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Sequential Henstock integral for $L^p[0,1]$-interval valued functions

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Abstract. In this paper, we introduce the concept of Sequential Henstock integrals for $L^p[0,1]$- interval valued functions and discuss some of their properties.

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Contents

1. Introduction and Background

Henstock integral of a function introduced in the mid-1950s by R. Henstock and J. Kursweil is a powerful generalisation of the Riemann integral, which can handled nowhere-continuous functions which gives a simpler and more satisfactory version of the fundamental theorem of calculus. Simply put, the Henstock integral includes the Riemann, Improper Riemann, Newton and Lebesgue integrals and is equivalent to the Denjoy and Perron integrals(see [1-9]). While the standard definition of the Henstock integral uses the $\varepsilon - \delta$ definition, then the Sequential Henstock integral was introduced, by employing sequences of guage functions. Many authors have worked on the application of the Henstock integral to functions taking real values and have made generalisations on a number of its' properties, see [1-16].

For instance, Cao [3] gave a generalization of the definition of the Henstock integral for Banach space-valued function, and then established some of its properties. Macalalag and Paluga [9] studied the Henstock-type integral for l_p-valued functions with $0 < p < 1$ and obtained its basic properties. The authors have studied the Sequential characterization of the Henstock integral and obtained equivalence results between Henstock and Certain Sequential Henstock Integrals when dealing with real-valued functions.(see [6]). Wu and Gong[15] introduced the notion of the Henstock (H) integral of interval valued functions and Fuzzy number-valued functions and obtained a number of properties. Hamid and Elmuiz[5] established the concept of the Henstock Stieltjes (HS) integrals of interval valued functions and Fuzzy number-valued functions and obtained some number of properties of these integrals. It is well known that the class of $L^p[0,1]$-valued functions with $0 < p < 1$ is a Banach Space with the norm denoted by $\|.\|_{L^p}$.

In this paper, we introduce the notion of Sequential Henstock integral for $L^p[0,1]$-interval valued functions with $0 < p < 1$, and investigate some of its basic properties.

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2. Main Results

Let $\mathbb{R}$ denote the set of real numbers, $F(X)$ as an interval valued function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a,b]$, X, as non empty interval in $\mathbb{R}$ and $d(X) = X^+ - X^-$, as width of the interval X and $\ll$ as much more smaller and consider the integral of interval functions defined on the compact interval and ranging in a quasi-Banach $L^p[0,1]$-space which carries a quasi-norm denoted by $\|.\|_{L^p}$.

Let E a Lebesgue measurable set in any euclidean space, and q any positive number, we define $L^p(E)$ to be the class of all real valued Lebesgue measurable functions f on E for which $\int_E |f|^q < \infty$. As it's well known, whenever $q < 1$, this class of functions is a Banach Space with the norm $\|f\|_q = (\int_E |f|^q)^{\frac{1}{q}}$. When $0 < p < 1$, the function $\|f\|_p$ no longer satisfies the triangular inequality but only the weaker condition

$$\|f_1 + f_2\| \leq 2^{\gamma}(\|f_1\| + \|f_2\|)$$

where $\gamma = \dfrac{(1-p)}{p}$(see [3]).

Definition 2.1[10,12] A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

Definition 2.2[10,12] A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} <, ..., < u_{m_n} = b$.

Definition 2.3 [12] A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable to α on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that if $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}\, u_i), t_i\}_{i=1}^{n}$, we have

$$|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon.$$

where the number α is the Henstock integral of f on $[a,b]$. The family of all Henstock integrals function on $[a,b]$ is denoted by $H[a,b]$ with $\alpha = (H)\int_{[a,b]} f(x)dx$ and $f \in H[a,b]$.

Definition 2.4 [12] A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable to $\alpha \in \mathbb{R}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\delta_\mu(x) = \{\delta_n(x)\}_{n=1}^{\infty}$ such that for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

where the sum $\sum$ is over P_n, we write $\alpha = (SH)\int_{[a,b]} f(x)dx$ and $f \in SH[a,b]$.

Lemma 2.5[5] Let f, k be Sequential Henstock (SH)integrable functions on $[a,b]$, if $f \leq k$ is almost everywhere on $[a,b]$, then

$$\int_a^b f \leq \int_a^b k.$$

Definition 2.6 [11 and 15]
Let $I_{\mathbb{R}} = \{I = [I^-, I^+]$: I is a closed bounded interval on the real line $\mathbb{R}\}$.

For $X, Y \in I_{\mathbb{R}}$, we define
i. $X \leq Y$ if and only if $Y^- \leq X^-$ and $X^+ \leq Y^+$,
ii. $X + Y = Z$ if and only if $Z^- = X^- + Y^-$ and $Z^+ = X^+ + Y^+$,
iii. $X.Y = \{x.y : x \in X, y \in Y\}$, where

$$(X.Y)^- = \min\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}$$

and

$$(X.Y)^+ = \max\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}.$$

Define $d(X, Y) = \max(|X^- - Y^-|, |X^+ - Y^+|)$ as the distance between intervals X and Y.

Definition 2.7 [5]
An interval valued function $F : [a, b] \to L^p$ is Henstock integrable(l_p-$IH[a, b]$) to $I_0 \in L^p[0, 1]$ on $[a, b]$ if for every $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ on $[a, b]$ such that for every $\delta(x) - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$, we have

$$\|\sum_{i=1}^{n\in\mathbb{N}} F(t_i)(u_i - u_{i-1}) - I_o\|_{L^p} < \varepsilon$$

We say that I_0 is the Henstock integral of F on $[a, b]$ with $(L^p[0, 1]\text{-}IH) \int_{[a,b]} F = I_0$ and $F \in L^p[0, 1]$-$IH[a, b]$.

Now, we will define the Sequential Henstock integral of $L^p[0, 1]$-interval valued function and then discuss some of the properties of the integral.

Definition 2.8
An interval valued function $F : [a, b] \to L^p$ is Sequential Henstock integrable($L^p[0, 1]$-$ISH[a, b]$) to $I_0 \in L^p[0, 1]$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^\infty$ such that for every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - l_o\|_{L^p} < \varepsilon.$$

We say that $L^p[0, 1]$ is the Sequential Henstock integral of F on $[a, b]$ with $(L^p[0, 1]\text{-}ISH) \int_{[a,b]} F = \alpha$ and $F \in L^p[0, 1]$-$ISH[a, b]$.

In this section, we discuss some of the basic properties of the $L^p[0, 1]$-interval valued Sequential Henstock integrals.

Theorem 2.9
If $F \in L^p[0, 1]$-$ISH[a, b]$, then there exists a unique integral value.

Proof. Suppose the integral value are not unique. Let $\alpha_1 = (L^p[0, 1]\text{-}ISH) \int_{[a,b]} F$ and $\alpha_2 = (L^p[0, 1]\text{-}ISH) \int_{[a,b]} F$ with $\alpha_i \neq \alpha_2$. Let $\varepsilon > 0$ then there exists a $\{\delta_n^1(x)\}_{n=1}^\infty$ and $\{\delta_n^2(x)\}_{n=1}^\infty$ such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a, b]$ and for each $\delta_n^2(x)$-fine tagged partitions P_n^2 of $[a, b]$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 \|_{L^p} < \frac{\varepsilon}{2},$$

and

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p} < \frac{\varepsilon}{2}.$$

respectively.
Define a positive gauge function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$ and let $\varepsilon = \dfrac{\|\alpha_1 - \alpha_2\|_p}{2^{\frac{1}{p}}}$. Then we have

$$\|\alpha_1 - \alpha_2\|_{L^p} = \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 + \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p}$$

$$\leq \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_1 \|_{L^p} + \| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha_2 \|_{L^p}$$

$$< 2^{\frac{1}{p}} (\frac{\varepsilon}{2} + \frac{\varepsilon}{2}) = 2^{\frac{1}{p}} \varepsilon = \|\alpha_1 - \alpha_2\|_{L^p},$$

This is a contradiction. Thus $\alpha_1 = \alpha_2$. This completes the proof. ■

Theorem 2.10
An interval valued function $F \in L^p[0,1]$-$ISH[a,b]$ if and only if $F^-, F^+ \in L^p[0,1]$-$SH[a,b]$ and

$$(L^p[0,1]\text{-}ISH) \int_{[a,b]} F = [(l_p\text{-}SH) \int_{[a,b]} F^-, (L^p[0,1]\text{-}SH) \int_{[a,b]} F^+] \tag{2.1}$$

Proof. Let $F \in L^p[0,1]$-$ISH[a,b]$, from Definition 2.8 there is a unique interval number $I_o = [I_0^-, I_0^+]$ in the property, then for any $\varepsilon > 0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$, $n \geq \mu$ on $[a,b] \in \mathbb{R}$ such that for any $\delta_n(x)$-fine tagged partition P_n, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0 \|_{L^p} < \varepsilon.$$

Observe that

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0 \|_{L^p} = \max(\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^- \|_{L^p},$$
$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^+ \|_{L^p}.$$

Since $u_{i_n} - u_{(i-1)_n} \geq 0$ for $1 \leq i_n \leq m_n$, hen it follows that

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - I_0^- \|_{L^p} < \varepsilon, \| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n} - I_0^+)\|_p < \varepsilon.$$

for every $\delta_n(x)$-tagged partition $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$. Thus, by Definition 2.8, we obtain $F^+, F^- \in L^p[0,1]$-$SH[a,b]$ and

$$I_o^- = (L^p[0,1]\text{-}SH) \int_{[a,b]} F^-(x)dx$$

and

$$I_o^+ = (L^p[0,1]\text{-}SH) \int_{[a,b]} F^+(x)dx.$$

Conversely, Let $F^- \in L^p[0,1]$-$SH_{[a,b]}$. Then there exist a unique $\beta_1 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^1(x)$-fine tagged partitions P_n^1 we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \beta_1 \|_{L^p} < \varepsilon.$$

Similarly,
Let $F^+ \in L^p[0,1]$-$SH[a,b]$. Then there exist a unique $\beta_2 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^2(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^2(x)$-fine tagged partitions P_n^2 we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \beta_2) \|_{L^p} < \varepsilon.$$

Let $\beta = [\beta_1, \beta_2]$. If $F^- \leq F^+$, then $\beta_1 \leq \beta_2$. We define $\delta_n(x) = \min(\delta_n^1(x), \delta_n^2(x))$, then for any $\delta_n(x) - fine$ tagged partitions P_n we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \beta \|_{L^p} < \varepsilon.$$

Hence, $F : [a,b] \to L^p$ is Sequential Henstock integrable on $[a,b]$.
This completes the proof.

■

Theorem 2.11
Let $F, K \in L^p[0,1]$-$ISH[a,b]$ with $F = [F^-, F^+]$ and $H = [K^-, K^+]$ and $\gamma, \xi \in \mathbb{R}$. Then $\gamma F, \xi K \in L^p[0,1]$-$ISH[a,b]$ and

$$(L^p[0,1]\text{-}ISH) \int_{[a,b]} (\gamma F + \xi K)dx = \gamma (L^p[0,1]\text{-}ISH) \int_{[a,b]} F dx + \xi (L^p[0,1]\text{-}ISH) \int_{[a,b]} K dx$$

Proof. (i) If $F, K \in L^p[0,1]$-$ISH[a,b]$, then $[F^-, F^+], K = [K^-, K^+] \in L^p[0,1]$-$SH[a,b]$ by Theorem 2.10. Hence, $\gamma F^- + \xi K^-, \gamma F^- + \xi K^+, \gamma F^+ + \xi K^-, \gamma F^+ + \xi K^+ \in L^p[0,1]$-$SH[a,b]$.
1) If $\gamma > 0$ and $\xi > 0$, then

$$\begin{aligned}(L^p[0,1]\text{-}SH) \int_{[a,b]} (\gamma F + \xi K)^- dx &= (L^p[0,1]\text{-}SH) \int_{[a,b]} (\gamma F^- + \xi K^-)dx \\ &= \gamma (L^p[0,1]\text{-}SH) \int_{[a,b]} F^- dx + \xi (L^p[0,1]\text{-}SH) \int_{[a,b]} K^- dx\end{aligned}$$

$$= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^- + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-$$
$$= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.$$

2) If $\gamma < 0$ and $\xi > 0$, then

$$\begin{aligned}(L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F^+ + \xi K^+)dx\\ &= \gamma(L^p[0,1]\text{-}SH)\int_{[a,b]} F^+dx + \xi(L^p[0,1]\text{-}SH)\int_{[a,b]} K^+dx\\ &= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^+ + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+\\ &= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.\end{aligned}$$

3) If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$\begin{aligned}(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (L^p[0,1]\text{-}SH)\int_{[a,b]}(\gamma F^- + \xi K^+)dx\\ &= \gamma(L^p[0,1]\text{-}SH)\int_{[a,b]} F^-dx + \xi(L^p[0,1]\text{-}SH)\int_{[a,b]} K^+dx\\ &= \gamma((L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx)^- + \xi((L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+\\ &= (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^-.\end{aligned}$$

Similarly, for four cases above, we have

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)^+ dx = (\gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx)^+$$

Hence, by Theorem 2.10, $\gamma F, \xi K \in L^p[0,1]\text{-}ISH[a,b]$ and

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]}(\gamma F+\xi K)dx = \gamma(L^p[0,1]\text{-}ISH)\int_{[a,b]} Fdx + \xi(L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx.$$

This completes the proof. ■

Theorem 2.12
Let $F, K \in L^p[0,1]\text{-}ISH[a,b]$ and $F(x) \leq K(x)$ nearly everywhere on $[a,b]$, then

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F(x)dx \leq (L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx$$

Proof. If $F(x) \leq K(x)$ nearly everywhere on $[a,b]$ and $F, K \in L^p[0,1]\text{-}ISH[a,b]$, then $F^-, F^+, K^-, K^+ \in L^p[0,1]\text{-}SH[a,b]$ and $F^- \leq F^+, K^- \leq K^+$ nearly everywhere on $[a,b]$. By Lemma 2.5

$$(L^p[0,1]\text{-}SH)\int_{[a,b]} F^-(x)dx \leq (L^p[0,1]\text{-}SH)\int_{[a,b]} K^-dx$$

and

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F^+(x)dx \le (L^p[0,1]\text{-}ISH)\int_{[a,b]} K^+dx.$$

Hence by Theorem 2.10, we have

$$(L^p[0,1]\text{-}ISH)\int_{[a,b]} F(x)dx \le (L^p[0,1]\text{-}ISH)\int_{[a,b]} Kdx.$$

This completes the proof. ■

Theorem 2.13 Let $k \in \mathbb{R}$.
1. If $F \in L^p[0,1]\text{-}ISH[a,b]$, then $kF \in L^p[0,1]\text{-}ISH[a,b]$. Moreover,

$$\int_a^b kF = k\int_a^b F.$$

2. If $F \in L^p[0,1]\text{-}ISH[a,b]$ and $G \in L^p[0,1]\text{-}ISH[c,b]$, then $(F+G) \in L^p[0,1]\text{-}ISH[a,b]$. Moreover

$$\int_a^b (F+G) = \int_a^b F + \int_a^b G.$$

Proof. (1) Suppose $F \in L^p[0,1]\text{-}ISH[a,b]$. The case $k = 0$ is obvious. Suppose $k \neq 0$ and $F \in L^p[0,1]\text{-}ISH[a,b]$, there exists a sequence of positive functions$\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \int_a^b F\|_{L^p} < \frac{\varepsilon}{|k|_{L^p}}$$

whenever P_n is $\delta_n(x)-fine$ tagged partitions of $[a,b]$. Then, exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,c]$ such that

$$\begin{aligned}\|\sum_{i=1}^{m_n\in\mathbb{N}} kF(t_{i_n})(u_{i_n} - u_{(i-1)_n} - k\int_a^b F\|_{L^p} &= \|k\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - k\int_a^b F\|_{L^p}\\ &< |k|_{L^p}\frac{\varepsilon}{|k|_{L^p}}\\ &= \varepsilon.\end{aligned}$$

(2) Let $\varepsilon > 0$ Suppose $\int_a^b F = \alpha_1$ and $\int_a^b G = \alpha_2$. Then there exists a sequence of positive functions$\{\delta_n^1(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_1\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^1 is $\delta_n^1(x) - fine$ tagged partitions of $[a,b]$. Also, there exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_2\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^2 is $\delta_n^2(x) - fine$ tagged partitions of $[a,b]$.
Define a positive gauge function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$. Then

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} (F+G)(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - (\alpha_1+\alpha_2)\|_{L^p}$$

$$= (\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}) + \sum_{i=1}^{m_n \in \mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - (\alpha_1 + \alpha_2)\|_{L^p})$$
$$\leq 2^{\frac{1}{p}}(\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_1\|_{L^p}) + 2^{\frac{1}{p}}(\| \sum_{i=1}^{m_n \in \mathbb{N}} G(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \alpha_2\|_{L^p})$$
$$< 2^{\frac{1}{p}}(\frac{\varepsilon}{2(2\frac{1}{p})} + \frac{\varepsilon}{2(2\frac{1}{p})})$$
$$= \varepsilon.$$

This completes the proof. ■

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