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Importance of $p-KdV$ equation in target tracking and its applications

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Abstract
In this paper we use Adomian Decomposition Method to solve time fractional $p-KdV$ equation.

Keywords
Time fractional potential KdV equation, Target tracking in sensor, Caputo fractional derivative, Adomian Decomposition Method, Mathematica.

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Contents

1. Introduction

The population will demand for bare essentials of life such as food, water and shelter. There essentials can be provided by infrastructure that can sustain them for the long-term. If continuously fossil fuels used instead of sustainable options, then the availability of basic needs will become difficult. Therefore our demand for green products and services lead to sustainabilty.

To increase the sustainability of the product , many companies are involved in sustainable production without using fossil fuels. During sustainable production sensors are designed to create data, which can be used to analyze and identify flaws in the product. By reducing spoilage and identifying the flaws, products can create more sustainable supply chain. Therefore sensors will play an important role in the creation of more sustainable society, as sensors are useful in safety, security, surveillance, monitoring and awareness.

Typical sensor systems such as radar, Infrared(IR) and Sonar report measurements from various resources targets background noise sources or internal error sources. The target tracking objective is to collect sensor data and then partition the sensor data in the tracks. Once tracks are formed, we can estimate number of targets, target velocity, future predicted position and target classification characteristics [1].

Therefore in recent years, researchers are working to find accurate and fast method to track real-world position and orientation of moving targets. Tracking is a process of estimating the current and future state of target [2].

Target tracking process can be defined as a set of algorithm and the algorithm is based on a nonlinear KdV equation as a moving target detector. In the paper [3], it is proved that the solutions of inhomogeneous KdV equation helps to get the right information about moving targets by using soliton resonance method. A novel neural architecture named ”Spectral network” is being proposed for detecting targets in a cluttered background and results can be interpreted in terms of resonances by KdV equations [2].

Therefore we observe the importance of KdV equation in target tracking process which is important for sensor data analysis and leads to sustainability. In this paper we will study time fractional Potential KdV (p-KdV) equation which is very important form of KdV and also useful in plasma physics, mechanics, lattice dynamics etc. We reviewed that in the paper [4], p-KdV was solved by $(\frac{G'}{G}, \frac{1}{G})$ expansion method and obtained soliton solutions are designated in terms of kink, bell-shaped solitary waves, periodic and singular periodic wave solutions. In [5] the kink solution and travelling wave solution of the p-KdV equation was obtained by using tanh-coth

method. In paper [6] p-KdV equation was solved by ansatz method.
During last decades, fractional calculus has been used in viscoelasticity, rheology, electrical engineering, biology, image processing, physics etc. Several methods are used to solve fractional differential equations such as Laplace transform method [16], Fourier transform method [1], Perturbation method, Iteration method [18]etc. In this paper we use Adomian Decomposition Method to solve time fractional p-KdV equation.
The KdV equation is given by-

$$\frac{\partial w(x,t)}{\partial t}+\varepsilon w\frac{\partial w(x,t)}{\partial x}+\mu\frac{\partial^3 w(x,t)}{\partial^3 x}=0$$

Here $w(x,t)$ is the dependent variable, x and t are independent variables. The parameters ε and μ are real constants.
The p-KdV is given by [19] -

$$\frac{\partial w(x,t)}{\partial t}+\varepsilon\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\mu\frac{\partial^3 w(x,t)}{\partial^3 x}=0$$

Here first term is the evolution term, second term is nonlinear term and third term is dispersion term. Now we will consider time fractional p-KdV equation as under-

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}+\varepsilon\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\mu\frac{\partial^3 w(x,t)}{\partial^3 x}=0,\ 0<\alpha\leq 1,t>0$$

The aim of this paper is to solve time fractional p-KdV equation by using Adomian Decomposition Method. We will give some formulae and theorem in Section 2 which are used in our calculations. Section 3 is devoted for ADM to solve time fractional $p-KdV$ alongwith uniqueness and convergence of solution. In section 4 some numerical problems are solved and presented graphically by using Mathematica software.

2. Basic Preliminaries and Properties of Fractional Derivatives

In this section, we study some definitions and properties of fractional calculus.

Definition 2.1. *The Caputo fractional derivative of the function $f(x)$ is defined as*

$$D_*^\beta f(x)=J^{(m-\beta)}D^m f(x)$$
$$=\frac{1}{\Gamma(m-\beta)}\int_0^x\frac{1}{(x-t)^{(1-m+\beta)}}f^{(m)}(t)dt,$$
$$\text{for } m-1<\beta\leq m,\ m\in N, x>0,\ f\in C_{-1}^m$$

Properties:
For $f(x)\in C_\mu$, $\mu\geq -1$, $\alpha,\beta\geq 0$ and $\gamma>-1$, [6] we have

(i) $J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x)$,

(ii) $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$,

(iii) $J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{(\alpha+\gamma)}$.

Lemma 2.2. *If $m-1<\alpha\leq m$, $m\in N$ and $f\in C_\mu^m$, $\mu\geq -1$, then*

$$D_*^\alpha J^\alpha f(x)=f(x)$$
$$J^\alpha D_*^\alpha f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{x^k}{k!},x>0.$$

3. The Fractional Adomian Decomposition Method (FADM)

To demonstrate Time Fractional ADM to solve p-KdV, we consider following time fractional p-KdV equation-

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}+\varepsilon\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\mu\frac{\partial^3 w(x,t)}{\partial^3 x}=0,$$

$0<\alpha\leq 1, t>0$, initial condition: $w(x,0)=f(x)$.
Operating with the operator J^α on both sides of equation, we have

$$J^\alpha\left[\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}+\varepsilon\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\mu\frac{\partial^3 w(x,t)}{\partial^3 x}\right]=0,$$

$0<\alpha\leq 1, t>0$. Now, we decompose the unknown function $w(x,t)$ into sum of an infinite number of components given by the decomposition series

$$w(x,t)=\sum_{n=0}^{\infty}w_n(x,t) \tag{3.1}$$

The nonlinear terms $Nu(x,t)$ are decomposed in the following form:

$$Nw(x,t)=\sum_{n=0}^{\infty}A_n \tag{3.2}$$

where the Adomian polynomial can be determined as follows:

$$A_n=\frac{1}{n!}\left[\frac{d^n N}{d\lambda^n}(\sum_{k=0}^{n}\lambda^k u_k)\right]_{\lambda=0} \tag{3.3}$$

where A_n is called Adomian polynomial and that can be easily calculated by Mathematica software. Substituting the decomposition series and using lemma (2.1), we get

$$\sum_{n=0}^{\infty}w_n(x,t)=\sum_{k=0}^{m-1}\frac{\partial^k w(x,0)}{\partial t^k}\frac{t^k}{k!}-J^\alpha\left[\mu\sum_{n=0}^{\infty}D_x^3 w_n(x,t)-\varepsilon\sum_{n=0}^{\infty}A_n\right],\ x>0 \tag{3.4}$$

The components $w_n(x,t)$, $n\geq 0$ of the solution $w(x,t)$ can be recursively determined by using the relation as follows:

$$w_0(x,t)=w(x,0)=f(x) \tag{3.5}$$

$$w_{n+1}(x,t)=-J^\alpha\left[\mu D_x^3 w_n(x,t)-\varepsilon A_n\right],\ x>0 \tag{3.6}$$

where each component can be determined by using the preceding components and we can obtain the solution in a series form by calculating the components $w_n(x,t)$, $n \geq 0$. Finally, we approximate the solution $w(x,t)$ by the truncated series.

$$\phi_N(x,t) = \sum_{n=0}^{N-1} w_n(x,t)$$
$$\lim_{N\to\infty} \phi_N = w(x,t)$$

Theorem 3.1 (Uniqueness Theorem [20]). *Consider time fractional p-KdV as follow. Taking $\varepsilon = 1$ and $\mu = 1$, we have*

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} + \left(\frac{\partial w(x,t)}{\partial x}\right)^2 + \frac{\partial^3 w(x,t)}{\partial^3 x} = 0,$$

$0 < \alpha \leq 1, t > 0$, *tinitial condition:* $w(x,0) = f(x)$.
The equation has a unique solution whenever $0 < \gamma < 1$ *where*

$$\gamma = \frac{(C_1+C_2)t^\alpha}{\Gamma\alpha+1}.$$

Proof. Let $X = (C(I), \|.\|)$ be the Banach space of all continuous functions on $I = [0,T]$ with norm $\|w(t)\| = \max_{t\in I} \mid w(t) \mid$. We define a mapping M : $X \to X$, where $M(w(t)) = f(x) - J^\alpha N(w(t)) - J^\alpha D(w(t))$
N(w(t)) denotes nonlinear term and D(w(t)) denotes dispersive term. Also nonlinear term $N(w(t))$is Lipschitzian i.e.
$\mid N(w) - N(p) \mid \leq C_1 \mid w - p \mid$, where C_1 is Lipschitz constant.
Let $w, w' \in X$, we have-

$$\begin{aligned}
&\| M(w) - M(w') \| \\
&= \max_{t\in I} \mid -J^\alpha N(w(t)) - J^\alpha D(w(t) + J^\alpha N(w(t)) + J^\alpha D(w(t) \mid \\
&= \max_{t\in I} \mid -J^\alpha (Dw - Dw') - J^\alpha (Nw - Nw') \mid \\
&= \max_{t\in I} \mid J^\alpha (Dw - Dw') + J^\alpha (Nw - Nw') \mid \\
&\leq \max_{t\in I} \mid J^\alpha (Dw - Dw') \mid + \mid J^\alpha (Nw - Nw') \mid
\end{aligned}$$

Now suppose $D(w(t))$is also Lipschitzian i.e.
$\mid D(w) - D(p) \mid \leq C_2 \mid w - p \mid$, where C_2 is Lipschitz constant.
Therefore-

$$\begin{aligned}
\| M(w) - M(w') \| &\leq \max_{t\in I}(C_1 J^\alpha \mid w - w' \mid + C_2 J^\alpha \mid w - w' \mid) \mid) \\
&\leq (C_1 + C_2) \| w - w' \| \frac{t^\alpha}{\Gamma\alpha+1}
\end{aligned}$$

$$\| M(w) - M(w') \| \leq \gamma \| w - w' \|, where\ \gamma = \frac{(C_1+C_2)t^\alpha}{\Gamma\alpha+1}$$

Under the condition $0 < \gamma < 1$, the mapping is contraction, therefore by Banach fixed point theorem for contraction, there exist a unique solution to equation. □

Theorem 3.2 (Convergence Theorem [20,3]). *Let Q_n be the n^{th} partial sum, i.e.*

$$Q_n = \sum_{i=0}^{n} w_i(x,t)$$

Proof. we shall prove that (Q_n) is a Cauchy sequence in Banach space X.

$$\begin{aligned}
&\| Q_{n+p} - Q_n \| = \max_{t\in I} \mid Q_{n+p} - Q_n \mid = \max_{t\in I} \Big| \sum_{i=n+1}^{n+p} w_i(x,t) \Big| \\
&= \max_{t\in I} \Big| -J^\alpha \sum_{i=n+1}^{n+p} Dw_{i-1}(x,t) - J^\alpha \sum_{i=n+1}^{n+p} A_{i-1}(x,t) \Big| \\
&= \max_{t\in I} |J^\alpha DQ_{n+p-1} - DQ_{n-1} + J^\alpha NQ_{n+p-1} - NQ_{n-1}| \\
&\leq \max_{t\in I} J^\alpha (|(DQ_{n+p-1} - DQ_{n-1}|) + \max_{t\in I} J^\alpha |NQ_{n+p-1} - NQ_{n-1}| \\
&\leq C_2 \max_{t\in I} J^\alpha (|(Q_{n+p-1} - Q_{n-1}|) + C_1 \max_{t\in I} J^\alpha |Q_{n+p-1} - Q_{n-1}| \\
&\leq (C_1 + C_2) \frac{t^\alpha}{\Gamma\alpha+1} \| Q_{n+p-1} - Q_{n-1} \|
\end{aligned}$$

$$\| Q_{n+p} - Q_n \| \leq \gamma \| Q_{n+p-1} - Q_{n-1} \|$$
$$\text{where } \gamma = (C_1 + C_2)\frac{t^\alpha}{\Gamma\alpha+1}$$
$$\| Q_{n+p} - Q_n \| \leq \gamma \| Q_{n+p-1} - Q_{n-1} \|$$

Similarly we have

$$\begin{aligned}
\| Q_{n+p} - Q_n \| &\leq \gamma^2 \| Q_{n+p-2} - Q_{n-2} \| \\
&\vdots \\
&\leq \gamma^n \| Q_p - Q_0 \| \\
&\leq \gamma^n \| Q_1 - Q_0 \|,\ for p = 1 \\
&\leq \gamma^n \| w_1 \|
\end{aligned}$$

Now for $n > m$, where $n, m \in N$,

$$\begin{aligned}
&\| Q_n - Q_m \| \\
&\leq \| Q_{m+1} - Q_m \| + \| Q_{m+2} - Q_{m+1} \| + \cdots + \| Q_n - Q_{n-1} \| \\
&\leq (\gamma^m + \gamma^{m+1} + \cdots + \gamma^{n-1}) \| w_1 \| \\
&\leq \gamma^m \left[\frac{1-\gamma^{n-m}}{1-\gamma}\right] \| w_1 \|
\end{aligned}$$

Since $0 < \gamma < 1$, then $1 - \gamma^{n-m} < 1$, so we have,

$$\| Q_n - Q_m \| \leq \frac{\gamma^m}{1-\gamma} \| w_1 \|$$

Since $w(t)$ is bounded, therefore $\| w_1 \| < \infty$

$$\lim_{n\to\infty} \| Q_n - Q_m \| \to 0$$

Hence (Q_n) is a Cauchy sequence in X. Therefore the solution is convergent.
In the next section, we illustrate some examples and their solutions are represented graphically by mathematica software. □

4. Applications

Consider following time fractional linear partial differential equation:

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}-6\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\frac{\partial^3 w(x,t)}{\partial x^3}=0$$
$$\text{initial condition}: w(x,0)=tanhx,\ 0<\alpha\le 1$$

The operator form of the above equation can be written as

$$L_t^\alpha w(x,t)-6\left(D_x w(x,t)\right)^2-D_x^3 w(x,t)=0$$

initial condition: $w(x,0)=tanhx,\ 0<\alpha\le 1$.
Using equation 3.5 and 3.6, we have

$$\begin{aligned}
w_0(x,t)&=w(x,0)\\
&=tanhx\\
w_1(x,t)&=-J^\alpha D_x^3 w_0(x,t)+6J^\alpha A_0\\
&=(8sech^4x-4sech^2xtanh^2x)\frac{t^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

$$\begin{aligned}
w_2(x,t)=&-J^\alpha D_x^3 w_1(x,t)+6J^\alpha A_1\\
=&(864sech^4xtanh^3x-1152sech^6xtanhx\\
&-96sech^8xtanhx+544sech^6xtanh^3x\\
&-256sech^4xtanh^5x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\vdots
\end{aligned}$$

Therefore, the series solution for the IBVP is given by

$$w(x,t)=w_0(x,t)+w_1(x,t)+w_2(x,t)+w_3(x,t)+....$$

Substituting values of components in above equation, we get the solution as follow

$$\begin{aligned}
w(x,t)=&tanhx+(8sech^4x-4sech^2xtanh^2x)\frac{t^\alpha}{\Gamma(\alpha+1)}\\
&+(864sech^4xtanh^3x-1152sech^6xtanhx\\
&-96sech^8xtanhx+544sech^6xtanh^3x\\
&-256sech^4xtanh^5x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...
\end{aligned}$$

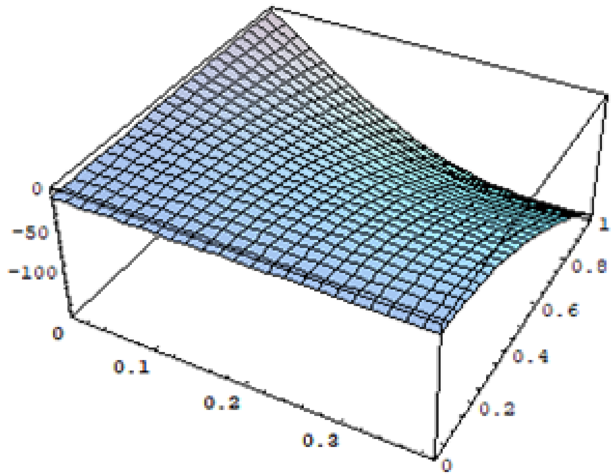

Fig. 1 : Graphical presentation of time fractional p-KdV equation with $\alpha=0.9$

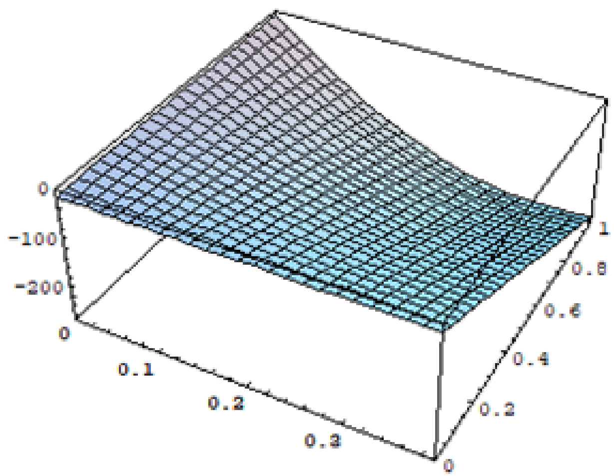

Fig. 2 : Graphical presentation of time fractional p-KdV equation with $\alpha=0.5$

Test Problem (ii): Consider following time fractional linear partial differential equation:

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha}-6\left(\frac{\partial w(x,t)}{\partial x}\right)^2+\frac{\partial^3 w(x,t)}{\partial x^3}=0$$

initial condition: $w(x,0)=1-3tanh^2\left(\frac{x}{2}+1\right),\ 0<\alpha\le 1$.
The operator form of the above equation can be written as

$$L_t^\alpha w(x,t)-6\left(D_x w(x,t)\right)^2-D_x^3 w(x,t)=0$$

initial condition: $w(x,0)=1-3tanh^2\left(\frac{x}{2}+1\right),\ 0<\alpha\le 1$.
Using equation 3.5 and 3.6 and considering $\mu=1$, $\varepsilon=1$,we

have

$$\begin{aligned} w_0(x,t) &= w(x,0) \\ &= 1-3tanh^2\left(\frac{x}{2}+1\right) \\ w_1(x,t) &= -J^\alpha D_x^3 w_0(x,t) + 6J^\alpha A_0 \\ &= \left[6-6tanh\left(\frac{x}{2}+1\right) - 36tanh^2\left(\frac{x}{2}+1\right)\right. \\ &+ 15tanh^3\left(\frac{x}{2}+1\right) + 54tanh^4\left(\frac{x}{2}+1\right) \\ &\left.- 9tanh^5\left(\frac{x}{2}+1\right)\right]\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\vdots \end{aligned}$$

Therefore, the series solution for the IBVP is given by

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots.$$

Substituting values of components in above equation, we get the solution as follow

$$\begin{aligned} w(x,t) = 1-3tanh^2\left(\frac{x}{2}+1\right) + \left[6-6tanh\left(\frac{x}{2}+1\right)\right. \\ - 36tanh^2\left(\frac{x}{2}+1\right) + 15tanh^3\left(\frac{x}{2}+1\right) \\ \left.+ 54tanh^4\left(\frac{x}{2}+1\right) - 9tanh^5\left(\frac{x}{2}+1\right)\right]\frac{t^\alpha}{\Gamma(\alpha+1)} + \ldots \end{aligned}$$

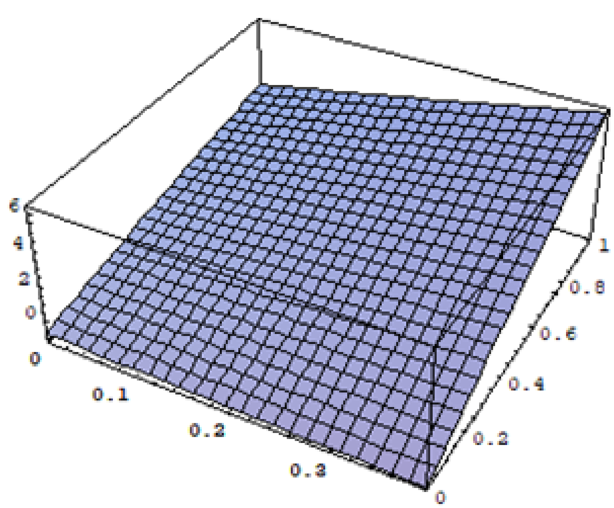

Fig. 3 : Graphical presentation of time fractional p-KdV equation with $\alpha = 0.9$

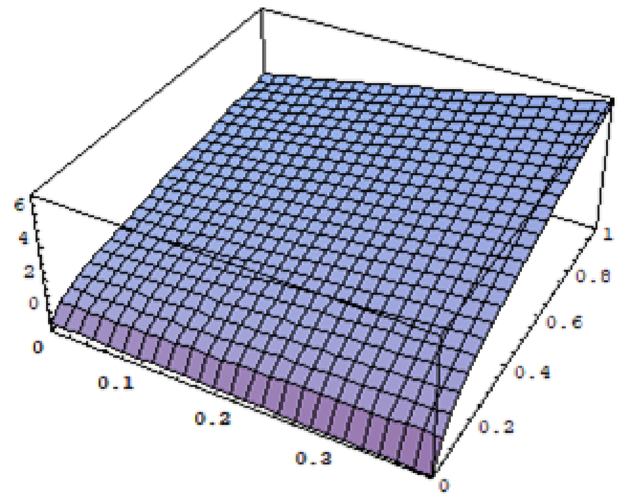

Fig. 4 : Graphical presentation of time fractional p-KdV equation with $\alpha = 0.5$

5. Conclusion

1. Time fractional p-KdV equation is solved by using ADM and it is found that ADM is very efficient and powerful technique to find solution of nonlinear fractional partial differential equation.

2. The obtained results demonstrate the reliability of the algorithm and its wider applicability to linear and nonlinear fractional partial differential equations.

3. We also developed uniqueness theorem and convergence theorem for the solution of time fractional p-KdV equation.

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