(s,t)-Dominating polynomials in graphs

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Abstract

In this paper, We discuss some properties of dominantly open, and dominantly closed graphs, and polynomial representation of such graphs.

Keywords

Dominating, Polynomials, simple graph.

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1. Introduction

A simple graph G is a triple consisting of a vertex set V(G), an edge set E(G) and a relation χ that associates each edge with two distinct vertices. An edge e $\in$ E(G) that associates two vertices u,v $\in$ V(G) is simply denoted as $e = uv$. The number of vertices in V(G) is called the order of G and the number of edges in E(G) is called the size of G. A set $S \subseteq$ V(G) is a dominating set of G if for each $v \in v(G)$, either $v \in S$ or there exists an u $\in$ S which is adjacent to v. The minimum cardinality of such a subset S in V(G) is called the domination number of G, and the set S is called the minimum dominating set. The domination number is denoted by $\gamma(G)$. A dominating set S is a minimal dominating set if S-v is not a dominating set for every v $\in$ S. Since, every minimum dominating set is also a minimal dominating set, but not the converse. The minimal dominating set which is not a minimum dominating set is called a strictly minimal dominating set. In other words, a set S is a minimal dominating set if no proper subset of S is a dominating set. A minimal dominating set in a graph G with cardinality more than the domination number $\gamma(G)$ is called the strictly minimal dominating set. A graph G is called dominantly open if G has a strictly minimal dominating set, otherwise G is called dominantly closed. In this paper, We discuss some properties of dominantly open, and dominantly closed graphs, and polynomial representation of such graphs.

2. Some property of dominating open and closed graphs

The notion of domination was introduced by Berge[5], and Ore[9] coined the terminology and the notation. In general, domination in graphs is a graph parameter in which a set of minimum number of vertices that cover the whole graph is identified. T.W.Haynes, S.T. Hedetniemi, and P.J. Slater,[8] studied the fundamentals of domination in graphs. In some situations when such a minimum dominating set fails to accomplish the expected task, it is natural to search for the next minimum dominating set. The strictly minimal dominating set plays the role of the next minimum dominating set. In this section, we discuss some of the properties of the graphs which posses the next minimum dominating sets i.e., strictly minimal dominating sets. S. Akbari, S. Alikhani, and Y.H. Peng,[1] studied characterization of graphs using domination polynomial. Now here an attempt is made to characterize the graphs combining the minimum dominating sets and strictly minimal dominating sets and respective polynomials.

Let the minimum dominating set, and the strictly minimal dominating set of a graph G be represend respectively as D(G) and $D_i(G)$, and the corresponding numbers be γ(G) and $\gamma_l(G)$.

It can be easily verified that the path graph P_5 is dominantly open and the cycle C_5 is dominantly closed. In C_5, any three-elements vertex set is a dominating set, but it is not minimum and also it is not minimal.The lollipop graph is the graph obtained by joining a complete graph to a path graph P_1, with a bridge and is denoted by $L_{n,1}$ is a dominantly open graph. For basic concepts and notations in graph theory one

can refer [6,7].

Definition 2.1. *The minimum cardinality of a strictly minimal dominating set is the minimal domination number of the graph. It is denoted as $\gamma_l(G)$.*

Example 2.2. *In the graph P_5, given in the fig 1,the set $\{v_2, v_4\}$ is the minimum dominating set and the set $\{v_1, v_3, v_5\}$ is the strictly minimal dominating set. Hence $\gamma(P_5) = 2$ and $\gamma_l(P_5) = 3$.*
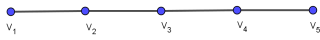

Figure 1 The path P_5

From the definition of minimum, and minimal dominating sets it is easy to observe the following

Observations 2.3. *(i) $\gamma(G) \leq \gamma_l(G)$, if $\gamma_l(G) \neq 0$.*

(ii) The sets $D(G)$ and $D_l(G)$ may be overlapping sets, but $D(G)$ can never be a subset of $D_l(G)$.

(iii) For complete graph K_n, $\gamma(G) = 1$ and $\gamma_l(G) = 0$.

Alikhani,S., and Peng,Y.H.,[3] here introduced domination polynomial of a graph. In general, while studying domination polynomial of a graph, not only the number of minimum dominating sets, the dominating sets of all sizes are considered. But in this attempt, only the number of minimum dominating sets, and the number of strictly minimum dominating sets are considered to represent the domination polynomial of the graph. In the following definition, we define the separate polynomial representation each for minimum dominating sets and strictly minimum domination sets. And later, we give combined polynomial representation for the given graph.

Definition 2.4. *The dominating polynomial of a graph G is $P_s(G) = \sum_i d_i s^i$, where d_i is the number of dominating sets of size i in G. It can be easily verified that*

$$P_s(P_1) = s$$
$$P_s(P_2) = 2s + s^2$$
$$P_s(P_3) = s + 3s^2 + s^3$$
$$P_s(P_4) = 4s^2 + 4s^3 + s^4$$
$$P_s(P_5) = 3s^2 + 8s^3 + 5s^4 + s^5$$
$$P_s(P_6) = s^2 + 10s^3 + 13s^4 + 6s^5 + s^6$$

Definition 2.5. *The strictly dominating polynomial of a graph G is $P_t(G) = \sum d_i t^i$, where d_i is the number of strictly dominating sets of size i in G. It can also been verified that the paths P_1, P_2 and P_4 have no strictly dominating sets and the strictly dominating polynomial of P_3, P_5 and P_6 are as follows $P_t(P_3) = t^2; P_t(P_5) = t^3; P_t(P_6) = 6t^3$.*

Lemma 2.6. *In a graph G, the dominating sets $D(G)$ and $D_l(G)$ are either disjoint or differ by atleast one element.*

Proof. From the definition of strictly dominating set, it is quite clear that $|D_l(G)| > |D(G)|$. Hence $D_l(G)$ has at least one element more than the number of elements in $D(G)$. Also $D(G)$ cannot be a proper subset of G. Hence D(G) and $D_l(G)$ are either disjoint or differ by at least one element. □

Corollary 2.7. *If $D(G) \subset B$, a subset of $V(G)$, then B cannot be a $D_l(G)$.*

3. Dominating sets in Paths

S.Alikhani, E.Mahmoudi, and M.R. Oboudi [2] discussed new approaches to compute the domination polynomial of some specific graphs. One such new approach is attempted in this paper in the name of (s,t)-polynomial for some special graphs. It can be easily observed that the dominating set in any path P_n where $V(P_n) = \{v_1, v_2, ...v_n\}$,such that $v_i \sim v_{i+1}$ and v_1, v_n are pendent vertices, must contain either v_1 or v_2. If it contains both v_1 and v_2 then it cannot be a dominating set. Let $V_0 = \{v_1, v_3....\}$; $V_e = \{v_2, v_4....\}$ are the subsets of the vertex set of the path P_n. Then the sets V_0 and V_e are always dominating set sets.

Lemma 3.1. *The set V_0 is $D(P_n)$ when $n = 2, 4$ and $D_l(P_n)$ when $n = 3, 5$.*

On generalizing the above lemma it can be observed that the sets V_0, V_e are always dominating sets in P_n. The following theorem gives when the above sets are strictly minimal dominating sets.

Theorem 3.2.

$$V_e = \begin{cases} D(P_n) & when & 2 \leq n \leq 5 & and & n = 7 \\ D_l(P_n) & when & n = 6, n > 7 \end{cases}$$

and $V_0 = D_l(P_n)$ when $n > 2$.

Proof. In path P_n, Let $V(P_n) = \{v_1...v_n\}$ and $v_i \sim v_{i+1}$, i= 1,2....n-1, and also v_1, v_n are pendent vertices. The graphs P_1 and P_2 have no strictly minimal dominating sets. The set V_0 in P_3 is a strictly minimal dominating set, but V_e is a minimum dominating set. The sets V_0, V_e in P_4 are minimum dominating sets, where as in P_5, V_0 is a strictly minimal dominating set and V_e is a minimum dominating set. Similarly we can check the theorem easily for P_6 and P_7. So, we discuss the nature of domination of V_0 and V_e in P_n only for $n > 7$.
If n is even, $|V_0| = |V_e| = \frac{n}{2}$ and if n is odd $|V_0| = \lceil \frac{n}{2} \rceil$ and $|V_e| = \lceil \frac{n}{2} \rceil$. More over the sets V_0 and V_e are independent dominating sets. i.e., every v_i in V_0 dominates at most two vertices in V_e, but not any of the vertices in V_0, and vice versa, we also know that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.
Hence the theorem is proved once if we prove that

(i) $|V_0| = |V_e| = \frac{n}{2} > \lceil \frac{n}{3} \rceil$ when n is even

(ii) $|V_0| = \lceil \frac{n}{2} \rceil = \frac{n+1}{2} > \lceil \frac{n}{3} \rceil$ and $|V_e| = \lceil \frac{n}{2} \rceil = \frac{n-1}{2} > \lceil \frac{n}{3} \rceil$ when n is odd and $n > 7$.

First we prove (i) by using mathematical induction principle by applying induction on 'n'. If $n = 2$, we can easily check the statement (i). So, by induction hypothesis assume that $\frac{k}{2} > \lceil\frac{k}{3}\rceil$ for some even number k. Let $n = k+2$. If $k+2$ is divisible by 3

$$\lceil\frac{k+2}{3}\rceil = \lceil\frac{k}{3}\rceil + \lceil\frac{2}{3}\rceil \leq \frac{k}{2}+1 = \frac{k+2}{2}$$

If k+2 is not divisible by 3

$$\lceil\frac{k+2}{3}\rceil = \lceil\frac{k}{3}\rceil + \frac{2}{3} \leq \frac{k}{2}+1 = \frac{k+2}{2}.$$

Hence, by induction principle we can conclude that $\frac{n}{2} > \lceil\frac{n}{3}\rceil$ for all even numbers n. This proves (i).
We prove (ii), also by induction principle. In particular we prove that $\frac{n-1}{2} > \lceil\frac{n}{3}\rceil$. Let $n = 9$; It can be easily verified that $\frac{n-1}{2} > \lceil\frac{n}{3}\rceil$, when n=9. As induction hypothesis, let k be an odd number greater than 9, and $\frac{k-1}{2} > \lceil\frac{k}{3}\rceil$. Let $k = 2r+1$, where $r > 4$. Then

$$\lceil\frac{2r+1}{3}\rceil \leq \frac{2r+1-1}{2} = m.$$

Let

$$k = 2r + 3\lceil\frac{k}{3}\rceil = \lceil\frac{2r+1}{3}\rceil = \lceil\frac{2r+1}{3}\rceil + \lceil\frac{2}{3}\rceil$$
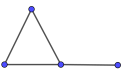
$$< m+1 = \frac{k-1}{2}+1 = \frac{k+1}{2}$$
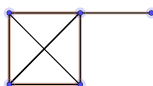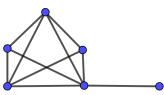
Hence by induction principle (ii) is true for all values of n. $\square$

4. (s,t)-**Polynomials**

Definition 4.1. *In a graph G, if $\gamma(G) = \alpha$ and $\gamma_l(G) = \beta$, and there are k,l number of minimum dominating sets and strictly minimal number of dominating sets respectively then $ks^\alpha + lt^\beta$ is the (s,t)-dominating polynomial of G.*

The following table gives the (s,t)-polynomial of some lollypop graphs.

Table 1.

|		
$L_{3,1}$		$s + 2t^2$
$L_{4,1}$		$s + 3t^2$
$L_{5,1}$		$s + 3t^2$

On generalization, we get the following lemma

Lemma 4.2. *The (s,t)-Polynomial of the lollypop graph $L_{n,1}$ is s+(n-1)t^2; where s is the size of minimum dominating set and t is the size of strictly minimal dominating set.*
In all the graphs, the sets $D(G)$ and $D_l(G)$ are disjoint. Hence, each term of the (s,t) polynomial contains only one variable. Also note that the coefficients (k,l) of the variables represent the number of respective dominating sets, and the powers (α, β) of the variable represent the size of the respective dominating sets.

In the following table, (s,t)-polynomials of some graphs are given

Table 2.

Graph	$(s,t) - Polynomials$
	$2s$
	$s + t^2$
	$3s$
	$s + t^3$
	$4s^2$
	$s + 2t^2$
	$6s^2$
	$2s + t^2$
	$4s$

Definition 4.3. *If G is a graph with minimum dominating sets $D_1(G), D_2(G)....D_p(G)$ and strictly minimum dominating sets $D_{l1}(G), D_{l2}(G)....D_{lq}(G)$ such that $D_i(G) \cap D_{lj}(G) = \emptyset$, and there are $u_1, u_2....u_x \in D_i(G)$ and $v_1, v_2....v_y \in D_{lj}(G); i = 1, 2.....p; j = 1, 2....q$, so that $\{u_1,....u_x, v_i....v_y\}$ is a strictly dominating set then $s^\alpha t^\beta$ is one of the terms in the (s,t)-polynomial of G. If there are k such sets, then the term is*

exactly $ks^{\alpha}t^{\beta}$. A graph with this property is called a separable graph.

Example 4.4.

Consider a separable graph given in Fig 2.

In the separable graph, given in fig 4.4 the subsets $\{m_{11}, m_{22}\}$, $\{m_{12}, m_{21}\}$ are minimum dominating sets, and the subsets $\{L_{11}, L_{12}, L_{13}, L_{14}, L_{15}\}$ and $\{L_{21}, L_{22}, L_{23}, L_{24}, L_{25}\}$ of $V(G)$ are strictly minimum dominating sets. i.e.,

The four subsets are the separable dominating sets of G. There are no other mixing minimum dominating sets and hence the corresponding term is $2s^2$. But there are mixing strictly dominating sets, These sets are
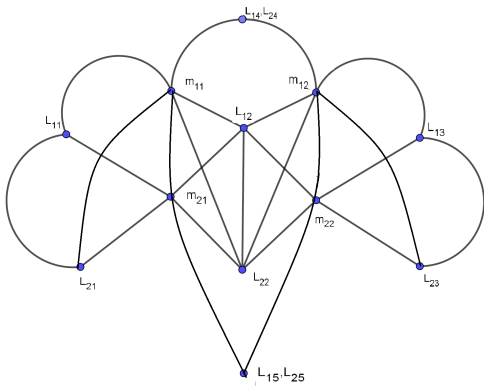

Figure 2. A Seperable graph G

(i) $\{L_{11}, L_{12}, L_{23}, L_4, L_5\}$

(ii) $\{L_{21}, L_{22}, L_{13}, L_4, L_5\}$

(iii) $\{L_{11}, L_{13}, L_{22}, L_4, L_5\}$

(iv) $\{L_{12}, L_{13}, L_{21}, L_4, L_5\}$

(v) $\{L_{21}, L_{23}, L_{12}, L_4, L_5\}$

(vi) $\{L_{11}, L_{22}, L_{23}, L_4, L_5\}$

where L_{14}, L_{24} are denoted simply as L_4 and L_{15}, L_{25} are denoted as L_5. Hence the corresponding term in the (s,t)- polynomial is $8t^5$. The other mixing strictly dominating sets and the corresponding terms in the (s,t)-polynomial are given in the table 3.

Therefore the (s,t)-polynomial of the graph given in fig 2. is

$$P_{s,t}(G) = 2s^2 + 8t^5 + 8st^2 + 8st^3 + 4s^2t^2 + 4s^2t.$$

5. Conclusion

In this paper, the minimum dominating sets and strictly minimum dominating sets and a polynomial representing these two dominating sets have been discussed.

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Table 3.

S.No	mixing strictly dominating sets	term in (s,t) -polynomial
i)	$\{m_{11}, L_{23}, L_5\}$ $\{m_{11}, L_{13}, L_5\}$ $\{m_{12}, L_{21}, L_5\}$ $\{m_{12}, L_{11}, L_5\}$ $\{m_{21}, L_{23}, L_4\}$ $\{m_{21}, L_{23}, L_4\}$ $\{m_{22}, L_{21}, L_4\}$ $\{m_{22}, L_{11}, L_4\}$	$8st^2$
ii)	$\{m_{12}, L_{11}, L_{12}, L_5\}$ $\{m_{22}, L_{11}, L_{12}, L_4\}$ $\{m_{21}, L_{11}, L_{13}, L_4\}$ $\{m_{22}, L_{11}, L_{13}, L_4\}$ $\{m_{11}, L_{11}, L_{13}, L_5\}$ $\{m_{12}, L_{11}, L_{13}, L_5\}$ $\{m_{11}, L_{12}, L_{13}, L_5\}$ $\{m_{21}, L_{12}, L_{13}, L_4\}$	$8st^3$
iii)	$\{m_{12}, m_{22}, L_{21}, L_{22}\}$ $\{m_{12}, m_{22}, L_{21}, L_{23}\}$ $\{m_{11}, m_{21}, L_{21}, L_{23}\}$ $\{m_{11}, m_{21}, L_{22}, L_{23}\}$	$4s^2t^2$
iv)	$\{m_{11}, m_{21}, L_{13}\}$ $\{m_{11}, m_{21}, L_{23}\}$ $\{m_{12}, m_{22}, L_{11},\}$ $\{m_{12}, m_{22}, L_{11},\}$	$4s^2t$