for$u, v \in V(C_n \odot K_1)$. Hence all comb graphs admit graceful distance labeling. Hence all hairy cycle graphs admit graceful distance labeling. Consider a map $f: V(C_n \odot K_1) \rightarrow \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq d(u,v)$, where $d(u,v)$ denotes the distance between any two vertices u and $v$in$C_n \odot K_1$

$$f(u_i) = \begin{cases} 3i-4; & 2 \leq i \leq n \\ 1; & i = 1 \end{cases}$$

$$f(v_i) = 3(i-1); \quad i = 1,2,\ldots,n$$

Here,

$$\text{Max}\{f(u): u \in V(C \odot K_1)\} = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

And, $\text{Min}\{f(u): u \in V(C_n \odot K_1)\} = 0$

$$\therefore \lambda_d(C_n \odot K_1) = k = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

Hence,

$$\lambda_d(C_n \odot K_1) = k = \begin{cases} 3(n-1); & n > 1 \\ 1; & n = 1 \end{cases}$$

□

For example, Hairy cycle graph $C_n \odot K_1$ and its GDL is shown in following figure;

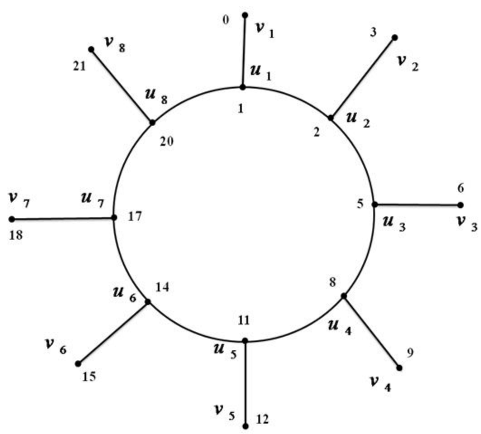

Figure 8. GDL of hairy cycle graph $C_g \odot K_1$

Double graph: Let G' be a copy of simple graph G, let u_i be the vertices of G and v_i be the vertices of G' correspond with u_i. A new graph denoted by $D(G)$ is called the double graph of G [7] if $V(D(G)) = V(G) \cup V(G)$
And

$$E(D(G)) = E(G) \cup E(G') \cup \{u_i v_j : u_i \in V(G), v_j \in V(G')\}$$

and $u_i u_j \in E(G)$.
Figures 9, 10 and 11 shows the path P_q, P_9' and double graph $D(P_9)$ respectively.

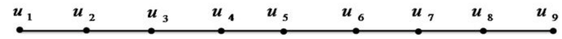

Figure 9. Path P_9

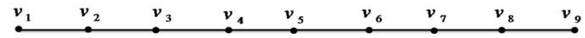

Figure 10. Path P_9'

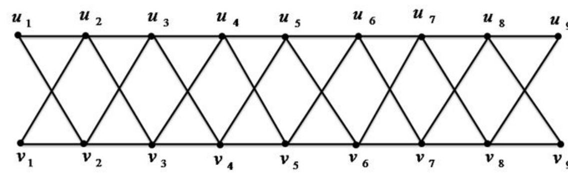

Figure 11. Double graph $D(P_9)$

Theorem 3.3. *All double graphs admit GDL and GDLN of double graph of path $D(P_n)$ with $2n$ vertices are*

$$\lambda_d(D(P_n)) = \begin{cases} 2n; & n \text{ is odd} \\ 2n-1; & n \text{ is even} \end{cases}$$

Proof. Let us consider the initial vertex of the double graph $D(P_n)$ has minimum label and one of the remaining vertex has maximum label. Then the distance between the smallest and the largest vertices is maximum i.e $|f(u)-f(v)|$ has maximum label. Also, we observe that $|f(u)-f(v)| \geq d(u,v)$ for$u, v \in V(D(P_n))$. Hence all double graphs admit graceful distance labeling.
Case I: When n is odd.
Consider a map $f: V(D(P_n)) \rightarrow \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq d(u,v)$, where $d(u,v)$ denotes the distance between any two vertices u and v in $D(P_n)$.

$$f(u_i) = \begin{cases} 2(i-1); & \text{i is odd and } i = 1,3,\ldots,n \\ 2i-3; & \text{i is even and } i = 2,4,\ldots,n-1 \end{cases}$$

$$f(v_i) = \begin{cases} 2i; & i \text{ is odd and } i = 1,3,\ldots,n \\ 2i-1; & i \text{ is even and } i = 2,4,\ldots,n-1 \end{cases}$$

Here, $\text{Max}\{f(u): u \in V(D(P_n))\} = 2n$.
And, $\text{Min}\{f(u): u \in V(D(P_n))\} = 0$.
$\therefore \lambda_d(D(P_n)) = k = 2n$.
Case II: When n is even.
Consider a map $f: V(D(P_n)) \rightarrow \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq d(u,v)$, where $d(u,v)$ denotes the distance between any two vertices u and v in $D(P_n)$.

$$f(u_i) = \begin{cases} 2(i-1); & i \text{ is odd and } i = 1,3,\ldots,n-1 \\ 2i-3; & i \text{ is even and } i = 2,4,\ldots,n \end{cases}$$

$$f(v_i) = \begin{cases} 2i; & i \text{ is odd and } i = 1,3,\ldots,n-1 \\ 2i-1; & \text{i is even and } i = 2,4,\ldots,n \end{cases}$$

Here, $\text{Max}\{f(u): u \in V(D(P_n))\} = 2n-1$.
And $\text{Min}\{f(u): u \in V(D(P_n))\} = 0$.
$\therefore \lambda_d(D(P_n)) = k = 2n-1$. Hence

$$\lambda_d(D(P_n)) = \begin{cases} 2n; & n \text{ is odd} \\ 2n-1; & \text{nis even} \end{cases}$$

□

For example, GDL of double graphs $D(P_9)$ and $D(P_8)$ are shown in following figure:

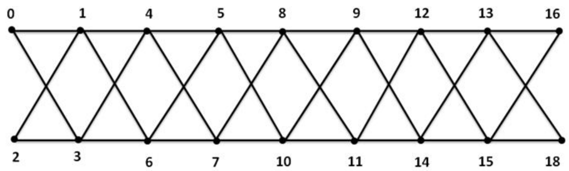

Figure 12.GDL of the double graph $D(P_9)$

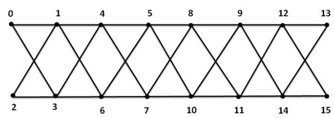

Figure 13. GDL of the double graph $D(P_8)$.

4. Conclusion

We find out the graceful distance labeling number for various graphs. The results obtained here are new and of very general nature. We also provide some illustrations for better understanding of the derived results. we also apply this result for other graphs for future investigation that mean by using above result we also find the graceful distance labeling number of some other graphs also. So this algorithm is useful for find the graceful distance labeling number of different graphs and we use it in future for further investigation.

4.1 Future Scopes

This review paper addressed the various challenges in mathematical fields like as coding theory, radar, crystallography, and traffic signal etc. in this paper we find the graceful distance labeling number for some particular graphs which is very useful in science field. In the field of graph theory, graceful labeling use in different field of sciences like as traffic signal problems. By using the graceful distance labeling number we find the solution of traffic signal problem so this review paper has the future scope in the field of science.

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