Malaya Journal of Matematik, Vol. 9, No. 1, 608-611, 2021
https://doi.org/10.26637/MJM0901/0104

Incomparable integer quintuple in arithmetic progression with prominent condition

P. Sandhya[1*] and V. Pandichelvi[2]

Abstract
In this document, the incomparable integer quintuple (p,q,r,s,t) in such a way that the components with the renowned property in algebra named as arithmetic progression with the postulation that the addition of three consecutive terms shows a perfect square is established.

Keywords
Diophantine m-tuple, quintuple in arithmetic progression.

[1] *Department of Mathematics, SRM Trichy Arts and Science College, Trichy-621105, Tamil Nadu, India.*
[2] *PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Trichy-620019, Tamil Nadu, India.*
***Corresponding author**: [1] sandhyaprasad2684@gmail.com
Article History: Received **27** December **2020**; Accepted **10** February **2021**

Contents

1. Introduction

Let n be an integer. A set of positive integers $(a_1,a_2,a_3,\dots a_m)$ is said to have the property $D(n)$ if a_ia_j+n is a perfect square for all $1\le i<j\le m$; such a set is called a Diophantine $m-$ tuple $[1-6]\cdot$ In [7], the authors were evaluated the triplesin Arithmetic Progression $(a-d,a,a+d)$ such that2 a - $d=\alpha^2, 2a+d=\beta^2, 2a=\chi^3$ and $2a-d=\alpha^2, 2a+d=\beta^2, 2a=\chi^4$ where a and d be two non- zero distinct integer.In [8] , triples were procured in Arithmetic Progression such that the sum of any two is a perfect square. In [9], the authors found the triples in Arithmetic Progression $(a-d,a,a+d)$ such that each of the expression $a^2-ad, 2a+d, 2a$ is a perfect square. In [10], the authors found the quadruples of the form (x,y,z,w) where the elements arein Arithmetic Progression satisfying the conditions $x+y=\alpha^2 z+w=\beta^2$ and $x+y+z+w=\gamma^3$.

In this manuscript, three unlike integer quintuples with the elements in Arithmetic Progression rewarding the condition that that the sum of three consecutive integers indicates a perfect square is acquired.

2. Course of action for survey

Presume that p,q,r,s,t be five non-zero separate integers such that the elements in the quintuple (p,q,r,s,t) materialize in Arithmetic Progression.
To symbolize this proclamation, let a and d be two non-zero integers such that $p=a-2d, q=a-d, r=a, s=a+d, t=a+2d$.
For the exploration of the perception of the manuscript, imagine the sum of three consecutive elements in the already assumed quintuple is a square of an integer. The above declaration is replicated by the subsequent equations

$$p+q+r=3a-3d=\varphi^2 \tag{2.1}$$

$$q+r+s=3a=\eta^2 \tag{2.2}$$

$$r+s+t=3a+3d=\chi^2 \tag{2.3}$$

Addition of (2.1) and (2.3) endow with the proportion that

$$a=\frac{\varphi^2+\chi^2}{6} \tag{2.4}$$

Similarly, subtraction of (2.1) from (2.3) bestow as in the succeeding fraction

$$d=\frac{\chi^2-\varphi^2}{6} \tag{2.5}$$

Elucidationof (2.2) and (2.4) yields the following equation

$$\eta^2 = \frac{\varphi^2+\chi^2}{2} \tag{2.6}$$

To convert the above said value of η as in integer, launch the novel conversions

$$\eta = 3\lambda, \varphi = 6\mu, \chi = 6\omega \tag{2.7}$$

These translations imitate (2.5) and (2.6) as follows

$$d = 6\left(\omega^2-\mu^2\right) \tag{2.8}$$
$$\lambda^2 = 2\left(\mu^2+\omega^2\right) \tag{2.9}$$

The elements in the required quintuple are making into integers with the property looking for is portrayed by the three procedures as below.

Procedure 1: Decode the parameter λ as

$$\lambda = u^2+v^2$$

Then, the equation (2.9) can be altered by

$$\left(u^2+v^2\right)^2 = 2\left(\mu^2+\omega^2\right)$$
$$\Rightarrow (u+iv)^2(u-iv)^2 = (1+i)(1-i)(\mu+i\omega)(\mu-i\omega)$$

By escalating and balancing positive terms and thenequating real and imaginary parts on both sides, the resulting equations are revealed by

$$\mu-\omega = u^2-v^2$$
$$\mu+\omega = 2uv$$

Resolving the above equations the most plausible values of μ and ω are demonstrated by

$$\mu = \frac{1}{2}\left(u^2-v^2+2uv\right)$$
$$\omega = \frac{1}{2}\left(v^2-u^2+2uv\right)$$

The parametric values of λ, μ and ω in integers are created by selecting the options of $u = 2U$ and and $v = 2V$ as follows

$$\lambda = 4\left(U^2+V^2\right)$$
$$\mu = 2\left(U^2-V^2+2UV\right)$$
$$\omega = 2\left(V^2-U^2+2UV\right)$$

The replacement of the above value of λ in (2.7), endow with the value of η as

$$\eta = 12\left(U^2+V^2\right)$$

According to (2.2) and (2.8), the components in the essential quintuple are offered by

$$a = 48\left(U^2+V^2\right)^2$$
$$d = 192UV\left(V^2-U^2\right)$$

Subsequently, the necessary guintuple in which the elements form an Arithmetic progression is discovered by

$$(p,q,r,s,t) = \Big\{48\left(U^2+V^2\right)^2 - 384UV\left(V^2-U^2\right)$$
$$48\left(U^2+V^2\right)^2 - 192UV\left(V^2-U^2\right), 48\left(U^2+V^2\right)^2$$
$$48\left(U^2+V^2\right)^2 + 192UV\left(V^2-U^2\right), 48\left(U^2+V^2\right)^2$$
$$+384UV\left(V^2-U^2\right)\Big\}$$

2.1 Logical postulation is checked for certain values of U and V as tabulated below

Table 1.

U	V	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
2	1	(3504,2352,1200, 48,-1104)	84^2	60^2	12^2
5	7	(-59712,101568, 262848,424128, 585408)	552^2	888^2	1128^2
1	3	(-4416,192,4800, 9408,14016)	24^2	120^2	168^2

Procedure 2:

The same conversion of $\lambda = u^2+v^2$ supplies the alternative appearance of (2.9) as

$$(u+iv)^2(u-iv)^2 = \frac{(7+i)(7-i)}{25}(\mu+i\omega)(\mu-i\omega)$$

Replicate the same course of action as mentioned in procedure (2.1), the corresponding values of μ and ω satisfying the double equations $7\mu-\omega = 5\left(u^2-v^2\right), \mu+7\omega = 10uv$ are appraised by

$$\mu = \frac{1}{10}\left(7\left(u^2-v^2\right)+2uv\right)$$
$$\omega = \frac{1}{10}\left(v^2-u^2+14uv\right)$$

The chances of λ, μ and ω in integers by picking $u = 10U$ and $v = 10V are$ produced by

$$\lambda = 100\left(U^2+V^2\right)$$
$$\mu = 10\left(7U^2-7V^2+2UV\right)$$
$$\omega = 10\left(V^2-U^2+14UV\right)$$

Renovate the value of λ in (2.7), the value of η is calculated by

$$\eta = 300\left(U^2+V^2\right)$$

In sight of (2.2) and (2.8), the equivalent choices of a and d are pointed out by

$$a = 30000\left(U^2+V^2\right)^2$$
$$d = -4800\left(6U^4+6V^4+7U^3V-7UV^3-36U^2V^2\right)$$

Hence, the needed quintuple with desired property is exposed by

$$\begin{aligned}(p,q,r,s,t)=\Big\{&30000\left(U^2+V^2\right)^2+9600(6U^4+6V^4\\&+7U^3V-7UV^3-36U^2V^2),30000\left(U^2+V^2\right)^2\\&+4800\left(6U^4+6V^4+7U^3V-7UV^3-36U^2V^2\right)\\&30000\left(U^2+V^2\right)^2,30000\left(U^2+V^2\right)^2\\&-4800\left(6U^4+6V^4+7U^3V-7UV^3-36U^2V^2\right),\\&30000\left(U^2+V^2\right)^2-9600(6U^4+6V^4\\&+7U^3V-7UV^3-36U^2V^2)\Big\}\end{aligned}$$

2.2 Presumption is verified for definite values of U and V in the table given below

Table 2.

U	V	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
0	1	(87600,58800, 30000,1200, -27600)	420^2	300^2	60^2
1	2	(-56400,346800, 750000,1153200, 1556400)	1020^2	1500^2	1860^2
1	1	(-110400,4800,120000, 235200,350400)	120^2	600^2	840^2

Procedure 3:
Commencement of the fresh renovation $\lambda=2\,\mathrm{Ain}(9)$ declare the same equation as

$$\begin{aligned}2\,A^2=&\mu^2+\omega^2\\&\Rightarrow A^2-\mu^2=\omega^2-A^2\\&\Rightarrow(A+\mu)(A-\mu)=(\omega+A)(\omega-A)\\&\Rightarrow\left(1+\frac{\mu}{A}\right)\left(1-\frac{\mu}{A}\right)=\left(\frac{\omega}{A}+1\right)\left(\frac{\omega}{A}-1\right)\end{aligned}\tag{2.10}$$

Again, make use of the transformations $\frac{\mu}{A}=\theta$, $\frac{\omega}{A}=\rho$ in (2.10) produces the proportion as

$$\frac{(1+\theta)}{(1+\rho)}=\frac{(\rho-1)}{(1-\theta)}=\frac{m}{n},n\neq 0\tag{2.11}$$

Hereafter, calculate the values of θ and ρ from (2.11) by the process of cross multiplication and then substituting these values in the ultimate transformation, it is determined by

$$\begin{aligned}A=&m^2+n^2\Rightarrow\lambda=2\left(m^2+n^2\right)\\\mu=&m^2+2mn-n^2\\\omega=&n^2+2mn-m^2\end{aligned}\tag{2.12}$$

Interpretation (2.2) and (2.7) offers the relevant values of a and d as presented in the equations scripted below.

$$\begin{aligned}a=&12\left(m^2+n^2\right)^2\\d=&48mn\left(n^2-m^2\right)\end{aligned}$$

Hence, the essential quintuple in which the elements in Arithmetic progression is rendered by

$$\begin{aligned}&(p,q,r,s,t)\\&=\Big\{12\left(m^2+n^2\right)^2-96mn\left(n^2-m^2\right),12\left(m^2+n^2\right)^2\\&-48mn\left(n^2-m^2\right),12\left(m^2+n^2\right)^2,12\left(m^2+n^2\right)^2\\&+48mn\left(n^2-m^2\right),\ 12\left(m^2+n^2\right)^2+96mn\left(n^2-m^2\right)\Big\}\end{aligned}$$

2.3 Supposition is authenticated for specific values of U and V in the following table

Table 3.

m	n	(p, q, r, s, t)	p+q+r	q+r+s	r+s+t
2	1	(-276,12,300, 588,876)	6^2	30^2	42^2
5	7	(-14928,25392, 65712,106032, 146352)	276^2	444^2	564^2
1	3	(-1104,48,1200, 2352,3504)	12^2	60^2	84^2

The emerging C software shows verification of the numerical samples:

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
void main()
{
char ch;
clrscr();
do {
long long int x,u,v,m,n;
long long int U,V,M,N,a,d,p,q,r,s,t,A,B,C,E,F,G;
printf("\n Enter the case 1 or 2 or 3\n");
scanf("%lld",&x);
switch(x)
{
case 1:
printf("\n Enter integer values for u and v \n");
scanf("%lld%lld",&u,&v);
U=u*u;
V=v*v;
a=48*(U+V)*(U+V);
d=192*u*v*(V-U);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
case 2:
printf("\n Enter integer values for u and v \n");
scanf("%lld%lld",&u,&v);
U=u*u;
V=v*v;
```

```
a=30000*(U+V)*(U+V);
d=-4800*(6*U*U+6*V*V+7*U*u*v-7*u*v*V-36*U*V);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
case 3:
printf("\n interger values for m and n \n");
scanf("%lld%lld",&m,&n);
M=m*m;
N=n*n;
a=12*(M+N)*(M+N);
d=48*m*n*(N-M);
p=a-2*d;
q=a-d;
r=a;
s=a+d;
t=a+2*d;
break;
}
A=p+q+r;
B=q+r+s;
C=r+s+t;
E=sqrt(A);
F=sqrt(B);
G=sqrt(C);
printf("\n p+q+r=%lld=%lld2 \n q+r+s=%lld=%lld2\nr+s+t
=%lld =%lld2",A,E,B,F,C,G);
printf("\n Do you want to continue for different cases (y/n)?");
ch=getche();
}
while (ch=='y'——ch=='Y');
getch();
}
```

3. Conclusion

In this paper, an elegant integer quintuple (p,q,r,s,t) where the components make ensure in arithmetic progression with the conjecture that the sum of any three consecutive elements designates a perfect square is recognized. In this manner, one can search an integer quintuple (p,q,r,s,t) with elements in Geometric progression or Harmonic progression satisfying some other condition.

References

[1] Andre Weil, *Number Theory: An Approach through History*, From Hammurapito to Legendre, Birkahsuser, Boston, 1987.

[2] I.G.Bashmakova (ed.), *Diophantus of Alexandria,* Arithmetic's and the Book of Polygonal Numbers, Nauka, Moscow, 1974.

[3] A.F. Beardon, and M.N.Deshpande, Diophantine Triplets, *The Mathematical Gazette,* 86(2002), 258-260.

[4] L.E.Dickson, *History of the theory of Numbers,* Vol.2, Chelsea Publishing House, New York, 1966, 513-520.

[5] M.N.Deshpande, Families of Diophantine Triplets, *Bulletin of the Marathwada Mathematical Society,* 4(2003), 19-21.

[6] M.N.Deshpande, One Interesting Family of Diophantine Triplets, *Internet. J. Math. d.Sci.Tech.* 33(2002), 253-256.

[7] M. A. Gopalan, V. Geetha, V. Kiruthika, On Two Special Integer Triples in Arithmetic Progression, *Open Journal of Applied & Theoretical Mathematics (OJATM)*, 2(1)(2016), 01-07.

[8] M. A. Gopalan, V. Sangeetha, An Interesting Diophantine Problem, *Open Journal of Applied & Theoretical Mathematics (OJATM)*, 2(2)(2016), 42-47.

[9] K. Meena, S. Vidhyalakshmi, M. A. Gopalan, S. Aarthy Thangam, Special Integer Quadruple in Arithmetic Progression, *International Journal of Recent Trends in Engineering & Research,* 3(5)(2017), 108-112.

[10] S. Vidhyalakshmi, A. Kavitha. M. A. Gopalan, Diophantine Problem on Integer Triple in Arithmetic Progression, *Transactions on Mathematics TM,* 2(2)(2016), 36-42.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆