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Some notions on nano binary continuous

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Abstract

The purpose of this paper, we introduce and study the nano binary continuous functions in nano binary topological spaces. Also we introduce some nano binary continuous functions and the relationships between these functions are also studied.

Keywords

N_B-continuous, $N_B\alpha$-continuous, N_B semi continuous, N_B pre continuous, $N_B\beta$- continuous.

AMS Subject Classification

03E72.

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Contents

1. Introduction

M. Lellis Thivagar [1] introduced the concept of nano topological space with respect to a subset X of a universe U. S. Nithyanantha Jothi and P. Thangavelu [2] introduced the concept of binary topological spaces. By combining these two concepts Dr. G. Hari Siva Annam and J. Jasmine Elizabeth [3] introduced nano binary topological spaces. Njastad [4], Levine [5] and Mashhour et al [6] respectively introduced the notions of α-open, semi-open and pre-open sets. In this paper we have introduced a new class of functions on nano binary topological spaces called nano binary continuous functions and derived their characterizations in terms of nano binary closed, nano binary closure and nano binary interior. Also the relationships between some nano binary continuous functions are studied.

2. Preliminaries

Definition 2.1. *[3] Let (U_1,U_2) be a non-empty finite set of objects called the universe and R be an equivalence relation on (U_1,U_2) named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U_1,U_2,R) is said to be the approximation space. Let $(X_1,X_2) \subseteq (U_1,U_2)$*

1) *The lower approximation of (X_1,X_2) with respect to R is the set of all objects, which can be for certain classified as (X_1,X_2) with respect to R and it is denoted by $L_R(X_1,X_2)$.*
   That is, $L_R(X_1,X_2) = \cup_{(x_1,x_2)\in(U_1,U_2)}\{R(x_1,x_2) : R(x_1, x_2) \subseteq (X_1,X_2)\}$ Where $R(x_1,x_2)$ denotes the equivalence class determined by (x_1,x_2)

2) *The upper approximation of (X_1,X_2) with respect to R is the set of all objects, which can be possibly classified as (X_1,X_2) with respect to R and it is denoted by $U_R(X_1,X_2)$.*
   That is, $U_R(X_1,X_2) = \cup_{(x_1,x_2)\in(U_1,U_2)}\{R(x_1,x_2) : R(x_1, x_2) \cap (X_1,X_2) \neq \phi\}$

3) *The boundary region of (X_1,X_2) with respect to R is the set of all objects, which can be classified neither as (X_1,X_2) nor as not $-(X_1,X_2)$ with respect to R and it is denoted by $B_R(X_1,X_2)$.*
   That is, $B_R(X_1,X_2) = U_R(X_1,X_2) - L_R(X_1,X_2)$

Proposition 2.2. *[3] If (U_1,U_2,R) is an approximation space and $(X_1,X_2),(Y_1,Y_2) \subseteq (U_1,U_2)$, then*

1) $L_R(X_1,X_2) \subseteq (X_1,X_2) \subseteq U_R(X_1,X_2)$
2) $L_R(\phi,\phi) = U_R(\phi,\phi) = (\phi,\phi)$ and $L_R(U_1,U_2) = U_R(U_1,U_2) = (U_1,U_2)$
3) $U_R((X_1,X_2)\cup(Y_1,Y_2)) = U_R(X_1,X_2)\cup U_R(Y_1,Y_2)$
4) $U_R((X_1,X_2)\cap(Y_1,Y_2)) \subseteq U_R(X_1,X_2)\cap U_R(Y_1,Y_2)$
5) $L_R((X_1,X_2)\cup(Y_1,Y_2)) \supseteq L_R(X_1,X_2)\cup L_R(Y_1,Y_2)$
6) $L_R((X_1,X_2)\cap(Y_1,Y_2)) \subseteq L_R(X_1,X_2)\cap L_R(Y_1,Y_2)$
7) $L_R(X_1,X_2) \subseteq L_R(Y_1,Y_2)$ and $U_R(X_1,X_2) \subseteq U_R(Y_1,Y_2)$ whenever $(X_1,X_2) \subseteq (Y_1,Y_2)$
8) $U_R(X_1,X_2)^C = [L_R(X_1,X_2)]^C$ and $L_R(X_1,X_2)^C = [U_R(X_1,X_2)]^C$
9) $U_RU_R(X_1,X_2) = L_RU_R(X_1,X_2) = U_R(X_1,X_2)$
10) $L_RL_R(X_1,X_2) = U_RL_R(X_1,X_2) = L_R(X_1,X_2)$

Definition 2.3. *[3] Let (U_1,U_2) be the universe, R be an equivalence on (U_1,U_2) and $\tau_R(X_1,X_2) = \{(U_1,U_2),(\phi,\phi), L_R(X_1,X_2),U_R(X_1,X_2),B_R(X_1,X_2)\}$ where $(X_1,X_2) \subseteq (U_1,U_2)$. Then by the property $R(X_1,X_2)$ satisfies the following axioms*
1.(U_1,U_2) and $(\phi,\phi) \in (X_1,X_2)$
2. The union of the elements of any sub collection of $\tau_R(X_1,X_2)$ is in $\tau_R(X_1,X_2)$
3. The intersection of the elements of any finite sub collection of $\tau_R(X_1,X_2)$ is in $\tau_R(X_1,X_2)$. That is, $\tau_R(X_1,X_2)$ is a topology on (U_1,U_2) called the nano binary topology on (U_1,U_2) with respect to (X_1,X_2). We call $(U_1,U_2,\tau_R(X_1,X_2))$ as the nano binary topological spaces. The elements of $\tau_R(X_1,X_2)$ are called as nano binary open sets and it is denoted by N_B open sets. Their complement is called N_B closed sets.

Definition 2.4. *[3] If $(U_1,U_2,\tau_R(X_1,X_2))$ is a nano binary topological spaces with respect to (X_1,X_2) and if $(H_1,H_2) \subseteq (U_1,U_2)$, then the nano binary interior of (H_1,H_2) is defined as the union of all N_B open subsets of (A_1,A_2) and it is defined by $N_B^\circ(H_1,H_2)$*
That is, $N_B^\circ(H_1,H_2)$ is the largest N_B open subset of (H_1,H_2). The nano binary closure of (H_1,H_2) is defined as the intersection of all N_B closed sets containing (H_1,H_2) and it is denoted by $\overline{N_B}(H_1,H_2)$.
That is, $\overline{N_B}(H_1,H_2)$ is the smallest N_B closed set containing (H_1,H_2).

Definition 2.5. *[3] Let $(U_1,U_2,\tau_R(X_1,X_2))$ be a nano binary topological space and $(A_1,A_2),(B_1,B_2) \in P(X_1)\times P(X_2)$ then*

i) $N_B^\circ(\phi,\phi) = (\phi,\phi), \overline{N_B}(\phi,\phi) = (\phi,\phi)$
ii) $N_B^\circ(U_1,U_2) = (U_1,U_2), \overline{N_B}(U_1,U_2) = (U_1,U_2)$
iii) $N_B^\circ(A_1,A_2) \subseteq (A_1,A_2) \subseteq \overline{N_B}(A_1,A_2)$
iv) $(A_1,A_2) \subseteq (B_1,B_2)$ implies $N_B^\circ(A_1,A_2) \subseteq N_B^\circ(B_1,B_2)$ and $\overline{N_B}(A_1,A_2) \subseteq \overline{N_B}(B_1,B_2)$
v) $N_B^\circ((A_1,A_2)\cap(B_1,B_2)) \subseteq N_B^\circ(A_1,A_2)\cap N_B^\circ(B_1,B_2)$
vi) $\overline{N_B}((A_1,A_2)\cap(B_1,B_2)) \subseteq \overline{N_B}(A_1,A_2)\cap\overline{N_B}(B_1,B_2)$
vii) $N_B^\circ((A_1,A_2)\cup(B_1,B_2)) \supseteq N_B^\circ(A_1,A_2)\cup N_B^\circ(B_1,B_2)$
viii) $\overline{N_B}((A_1,A_2)\cup(B_1,B_2)) \supseteq \overline{N_B}(A_1,A_2)\cup\overline{N_B}(B_1,B_2)$
ix) $N_B^\circ(N_B^\circ(A_1,A_2)) \subseteq N_B^\circ(A_1,A_2)$
x) $\overline{N_B}(\overline{N_B}(A_1,A_2)) \supseteq \overline{N_B}(A_1,A_2)$
xi) $N_B^\circ(\overline{N_B}(A_1,A_2)) \supseteq N_B^\circ(A_1,A_2)$
xii) $\overline{N_B}(N_B^\circ(A_1,A_2)) \subseteq \overline{N_B}(A_1,A_2)$

Definition 2.6. *[3] A subset (H_1,H_2) of a nano binary topological spaces $(U_1,U_2,\tau_R(X_1,X_2))$ is called*
1. $N_B\alpha$-open if $(H_1,H_2) \subseteq N_B^\circ(\overline{N_B}(N_B^\circ(H_1,H_2)))$.
2. N_B semi- open set if $(H_1,H_2) \subseteq \overline{N_B}(N_B^\circ(H_1,H_2))$
3. N_B pre - open set if $(H_1,H_2) \subseteq N_B^\circ(\overline{N_B}(H_1,H_2))$
The complements of the above mentioned sets are called their respective N_B closed sets.

Result 2.7. *[3] 1. Every N_B open sets is $N_B\alpha$-open.*
2. Every $N_B\alpha$-open is N_B semi -open.
3. Every $N_B\alpha$-open is N_B pre -open.

3. Nano Binary Continuity

Definition 3.1. *Let $(U_1,U_2,\tau_R(X_1,X_2))$ and $(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ be nano binary topological spaces. Then a mapping $f:(U_1,U_2,\tau_R(X_1,X_2)) \to (V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is nano binary continuous on (U_1,U_2) if the inverse image of every N_B open in (V_1,V_2) is N_B open in (U_1,U_2) and it is denoted by N_B- continuous.*

Example 3.2. *Let $U_1 = \{a,b,c\}, U_2 = \{1,2\}$ with $(U_1,U_2)/R = \{(\{a,b\},\{2\}),(\{c\},\{1\})\}$ and $(X_1,X_2) = (\{a,c\},\{1\})$. Then $\tau_R(X_1,X_2) = \{(\phi,\phi),(U_1,U_2),(\{c\},\{1\}),(\{a,b\},\{2\})\}$. Let $V_1 = \{x,y,z\}, V_2 = \{e,f\}$ with $(V_1,V_2)/R' = \{(\{x,z\},\{e\}),(\{y\},\{f\})\}$ and $(Y_1,Y_2) = (\{x,y\},\{f\})$. Then $\tau_{R'}(Y_1,Y_2) = \{(\phi,\phi),(V_1,V_2),(\{x,z\},\{e\}),(\{y\},\{f\})\}$. Define $f:(U_1,U_2,\tau_R(X_1,X_2)) \to (V_1,V_2,\tau_{R'}(Y_1,Y_2))$ as $f(\{a\},\{1\}) = (\{x\},\{f\}), f(\{a\},\{2\}) = (\{x\},\{e\}), f(\{b\},\{1\}) = (\{z\},\{f\}), f(\{b\},\{2\}) = (\{z\},\{e\}), f(\{c\},\{1\}) = (\{y\},\{f\}), f(\{c\},\{2\}) = (\{y\},\{e\})$. Then $f^{-1}(\{y\},\{f\}) = (\{c\},\{1\})$ and $f^{-1}(\{x,z\},\{e\}) = (\{a,b\},\{2\})$. That is the inverse image of every N_B open in (V_1,V_2) is N_B open in (U_1,U_2). Therefore, f is N_B-continuous.*

Theorem 3.3. *A function $f:(U_1,U_2,\tau_R(X_1,X_2)) \to (V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous if and only if the inverse image of every N_B closed in (V_1,V_2) is N_B closed in (U_1,U_2).*

Proof. Let f be N_B- continuous and (F_1,F_2) be N_B closed in (V_1,V_2). That is $(V_1,V_2)-(F_1,F_2)$ is N_B open in (V_1,V_2). Since f is N_B- continuous, $f^{-1}((V_1,V_2)-(F_1,F_2))$ is N_B open in (U_1,U_2). That is $(U_1,U_2)-f^{-1}(F_1,F_2)$ is N_B open in (U_1,U_2). Therefore, $f^{-1}(F_1,F_2)$ is N_B closed in (U_1,U_2). Thus the inverse image of every N_B closed in (V_1,V_2) is N_B closed in (U_1,U_2), if f is N_B- continuous on (U_1,U_2).
Conversely, suppose the inverse image of every N_B closed is N_B closed. Let (G_1,G_2) be N_B open in (V_1,V_2). Then $(V_1,V_2)-(G_1,G_2)$ is N_B closed in (V_1,V_2). Then $f^{-1}((V_1,V_2)-(G_1,G_2))$ is N_B closed in (U_1,U_2). That is $(U_1,U_2)-f^{-1}(G_1,G_2)$ is N_B closed in (U_1,U_2). Therefore, $f^{-1}(G_1,G_2)$ is N_B open in (U_1,U_2). Thus the inverse image of every N_B open in (V_1,V_2) is N_B open in (U_1,U_2). Hence f is N_B- continuous on (U_1,U_2). □

Theorem 3.4. *A function $f:(U_1,U_2,\tau_R(X_1,X_2))\rightarrow(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous if and only if $f(\overline{N_B}(A_1,A_2))\subseteq\overline{N_B}(f(A_1,A_2))$ for every subset (A_1,A_2) of (U_1,U_2).*

Proof. Let f be N_B- continuous and $(A_1,A_2)\subseteq(U_1,U_2)$. Then $f(A_1,A_2)\subseteq(V_1,V_2)$. Therefore, $\overline{N_B}(f(A_1,A_2))$ is N_B closed in (V_1,V_2) Since f is N_B- continuous, $f^{-1}(\overline{N_B}(f(A_1,A_2))$ is N_B closed in (U_1,U_2). Since $f(A_1,A_2)\subseteq\overline{N_B}(f(A_1,A_2)),(A_1,A_2)\subseteq f^{-1}(\overline{N_B}(f(A_1,A_2))$. Thus $f^{-1}(\overline{N_B}(f(A_1,A_2))$ is a N_B closed set containing (A_1,A_2). Since $\overline{N_B}(A_1,A_2)$ is the smallest N_B closed set containing $(A_1,A_2),\overline{N_B}(A_1,A_2)\subseteq f^{-1}(\overline{N_B}(f(A_1,A_2)))$. That is, $f(\overline{N_B}(A_1,A_2))\subseteq\overline{N_B}(f(A_1,A_2))$. Conversely, suppose $f(\overline{N_B}(A_1,A_2))\subseteq\overline{N_B}(f(A_1,A_2))$ for every subset (A_1,A_2) of (U_1,U_2). If (F_1,F_2) is N_B closed in (V_1,V_2), since $f^{-1}(F_1,F_2)\subseteq(U_1,U_2)$, $f(\overline{N_B}(f^{-1}(F_1,F_2)))\subseteq\overline{N_B}(f(f^{-1}(F_1,F_2)))\subseteq\overline{N_B}(F_1,F_2)$. That is, $\overline{N_B}(f^{-1}(F_1,F_2))\subseteq f^{-1}(\overline{N_B}(F_1,F_2))=f^{-1}(F_1,F_2)$, since (F_1,F_2) is N_B closed. Thus $\overline{N_B}(f^{-1}(F_1,F_2))\subseteq f^{-1}(F_1,F_2)$. But $f^{-1}(F_1,F_2)\subseteq\overline{N_B}(f^{-1}(F_1,F_2))$. Therefore, $\overline{N_B}(f^{-1}(F_1,F_2))=f^{-1}(F_1,F_2)$. Therefore, $f^{-1}(F_1,F_2)$ is N_B closed in (U_1,U_2) for every N_B closed set (F_1,F_2) in (V_1,V_2). That is, f is N_B- continuous. □

Remark 3.5. *If a function $f:(U_1,U_2,\tau_R(X_1,X_2))\rightarrow(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous, then $f(\overline{N_B}(A_1,A_2))$ is not necessarily equal to $\overline{N_B}(f(A_1,A_2))$ where $(A_1,A_2)\subseteq(U_1,U_2)$.*

Example 3.6. *Let $U_1=\{1,2,3\},U_2=\{a,b,c\}$ with $(U_1,U_2)/R=\{(\{1,2\},\{a\}),(\{4\},\{b\}),(\{3\},\{c\})\}$ and $(X_1,X_2)=(\{2,4\},\{b,c\})$. Then $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{4\},\{b\}),(\{1,2,3\},\{a,c\})\}$. The closed sets are $(\phi,\phi),(U_1,U_2),(\{4\},\{b\}),(\{1,2,3\},\{a,c\})$. Let $V_1=\{e,f,g,h\},V_2=\{x,y,z\}$ with $(V_1,V_2)/R'=\{(\{e\},\{z\}),\ (\{f,g\},\{y\}),(\{h\},\{x\})\}$ and $(Y_1,Y_2)=(\{g,h\},\{x\})$. Then $\tau_{R'}(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{h\},\{x\}),(\{f,g,h\},\{x,y\}),(\{f,g\},\{y\})\}$. The closed sets are $(\phi,\phi),(V_1,V_2),(\{e,f,g\},\{y,z\}),(\{e\},\{z\}),(\{e,h\},\{x,z\})$. Define $f:(U_1,U_2,\tau_R(X_1,X_2))\rightarrow(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ as $f(\{1\},\{a\})=(\{e\},\{z\}),f(\{1\},\{b\})=(\{h\},\{y\}),f(\{1\},\{c\})=(\{e\},\{x\}),f(\{2\},\{a\})=(\{h\},\{x\}),f(\{2\},\{b\})=(\{h\},\{y\}),f(\{2\},\{c\})=(\{h\},\{z\}),f(\{3\},\{a\})=(\{e\},\{x\}),f(\{3\},\{b\})=(\{h\},\{y\}),f(\{3\},\{c\})=(\{h\},\{z\}),f(\{4\},\{a\})=(\{f\},\{x\}),f(\{4\},\{b\})=(\{g\},\{y\}),f(\{4\},\{c\})=(\{g\},\{z\})$. Then $f^{-1}(\{h\},\{x\})=(\{1,2,3\},\{a,c\}),f^{-1}(\{f,g,h\},\{x,y\})=((U_1,U_2))$ and $f^{-1}(\{f,g\},\{y\})=(\{4\},\{b\})$. That is the inverse image of every N_B open in (V_1,V_2) is N_B open in (U_1,U_2). Therefore, f is N_B- continuous. Let $(A_1,A_2)=(\{4\},\{b\})\subseteq(U_1,U_2)$. Then $f(\overline{N_B}(\{4\},\{b\}))=f(\{4\},\{b\})=(\{f,g\},\{y\})$. But $\overline{N_B}(f(\{4\},\{b\}))=\overline{N_B}(\{f,g\},\{y\})=(\{e,f,g\},\{y,z\})$. Thus $f(\overline{N_B}(A_1,A_2))\neq\overline{N_B}(f(A_1,A_2))$, even though f is N_B- continuous. That is, equality does not hold in the previous theorem when f is N_B- continuous.*

Definition 3.7. *The basis for the nano binary topology $\tau_R(X_1,X_2)$ with respect to (X_1,X_2) is given by $\mathscr{B}_R(X_1,X_2)=\{(U_1,U_2),L_R(X_1,X_2),B_R(X_1,X_2)\}$.*

Theorem 3.8. *Let (X_1,X_2) be a set. Let $\mathscr{B}$ be a basis for a nano binary topology $\tau_R(X_1,X_2)$. Then $\tau_R(X_1,X_2)$ equals the collection of all unions of elements of $\mathscr{B}$.*

Proof. Given a collection of elements of $\mathscr{B}$, they are also elements of $\tau_R(X_1,X_2)$. Because $\tau_R(X_1,X_2)$ is a topology, their union is in $\tau_R(X_1,X_2)$. Conversely, given $(U_1,U_2)\in\tau_R(X_1,X_2)$, choose for each $(x_1,x_2)\in(U_1,U_2)$ an element (B_{x_1},B_{x_2}) of $\mathscr{B}$ such that $(x_1,x_2)\in(B_{x_1},B_{x_2})\subset(U_1,U_2)$. Then $(U_1,U_2)=\cup_{(x_1,x_2)\in(U_1,U_2)}(B_{x_1},B_{x_2})$, so (U_1,U_2) equals a union of elements of $\mathscr{B}$. □

Theorem 3.9. *Let $(U_1,U_2,\tau_R(X_1,X_2))$ and $(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ be nano binary topological spaces where $(X_1,X_2)\subseteq(U_1,U_2)$ and $(Y_1,Y_2)\subseteq(V_1,V_2)$. Then $\tau_{R'}(Y_1,Y_2)=\{(V_1,V_2),(\phi,\phi),L_{R'}(Y_1,Y_2),U_{R'}(Y_1,Y_2),B_{R'}(Y_1,Y_2)\}$ and its basis is given by $\mathscr{B}_{R'}=\{(V_1,V_2),L_{R'}(Y_1,Y_2),B_{R'}(Y_1,Y_2)\}$. A function $f:(U_1,U_2,\tau_R(X_1,X_2))\rightarrow(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous if and only if the inverse image of every member of $\mathscr{B}_{R'}$ is N_B open in (U_1,U_2).*

Proof. Let f be N_B- continuous on (U_1,U_2). Let $(B_1,B_2)\in\mathscr{B}_{R'}$. Then (B_1,B_2) is N_B open in (V_1,V_2). That is, $(B_1,B_2)\in\tau_{R'}(Y_1,Y_2)$. Since f is N_B- continuous, $f^{-1}(B_1,B_2)\in\tau_R(X_1,X_2)$. That is, the inverse image of every member of $\mathscr{B}_{R'}$ is N_B open in (U_1,U_2). Conversely, suppose the inverse image of every member of $\mathscr{B}_{R'}$ is N_B open in (U_1,U_2). Let (G_1,G_2) be N_B open in (V_1,V_2). Then $(G_1,G_2)=\cup\{(B_1,B_2):(B_1,B_2)\in\mathscr{B}_1\}$, where $\mathscr{B}_1\subset\mathscr{B}_{R'}$. Then $f^{-1}(G_1,G_2)=f^{-1}(\cup\{(B_1,B_2):(B_1,B_2)\in\mathscr{B}_1\})=\cup\{f^{-1}(B_1,B_2):(B_1,B_2)\in\mathscr{B}_1\}$, where each $f^{-1}(B_1,B_2)$ is N_B open in (U_1,U_2) and hence their union $f^{-1}(G_1,G_2)$ is N_B open in (U_1,U_2). Thus f is N_B- continuous on (U_1,U_2). □

Theorem 3.10. *A function $f:(U_1,U_2,\tau_R(X_1,X_2))\rightarrow(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous if and only if $\overline{N_B}(f^{-1}(B_1,B_2))\subseteq f^{-1}(\overline{N_B}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2).*

Proof. If f is N_B- continuous and $(B_1,B_2)\subseteq(V_1,V_2)$, then $\overline{N_B}(B_1,B_2)$ is N_B closed in (V_1,V_2) and hence $f^{-1}(\overline{N_B}(B_1,B_2))$ is N_B closed in (U_1,U_2). Therefore, $\overline{N_B}(f^{-1}(\overline{N_B}(B_1,B_2)))=f^{-1}(\overline{N_B}(B_1,B_2))$. Since $(B_1,B_2)\subseteq\overline{N_B}(B_1,B_2),f^{-1}(B_1,B_2)\subseteq f^{-1}(\overline{N_B}(B_1,B_2))$. Therefore, $\overline{N_B}(f^{-1}(B_1,B_2))\subseteq\overline{N_B}(f^{-1}(\overline{N_B}(B_1,B_2)))=f^{-1}(\overline{N_B}(B_1,B_2))$. That is, $\overline{N_B}(f^{-1}(B_1,B_2))\subseteq$

$f^{-1}(\overline{N_B}(B_1,B_2))$. Conversely, suppose $\overline{N_B}(f^{-1}(B_1,B_2)) \subseteq f^{-1}(\overline{N_B}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2). Let (B_1,B_2) be N_B closed in (V_1,V_2). Then $\overline{N_B}(B_1,B_2)=(B_1,B_2)$. By our assumption $\overline{N_B}(f^{-1}(B_1,B_2)) \subseteq f^{-1}(\overline{N_B}(B_1,B_2)) = f^{-1}(B_1,B_2)$. Therefore, $\overline{N_B}(f^{-1}(B_1,B_2)) \subseteq f^{-1}(B_1,B_2)$. But $f^{-1}(B_1,B_2) \subseteq \overline{N_B}(f^{-1}(B_1,B_2))$. Therefore, $\overline{N_B}(f^{-1}(B_1,B_2))$ $= f^{-1}(B_1,B_2)$. That is, $f^{-1}(B_1,B_2)$ is N_B closed in (U_1,U_2) for every N_B closed (B_1,B_2) in (V_1,V_2). Therefore, f is N_B-continuous on (U_1,U_2). □

Theorem 3.11. *A function $f:(U_1,U_2,\tau_R(X_1,X_2)) \to (V_1,V_2,\tau_{R'}(Y_1,Y_2))$ is N_B- continuous on (U_1,U_2) if and only if $f^{-1}(N_B^{\circ}(B_1,B_2)) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2).*

Proof. Let f be N_B- continuous and $(B_1,B_2) \subseteq (V_1,V_2)$. Then $N_B^{\circ}(B_1,B_2)$ is N_B open in $(V_1,V_2,\tau_{R'}(Y_1,Y_2))$. Therefore, $(f^{-1}(N_B^{\circ}(B_1,B_2)))$ is N_B open in $(U_1,U_2,\tau_R(X_1,X_2))$. That is, $f^{-1}(N_B^{\circ}(B_1,B_2)) = N_B^{\circ}(f^{-1}(N_B^{\circ}(B_1,B_2)))$. Also $N_B^{\circ}(B_1,B_2) \subseteq (B_1,B_2)$ implies $f^{-1}(N_B^{\circ}(B_1,B_2)) \subseteq f^{-1}(B_1,B_2)$. Therefore, $N_B^{\circ}(f^{-1}(N_B^{\circ}(B_1,B_2))) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$ That is, $(f^{-1}(N_B^{\circ}(B_1,B_2))) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$. Conversely, suppose $f^{-1}(N_B^{\circ}(B_1,B_2)) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2). If (B_1,B_2) is N_B open in (V_1,V_2), then $N_B^{\circ}(B_1,B_2)=(B_1,B_2)$. Now $f^{-1}(N_B^{\circ}(B_1,B_2)) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$. That is, $f^{-1}(B_1,B_2) \subseteq N_B^{\circ}(f^{-1}(B_1,B_2))$. But $N_B^{\circ}(f^{-1}(B_1,B_2)) \subseteq f^{-1}(B_1,B_2)$. Therefore, $f^{-1}(B_1,B_2) = N_B^{\circ}(f^{-1}(B_1,B_2))$. Thus $f^{-1}(B_1,B_2)$ is N_B open in (U_1,U_2) for every N_B open subset (B_1,B_2) of (V_1,V_2). Therefore, f is N_B- continuous. □

Remark 3.12. *The above theorem equality does not hold if f is N_B- continuous.*

Example 3.13. *Let $U_1=\{a,b,c\}, U_2=\{1,2\}$ with $(U_1,U_2)/R=\{(\{a\},\{2\}),(\{b,c\},\{1\})\}$. Let $(X_1,X_2)=(\{a,b\},\{1,2\})$. Then $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{a\},\{2\}),(\{b,c\},\{1\})\}$. Let $V_1=\{x,y,z\}, V_2=\{e,f\}$ with $(V_1,V_2)/R'=\{(\{x,y\},\{f\}),(\{z\},\{e\})\}$. Let $(Y_1,Y_2)=(\{z\},\{e\})$. Then $\tau_{R'}(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{z\},\{e\})\}$. The N_B closed sets in (U_1,U_2) are $(\phi,\phi),(U_1,U_2),(\{a\},\{2\}),(\{b,c\},\{1\})$. The N_B closed sets in (V_1,V_2) are $(\phi,\phi),(V_1,V_2),(\{x,y\},\{f\})$. Define $f:(U_1,U_2,\tau_R(X_1,X_2)) \to (V_1,V_2,\tau_{R'}(Y_1,Y_2))$ as $f(\{a\},\{1\})=(\{z\},\{f\}), f(\{a\},\{2\})=(\{z\},\{e\}), f(\{b\},\{1\})=(\{x\},\{f\}), f(\{b\},\{2\})=(\{x\},\{e\}), f(\{c\},\{1\})=(\{y\},\{f\}), f(\{c\},\{2\})=(\{y\},\{e\})$. Then $f^{-1}(\{z\},\{e\})=(\{a\},\{2\})$ and $f^{-1}(V_1,V_2)=(U_1,U_2)$. That is the inverse image of every N_B open in (V_1,V_2) is N_B open in (U_1,U_2). Therefore, f is N_B- continuous. Let $(B_1,B_2)=(\{z\},\{e\}) \subseteq (V_1,V_2)$. Then $f^{-1}(\overline{N_B}(B_1,B_2))=f^{-1}(\overline{N_B}(\{z\},\{e\}))=f^{-1}(V_1,V_2)=(U_1,U_2)$. And $\overline{N_B}(f^{-1}(B_1,B_2))=\overline{N_B}(f^{-1}(\{z\},\{e\}))=\overline{N_B}(\{a\},\{2\})=(\{a\},\{2\})$. Thus $f^{-1}(\overline{N_B}(B_1,B_2)) \neq \overline{N_B}(f^{-1}(B_1,B_2))$. Also let $(A_1,A_2)=(\{x,y\},\{f\}) \subseteq (V_1,V_2)$. $f^{-1}(N_B^{\circ}(A_1,A_2)) = f^{-1}(N_B^{\circ}(\{x,y\},\{f\}))=f^{-1}(\phi,\phi)=(\phi,\phi), N_B^{\circ}(f^{-1}(A_1,A_2))=N_B^{\circ}(\{b,c\},\{1\})=(\{b,c\},\{1\})$ Thus $f^{-1}(N_B^{\circ}(A_1,A_2)) \neq N_B^{\circ}(f^{-1}(A_1,A_2))$*

Theorem 3.14. *Let $(U_1,U_2,\tau_R(X_1,X_2))$ and $(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ be nano binary topological spaces where $(X_1,X_2) \subseteq (U_1,U_2)$ and $(Y_1,Y_2) \subseteq (V_1,V_2)$, then for any function $f:(U_1,U_2) \to (V_1,V_2)$, the following statements are equivalent:*
1) f is N_B- continuous
2) The inverse image of every N_B closed in (V_1,V_2) is N_B closed in (U_1,U_2).
3) $f(\overline{N_B}(A_1,A_2)) \subset \overline{N_B}(f(A_1,A_2))$ for every subset (A_1,A_2) of (U_1,U_2).
4) The inverse image of every member of B_R' is N_B open in (U_1,U_2).
5) $\overline{N_B}(f^{-1}(B_1,B_2)) \subset f^{-1}(\overline{N_B}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2).
6) $f^{-1}(N_B^{\circ}(B_1,B_2)) \subset N_B^{\circ}(f^{-1}(B_1,B_2))$ for every subset (B_1,B_2) of (V_1,V_2).
The proof of the theorem follows from the previous theorems.

4. Weaker Forms of Nano Binary Continuous

Definition 4.1. *A subset (H_1,H_2) of a nano binary topological spaces $(U_1,U_2,\tau_R(X_1,X_2))$ is called $N_B\beta$-open if $(H_1,H_2) \subseteq \overline{N_B}(N_B^{\circ}(\overline{N_B}(H_1,H_2)))$.*

Result 4.2. *1) Every N_B pre –open is $N_B\beta$-open.*

2) Every N_B semi –open is $N_B\beta$-open.

3) Every N_B open is N_B semi -open.

4) Every N_B open is N_B pre -open.

5) Every $N_B\alpha$-open is $N_B\beta$-open.

6) Every N_B open is $N_B\beta$-open.

Proof. 1) If (H_1,H_2) is a N_B pre - open, then $(H_1,H_2) \subseteq N_B^{\circ}(\overline{N_B}(H_1,H_2))$. Now let us remind the monotonicity property, $(H_1,H_2) \subseteq \overline{N_B}(H_1,H_2)$. By making use of this we obtain that $\overline{N_B}(H_1,H_2) \subseteq \overline{N_B}(N_B^{\circ}(\overline{N_B}(H_1,H_2)))$ which obviously yields that $(H_1,H_2) \subseteq \overline{N_B}(N_B^{\circ}(\overline{N_B}(H_1,H_2)))$. Therefore, (H_1,H_2) is a $N_B\beta$-open. Hence every N_B pre-open is $N_B\beta$-open.

2) If (H_1,H_2) is a N_B semi- open, then $(H_1,H_2) \subseteq \overline{N_B}(N_B^{\circ}(H_1,H_2))$. Since $(H_1,H_2) \subseteq \overline{N_B}\ (H_1,H_2)$ implies $(H_1,H_2) \subseteq \overline{N_B}(N_B^{\circ}(\overline{N_B}(H_1,H_2)))$. Therefore, (H_1,H_2) is a $N_B\beta$-open. Hence every N_B semi-open is $N_B\beta$-open.

Proof 3, 4, 5 and 6 follows from the result 2.7 and above results. □

Note 4.3. *The converse of the above result is not true by the following example.*

Example 4.4. *Let $U_1=\{a,b,c,d,e\}$ and $U_2=\{1,2,3,4\}$ with $(U_1,U_2)/R=\{(\{a,b\},\{2\}),(\{c\},\{4\}),(\{d\},\{3\}),(\{e\},\{1\})\}$ and $(X_1,X_2)=\{(\{a,c,d\},\{2,3,4\})\}$. Then the nano binary topology $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{c,d\},\{3,4\}),(\{a,b,c,d\},\{2,3,4\}),(\{a,b\},\{2\})\}$.*
The N_B closed sets $=\{(\phi,\phi),(U_1,U_2),(\{a,b,e\},\{1,2\}),(\{e\},\{1\}),(\{c,d,e\},\{1,3,4\})\}$. In this example,

1. $(\{a,b,e\},\{2\})$ *is* $N_B\beta$*- open but not* N_B *pre – open.*
2. $(\{a,b,c,d,e\},\{2,3\})$ *is* $N_B\beta$*- open but not* N_B *semi- open.*

Definition 4.5. *Let* $(U_1,U_2,\tau_R(X_1,X_2))$ *and* $(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ *be nano binary topological spaces. A mapping* $f:(U_1,U_2,\tau_R(X_1,X_2))\to(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ *is said to be*

1) $N_B\alpha$*-continuous if* $(f^{-1}(B_1,B_2))$ *is* $N_B\alpha$*-open in* (U_1,U_2) *for every* N_B *open* (B_1,B_2) *in* (V_1,V_2)*.*

2) N_B *semi -continuous if* $(f^{-1}(B_1,B_2))$ *is* N_B *semi-open in* (U_1,U_2) *for every* N_B *open* (B_1,B_2) *in* (V_1,V_2)*.*

3) N_B *pre -continuous if* $(f^{-1}(B_1,B_2))$ *is* N_B *pre-open in* (U_1,U_2) *for every* N_B *open* (B_1,B_2) *in* (V_1,V_2)*.*

4) $N_B\beta$*-continuous if* $(f^{-1}(B_1,B_2))$ *is* $N_B\beta$*-open in* (U_1,U_2) *for every* N_B *open* (B_1,B_2) *in* (V_1,V_2)*.*

Result 4.6. 1) *Every* N_B*- continuous is* $N_B\alpha$*-continuous.*

2) *Every* $N_B\alpha$*- continuous is* N_B *semi - continuous.*

3) *Every* $N_B\alpha$*- continuous is* N_B *pre - continuous.*

4) *Every* N_B *pre – continuous is* $N_B\beta$*- continuous.*

5) *Every* N_B *semi – continuous is* $N_B\beta$*- continuous.*

Since result 2.7 and result 4.2, the above result is true but none of these implications is reversible as shown by the following examples

Example 4.7. *Let* $U_1=\{a,b,c\},U_2=\{1,2\}$ *with* $(U_1,U_2)/R=\{(\{a,b\},\{2\}),(\{c\},\{1\})\}$*. Let* $(X_1,X_2)=(\{b\},\{2\})$*. Then* $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{a,b\},\{2\})\}$*. Let* $V_1=\{x,y,z\},V_2=\{e,f\}$ *with* $(V_1,V_2)/R'=\{(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Let* $(Y_1,Y_2)=(\{z\},\{e\})$*. Then* $\tau_{R'}(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{x,z\},\{e\})\}$*. Define* $f:(U_1,U_2,\tau_R(X_1,X_2))\to(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ *as* $f(\{a\},\{1\})=(\{x\},\{e\}),f(\{a\},\{2\})=(\{x\},\{e\}),f(\{b\},\{1\})=(\{z\},\{e\}),f(\{b\},\{2\})=(\{z\},\{e\}),f(\{c\},\{1\})=(\{y\},\{e\}),f(\{c\},\{2\})=(\{y\},\{e\})$*.*
Here $(B_1,B_2)=(\{x,z\},\{e\}),f^{-1}(\{x,z\},\{e\})=(\{a,b\},\{1,2\})$ *then f is* $N_B\alpha$*- continuous but* N_B *continuous.*

Example 4.8. *Let* $U_1=\{a,b,c,d,e\},U_2=\{1,2,3,4\}$ *with* $(U_1,U_2)/R=\{(\{a,b\},\{2\}),(\{c\},\{4\}),(\{d\},\{3\}),(\{e\},\{1\})\}$*. Let* $(X_1,X_2)=(\{a,c,d\},\{2,3,4\})$*. Then* $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{c,d\},\{3,4\}),(\{a,b,c,d\},\{2,3,4\}),(\{a,b\},\{2\})\}$*. Let* $V_1=\{x,y,z\},V_2=\{e,f\}$ *with* $(V_1,V_2)/R'=\{(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Let* $(Y_1,Y_2)=(\{x,y\},\{f\})$*. Then* $\tau_{R'}(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Define* $f:(U_1,U_2,\tau_R(X_1,X_2))\to(V_1,V_2,\tau_{R'})(Y_1,Y_2))$ *as* $f(\{a\},\{1\})=(\{x\},\{e\}),f(\{a\},\{2\})=(\{x\},\{e\}),f(\{a\},\{3\})=(\{x\},\{f\}),f(\{a\},\{4\})=(\{x\},\{f\}),f(\{b\},\{1\})=(\{z\},\{e\}),f(\{b\},\{2\})=(\{z\},\{e\}),f(\{b\},\{3\})=(\{z\},\{f\}),f(\{b\},\{4\})=(\{z\},\{f\}),f(\{c\},\{1\})=(\{y\},\{e\}),f(\{c\},\{2\})=(\{y\},\{e\}),f(\{c\},\{3\})=(\{y\},\{f\}),\ f(\{c\},\{4\})=(\{y\},\{f\}),f(\{d\},\{1\})=(\{y\},\{e\}),\ f(\{d\},\{2\})=(\{y\},\{e\}),f(\{d\},\{3\})=(\{y\},\{f\}),\ f(\{d\},\{4\})=(\{y\},\{f\}),f(\{e\},\{1\})=(\{z\},\{e\}),f(\{e\},\{2\})=(\{z\},\{e\}),f(\{e\},\{3\})=(\{x\},\{f\}),f(\{e\},\{4\})=(\{x\},\{f\})$ *Here* $(B_1,B_2)=(\{x,z\},\{e\}),f^{-1}(\{x,z\},\{e\})=(\{a,b,e\},\{1,2\})$*. Then f is* N_B *semi-continuous but* $N_B\alpha$*- continuous. Also f is* $N_B\beta$*-continuous but not* N_B *pre- continuous.*

Example 4.9. *Let* $U_1=\{a,b,c,d,e\},U_2=\{1,2,3,4\}$ *with* $(U_1,U_2)/R=\{(\{a,b\},\{2\}),(\{c\},\{4\}),(\{d\},\{3\}),(\{e\},\{1\})\}$*. Let* $(X_1,X_2)=(\{a,c,d\},\{2,3,4\})$*. Then* $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{c,d\},\{3,4\}),(\{a,b,c,d\},\{2,3,4\}),(\{a,b\},\{2\})\}$*. Let* $V_1=\{x,y,z\},V_2=\{e,f\}$ *with* $(V_1,V_2)/R'=\{(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Let* $(Y_1,Y_2)=(\{x,y\},\{f\})$*. Then* $\tau'_R(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Define* $f:(U_1,U_2,\tau_R(X_1,X_2))\to(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ *as* $f(\{a\},\{1\})=(\{y\},\{f\}),f(\{a\},\{2\})=(\{y\},\{f\}),f(\{a\},\{3\})=(\{y\},\{e\}),f(\{a\},\{4\})=(\{y\},\{f\}),f(\{b\},\{1\})=(\{y\},\{f\}),f(\{b\},\{2\})=(\{y\},\{f\}),f(\{b\},\{3\})=(\{y\},\{e\}),f(\{b\},\{4\})=(\{y\},\{f\}),f(\{c\},\{1\})=(\{x\},\{f\}),f(\{c\},\{2\})=(\{x\},\{f\}),f(\{c\},\{3\})=(\{x\},\{e\}),f(\{c\},\{4\})=(\{x\},\{f\}),f(\{d\},\{1\})=(\{z\},\{f\}),f(\{d\},\{2\})=(\{z\},\{f\}),f(\{d\},\{3\})=(\{z\},\{e\}),f(\{d\},\{4\})=(\{z\},\{f\}),f(\{e\},\{1\})=(\{y\},\{f\}),f(\{e\},\{2\})=(\{y\},\{f\}),f(\{e\},\{3\})=(\{y\},\{e\}),f(\{e\},\{4\})=(\{y\},\{f\})$*. Here* $(B_1,B_2)=(\{x,z\},\{e\}),f^{-1}(\{x,z\},\{e\})=(\{c,d\},\{3\})$*. Then f is* N_B *pre- continuous but not* $N_B\alpha$*- continuous.*

Example 4.10. *Let* $U_1=\{a,b,c\},U_2=\{1,2\}$ *with* $(U_1,U_2)/R=\{(\{a,b\},\{2\}),(\{c\},\{1\})\}$*. Let* $(X_1,X_2)=(\{a,c\},\{1\})$*. Then* $\tau_R(X_1,X_2)=\{(\phi,\phi),(U_1,U_2),(\{a,b\},\{2\}),(\{c\},\{1\})\}$*. Let* $V_1=\{x,y,z\},V_2=\{e,f\}$ *with* $(V_1,V_2)/R'=\{(\{x,z\},\{e\}),(\{y\},\{f\})\}$*. Let* $(Y_1,Y_2)=(\{x,y\},\{f\})$*. Then* $\tau_{R'}(Y_1,Y_2)=\{(\phi,\phi),(V_1,V_2),(\{y\},\{f\}),(\{x,z\},\{e\})\}$*. Define* $f:(U_1,U_2,\tau_R(X_1,X_2))\to(V_1,V_2,\tau_{R'}(Y_1,Y_2))$ *as* $f(\{a\},\{1\})=(\{x\},\{e\}),f(\{a\},\{2\})=(\{x\},\{e\}),f(\{b\},\{1\})=(\{z\},\{e\}),f(\{b\},\{2\})=(\{z\},\{e\}),f(\{c\},\{1\})=(\{y\},\{e\}),f(\{c\},\{2\})=(\{y\},\{e\})$*. Here* $(B_1,B_2)=(\{x,z\},\{e\}),f^{-1}(\{x,z\},\{e\})=(\{a,b\},\{1,2\})$*. Then f is* $N_B\beta$*- continuous but not* N_B *semi- continuous.*

5. Conclusion

In this paper, we have defined N_B continuous functions in nano binary topological spaces and their characterizations were studied. Also we have explored some continuous functions in nano binary topological spaces and their features were discussed. The characterizations of weaker forms of N_B continuous functions are in future process.

References

[1] M. Lellis Thivagar and Carmel Richard, *On Nano forms of weakly open sets, International Journal of Mathematics and Statistics Invention*, 1(1) 2013, 31-37

[2] S. Nithyanantha Jothi and P. Thangavelu, *On Binary Topological Spaces, Pacific – Asian Journal of Mathematics.*

[3] Dr. G. Hari Siva Annam and J. Jasmine Elizabeth, *Cognition of Nano Binary Topological Spaces, Global*

Journal of Pure and Applied Mathematics, 6(2019), 1045-1054.

[4] Z. PAWLAK, *Rough Sets, International Journal of Computer and Information Sciences*, 11(1982), 341-356.

[5] M. LEVINE, *Semi-open sets and Semi continuity in topological spaces. Amer. Math. Monthly* 70(1963), 36-41.

[6] A. S. MASHHOUR, M. E. ABD EL - MONSEF AND I. A. HASANEIN, *on pre-topological spaces, BULL. MATH. de la Soc. Math. de la R. S. Roumanie* 28(76)(1984), 39-45.

[7] A. A. NASEF, A. I. AGGOUR, S. M. DARWESH ,*On some classes of nearly open sets in nano topological spaces, Journal of the Egyptian Mathematical Society* (2016)24, 585-589.

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