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Classification of an exquisite diophantine 4-tuples bestow with an order

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Abstract

In this text, a pattern of spectacular Diophantine 3-tuples $(l,m,n),(m,n,o),(n,o,p)$ etc concerning Gnomonic number is appraised with the condition that the product of any two elements of them augmented by four is a perfect square. Also, the above pattern of 3-tuple is protracted to a pattern of 4 -tuples by manipulating a distinct formula for the property $\mathrm{D}(4)$.

Keywords

Integer sequence, Diophantine quadruples, Pell equation.

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1. Introduction

In [1], a set of positive integers $(b_1, b_2, \dots b_m)$ is called a Diophantine m -tuple if $b_i b_j + n$ is a perfect square for all $1 \le i < j \le m$ with property $D(n)$. In[2], the Greek mathematician Diaphanous of Alexandria first studied the problem of finding four numbers such that the product of any two of them increased by unity is a perfect square. He found a set of four positive rational numbers $\left(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right)$. with the property $D(1)$. However, the first set of four positive integers (1,3,8,120) with the above property. Euler found the infinite family of such set $(a, b, a+b+2r, 4r(r+a)(r+b))$, where $ab+1=r^2$. For an extensive review of various articles one may refer $[3,4,6-10]$.

In this paper, a nice-looking categorization of Diophantine 3 -tuples comprising gnomonic number with an appropriate property $D(4)$ is inspected Also, the extension of all these 3 -tuples to 4 -tuples is deliberated by using a singular formula for distinguished the property $D(4)$.

2. Technique of classification of diophantine 4 - tuple

Contemplate $l=gm(x), m=gm(x+2)$ be x^{th} and $(x+2)^{th}$ elements in the sequence of Gnomonic numbers respectively composed with an adequate condition that $lm+4=a^2$ where $gm(x)=2x-1$. For the convenience to treasure the third element in the 3 -tuple satisfying the requirement that the product of any two of them added with four is square of a positive integer, choose n be an additional non-zero integer with the ensuing statement that

$$ngm(x)+4 = b^2 \tag{2.1}$$

$$ngm(x+2)+4 = c^2 \tag{2.2}$$

Eliminating n from (2.1) and (2.2), the expression relating gnomonic numbers of dissimilar orders is detected by

$$\left(b^2+4\right) gm(x+2)+4gm(x)=c^2 gm(x) \tag{2.3}$$

Acquaint the resulting linear adjustments in (2.3)

$$b = \eta + gm(x)\delta \tag{2.4}$$

$$c = \eta + gm(x+2)\delta \tag{2.5}$$

As a result, the notorious second-degree equation so-called Pell equation is exposed by

$$\eta^2 = D\delta^2 + 4, \tag{2.6}$$

where $D=gm(x)gm(x+2)$.

Put on the basic solutions $\eta = gm(x+1)$ and $\delta = 1$ of (2.6) in (2.4) and (2.5), the values of b and c in terms of Gnomonic numbers are premeditated by

$$b = gm(x+1) + gm(x) \tag{2.7}$$

$$c = gm(x+1) + gm(x+2) \tag{2.8}$$

Applications of either (2.7) or (2.8) in (2.1) or (2.2) delivers the notable third element in the essential 3 -tuples

$$n = gm(x) + 2gm(x+1) + gm(x+2).$$

It is authenticated that $(gm(x), gm(x+2), gm(x) + 2gm(x+1) + gm(x+2))$ is a Diophantine 3 - tuple in which the multiplication of any two of them plus four is a square of a number in Z^+, the set of all positive integers.

In what follows the concept of discovering infinite number of such 3-tuples with specific order, let us pick o, p, q be different non-zero integers along with the subsequent proclamations that

$$om + 4 = d^2 \text{ and } on + 4 = e^2 \tag{2.9}$$

$$pn + 4 = f^2 \text{ and } po + 4 = g^2 \tag{2.10}$$

$$qo + 4 = h^2 \text{ and } qp + 4 = i^2 \tag{2.11}$$

Also, deliberate the equivalent transformations for $d, e; f, g; h, i$ as

$$d = \eta + m\delta \text{ and } e = \eta + n\delta$$
$$f = \eta + n\delta \text{ and } g = \eta + o\delta$$
$$h = \eta + o\delta \text{ and } i = \eta + p\delta$$

Following the similar steps as described above, the third member in the needed sequences are attained by

$$o = gm(x) + 4gm(x+1) + 4gm(x+2)$$
$$p = 4gm(x) + 12gm(x+1) + 9gm(x+2)$$
$$q = 9gm(x) + 30gm(x+1) + 25gm(x+2)$$

Thus, an outline of remarkable Diophantine 3 -tuplestaken in a definite order $(l,m,n), (m,n,o), (n,o,p)$ etc satisfying the graceful property D(4) are created by

$$(gm(x), gm(x+2), gm(x) + 4gm(x+1) + 4gm(x+2))$$
$$(gm(x+2), gm(x) + 4gm(x+1) + 4gm(x+2),$$
$$gm(x) + 4gm(x+1) + 4gm(x+2))$$
$$(gm(x) + 4gm(x+1) + 4gm(x+2),$$
$$gm(x) + 4gm(x+1) + 4gm(x+2),$$
$$9(gm(x) + 30gm(x+1) + 25gm(x+2)) \text{ etc.}$$

3. Extension of the pattern of diophantine 3-tuples into diophantine 4-tuples.

Before to develop such 4 -tuples, let us note the conjecture specified in [5], 'If $\{a,b,c,d\}$ is a $D(4)-$ quadruple such that $a < b < c < d$, then $d(a+b+c) + \frac{1}{2}\{abc + \alpha\beta\gamma\}$ where $ab + 4 = \alpha^2, bc + 4 = \beta^2, ac + 4 = \gamma^2$. Every Diophantine 3 -tuple (l,m,n) composed with certain property $D(4)$ can be extendedto a Diophantine 4 -tuple (l,m,n,u) in which the fourth component u sustaining the identical property is given by

$$u = (l+m+n) + \frac{1}{2}\{\text{lm}n + abc\} \tag{3.1}$$

where $lm + 4 = a^2$, $ln + 4 = b^2$, $mn + 4 = c^2$. Here, the essential fourth factor concerning Gnomonic number in the above 4 -tuple is calculated by

$$\begin{aligned} u = &(2gm(x) + gm(x+1) + 2gm(x+2)) \\ &+ \frac{1}{2}\{(\text{gm}(x)(gm(x+2)(gm(x) \\ &+ 2gm(x+1) + gm(x+2)) + ((gm(x+1)(gm(x) \\ &+ gm(x+1))(gm(x+1) + gm(x+2))\} \end{aligned}$$

In the place of the 4-tuple (l,m,n,u), consider the shape of 4 -tuples like $(m,n,o,v), (n,o,p,w), (o,p,q,z)$ etc and interpret the succeeding formulae to construct them obeying the significant condition $D(4)$

$$v = (m+n+o) + \frac{1}{2}(mno + cde) \tag{3.2}$$

$$w = (n+o+p) + \frac{1}{2}(nop + efg) \tag{3.3}$$

$$z = (o+p+q) + \frac{1}{2}(opq + ghi) \tag{3.4}$$

where $om + 4 = d^2$, $on + 4 = e^2$, $pn + 4 = f^2$, $po + 4 = g^2$, $qo + 4 = h^2$, $qp + 4 = i^2$. Accordingly, the consistent values of v, w, z are perceived that

$$\begin{aligned} v = &(2gm(x) + 6gm(x+1) + 6gm(x+2)) \\ &+ \frac{1}{2}\{gm(x+2)(gm(x) + 2gm(x+1) \\ &+ gm(x+2)(gm(x) + 4gm(x+1) + 4gm(x+2)) \\ &+ (gm(x+1) + gm(x+2))(gm(x+1) \\ &+ 2gm(x+2))(gm(x) + 3gm(x+1) + 2gm(x+2))\} \end{aligned}$$

$$\begin{aligned} w = &(6gm(x) + 18gm(x+1) + 14gm(x+2)) \\ &+ \frac{1}{2}\{((gm(x) + 2gm(x+1) \\ &+ gm(x+2))(gm(x) + 4gm(x+1) + 4gm(x+2)) \\ &\times (4gm(x) + 12gm(x+1) \\ &+ 9gm(x+2)) + (gm(x) + 3gm(x+1) + 2gm(x+2)) \\ &\times (2gm(x) + 5gm(x+1) \\ &+ 3gm(x+2))(2gm(x) + 7gm(x+1) + 6gm(x+2))\} \end{aligned}$$

Hence, it is determined that an innovative arrangement of 4-tuples (l,m,n,u) $(m,n,o,v), (n,o,p,w), (o,p,q,x)$ etc connecting distinct Gnomonic numbers are plotted such that the multiplication of any two elements improved by four is a square of a positive integer. The above process of receiving a peculiar form of Diophantine 4 -tuples with an enhancing property $D(4)$ is substantiated by the following MATLAB programming.

Matlab programming:

```
clear all;
close all;
clc;
g m=[ ]
for i=1: 1: 13
b(i)=2 * i-1
end
g m=[g m b]
for x=1: 1: 11
l=g m(x)
m=g m(x+2)
n=g m(x)+2 * g m(x+1)+g m(x+2)
o=g m(x)+4 * g m(x+1)+4 * g m(x+2)
p=4 * g m(x)+12 * g m(x+1)+9 * g m(x+2)
q=9 * g m(x)+30 * g m(x+1)+25 * g m(x+2)
a=g m(x+1)
b=g m(x)+g m(x+1)
c=g m(x+1)+g m(x+2)
d=g m(x+1)+2 * g m(x+2)
e=g m(x)+3 * g m(x+1)+2 * g m(x+2)
f=2 * g m(x)+5 * g m(x+1)+3 * g m(x+2)
g=2 * g m(x)+7 * g m(x+1)+6 * g m(x+2)
h=3 * g m(x)+11 * g m(x+1)+10 * g m(x+2)
i=6 * g m(x)+19 * g m(x+1)+15 * g m(x+2)
u=(l+m+n)+0.5 *((l * m * n)+(a * b * c))
v=(m+n+o)+0.5 *((m * n * 0)+(c * d * e))
w=(n+o+p)+0.5 *((n * 0 * p)+(e * f * g))
z=(o+p+q)+0.5 *((0 * p * q)+(g * h * i))
fprint f ('l = %d\n',l)
fprint f ('m = %d\n',m)
fprint f ('n = %d\n',n)
fprint f ('o = %d\n',0)
fprint f ('p = %d\n',p)
fprint f ('q = %d\n',q)
fprint f ('u = %d\n',u)
fprint f ('v = %d\n',v)
fprint f ('w = %d\n',w)
fprint f ('z = %d\n',z)
end
```

Example 3.1. *Limited number of calculations of the necessary pattern of Diophantine 4-tuples along with an elegant property $D(4)$ for particular selections of x by operating the above MATLAB algorithm are tabularized below.*

Table 1.

x	(l, m, n, u)	(m, n, 0, v)	(n, 0, p, w)	(0, p, q, z)
1	(1,5,12,96)	(5,12,33,2080)	(12,33,85,33920)	(33,85,224,629004)
2	(3,7,20,480)	(7,20,51,7296)	(20,51,135,138112)	(51,135,352,2424596)
3	(5,9,28,1344)	(9,28,69,17600)	(28,69,185,357984)	(69,185,480,6128668)
4	(7,11,36,2880)	(11,36,87,34720)	(36,87,235,736736)	(87,235,608,12432420)

For all other choices of x, one can check the desired condition for the Diophantine 4-tuples by using MATHLAB algorithm.

4. Conclusion

In this manuscript, a new-fangled pattern of Diophantine 4-tuples comprising different Gnomonic numbers together with the feature $D(4)$ is assessed. In this way, one can pursuit so many Diophantine quadruples, quintuples, sex tuples etc satisfying some exciting properties none other than $D(4)$ involving other figurate numbers.

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