



Numerical accuracy of errors in Volterra integral equation by using quadrature methods

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Abstract

In this paper, we introduce new approaches of numerical solution and accuracy of Volterra integral equation of second kinds by Quadrature method. Some error estimates for the quadrature method in solving a class of Volterra integral equations are established. All calculation is calculated in MATLAB 13 versions. Numerical examples are given to show error analysis for Volterra integral equation of second kinds. Our results show that error analysis is simple and effective for the numerical solution of integral equation.

Keywords

Volterra integral equation, Trapezoidal rule, Simpson's rule, Weddle's rule.

AMS Subject Classification

45A05, 45D05, 65D32.

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1. Introduction

The many problems of Mathematics and Physics in the other disciplines are lead to linear and nonlinear integral equations. Integral equation is finding in special applicability within the scientific and mathematical disciplines [1]- [2]. Integral equation arises in the potential theory are more than any other area of the research. Integral equation arises also in conformal mapping, diffraction problems, scattering in quantum mechanics, water waves, and population growth models [4]-[5]. Integral equation arises in many application of wide range in the field of statistics, actuarial science, engineering, finance, fluid dynamics, Dirichlet problem, electrostatics, potential theory, astrophysics, reactor theory, diffusion problems and heat transfer problem etc [6]-[9]. Numerical integration is expressed in numerical value of the approximation. In this ex-

pression involve Trapezoidal rule, Simpson 1/3 rule, Simpson 3/8 rule, Weddle's rule, Boole's rule are estimated in different order and find the error. Many Researchers are in comparative studies in Trapezoidal rule & Simpson 1/3 rule, Simpson rule & Weddle's rule with the help of Newton Cote or Romberg integration methods [10]-[11]. In this "paper, we" present focus on linear integral equation with the help of Quadrature formulae by using Trapezoidal rule, Simpson rule, Weddle rule and it's comparative study with estimated errors [12]- [13].

Here, in this paper, we described in the solution of integral equations by using Trapezoidal rule, Simpson's rule and Weddle's rule. Moreover, we are comparing the numerical results with the help of an example. The many problems of Mathematics and Physics in the other disciplines are lead to linear and nonlinear integral equations. Integral equation is finding in special applicability within the scientific and mathematical disciplines [1]- [2]. Integral equation arises in the potential theory are more than any other area of the research. Integral equation arises also in conformal mapping, diffraction problems, scattering in quantum mechanics, water waves, and population growth models [4]-[5]. Integral equation arises in many application of wide range in the field of statistics, actuarial science, engineering, finance, fluid dynamics, Dirichlet problem, electrostatics, potential theory, astrophysics, reactor theory, diffusion problems and heat transfer problem etc [6]-

[9]. Numerical integration is expressed in numerical value of the approximation. In this expression involve Trapezoidal rule, Simpson 1/3 rule, Simpson 3/8 rule, Weddle’s rule, Boole’s rule are estimated in different order and find the error. Many Researchers are in comparative studies in Trapezoidal rule & Simpson 1/3 rule, Simpson rule & Weddle’s rule with the help of Newton Cote or Romberg integration methods [10]-[11]. In this “paper, we” present focus on linear integral equation with the help of Quadrature formulae by using Trapezoidal rule, Simpson rule, Weddle rule and it’s comparative study with estimated errors [12]-[13].

Here, in this paper, we described in the solution of integral equations by using Trapezoidal rule, Simpson’s rule and Weddle’s rule. Moreover, we are comparing the numerical results with the help of an example.

Composite, Newton- Cotes Formula:

Numerical integration which can be derived from the Lagrange interpolating method. Alternatively this formula is also deriving in the form of Taylor expansion. This idea is similar to way that obtains numerical integration schemes. This schemes which is develop in the assumption of equidistant points.

Composite, Newton- Cotes Formula:

This methods are composite since the repeatedly apply the simple formula derived previously to cover longer intervals. This idea allow for piecewise estimates of the integral thus improving the error of our integration.

2. Simple Newton-Cotes methods:

This idea is developed earlier on numerical differentiation and we use once again Lagrange interpolating polynomials as a starting point in obtaining numerical integration methods. In this fact we learn how to derive the well-known Trapezoidal, Simpson and Weddle’s rule. As usual we start with the Lagrange interpolation formula including the error terms,

$$P_n(x) = \sum_{i=0}^n g(x_i)L_i(x) + \frac{g^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x-x_i)$$

Simply integrating the above will produce a variety of numerical integration methods based on the number of nodes used.

Methods:

2.1 Trapezoidal Rule:

Volterra integral equation arises in the form of

$$g(x) = f(x) + \int_a^x k(x,t)y(t)dt \tag{2.1}$$

By using basic formula in Trapezoidal Rule

$$\int_a^{x_s} k(x_s,t)g(t)dt = \frac{h_i}{2} [k_{i,0}y_0 + 2 \sum_{j=0}^{i-1} k_{i,j}y_j + k_{i,i}y_i] + g_i \tag{2.2}$$

We suppose in 2.2 is divided in the interval [a, x] in ‘n’ parts so

$$a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n = x$$

Also we can write

$$\Delta y = \frac{(x_n - x_0)}{n} = \frac{(x - a)}{n}, \quad y_0 = a$$

$$y_i = y_0 + i\Delta y, \quad i = 1, 2, 3, \dots, n - 1$$

Whenever, we can find f(x) then we consider

$$x_0 = y_0 = a \quad \& \quad x = x_n = y_n, x_i = x_0 + i\Delta y = a + i\Delta y = y_i$$

i.e,

$$x_i = y_i, \quad 1 \leq i \leq n - 1$$

If using the following definitions

$$g(x_i) = g_i, \quad k(x_i, y_i) = k_{i,j}$$

Then we get

$$\int_a^x k(x,t)g(t)dt = \Delta y \left[\frac{1}{2}k(x,t_0)g(t_0) + k(x,t_1)g(t_1) + k(x,t_2)g(t_2) + \dots + k(x,t_{n-1})g(t_{n-1}) + \frac{1}{2}k(x,t_n)g(t_n) \right] \tag{2.3}$$

Where $\Delta y = \frac{(x-a)}{n}$

2.2 Simpsons’s Rule:

Suppose we can apply in Simpson’s rule

$$\int_a^x g(y)dy = \frac{h}{3} \left[g(a) + 2 \sum_{i=1}^{n/2-1} g(x_{2i}) + 4 \sum_{i=1}^{n/2} g(x_{2i-1}) + g(x) \right] \tag{2.4}$$

Using equation (2.3) results in (2.1) and we find approximate solutions in (2.1) for $a \leq y \leq x$. Then the solutions $g(x) = 0 \forall y > x$. Then error obtained in the form of the term $(n + 1)$ and an interval [a, x] into n segment. If rewrite the integral terms with $x = x_i$ Where $a = x_0 < x_1 < \dots < x_n = x$. We get

$$\int_a^{x_i} k(x,t)g(t)dt \approx h \sum_{j=0}^i w_{ij}k(x_i,t_j)g(t_j)$$

$$= h \sum_{j=0}^i w_{ij}k_{ij}g(t_j) + E_{i,t}(k(x_i,t),g(t)) \dots \tag{2.5}$$

If $x = x_i = a + ih$, $h = \frac{(x-a)}{n}$, $x_i = y_i$, $i = 1, 2, 3, \dots, n$. Where $E_{i,t}(k(x_i,t),g(t))$ errors are functions and w_{ij} represent a weight function. If Simpson’s 1/3 rule are aided them

$$g_n = f_n + \frac{h}{3} \sum_{i=0}^n w_{ij}k[nh, ih]g(i), \quad n = 2, 4, 6, \dots \tag{2.6}$$



2.3 Weddle’s Rule:

Suppose we can apply in Weddle’s rule

$$\int_a^x g(y)dy = \frac{3h}{10} \left[g(a) + 2\{g(x_6) + g(x_{12}) + \dots + g(x_{n-6})\} + \{g(x_2) + g(x_4) + g(x_8) + g(x_{10}) + \dots + g(x_{n-4}) + g(x_{n-2})\} + 5\{g(x_1) + g(x_5) + g(x_7) + g(x_{11}) + \dots + g(x_{n-5}) + g(x_{n-1})\} + 6\{g(x_3) + g(x_9) + \dots + g(x_{n-3})\} \right] \quad (2.7)$$

If rewrite the integral terms with $x = x_i$ Where $a = x_0 < x_1 < \dots < x_n = x$. We get

$$\int_a^{x_i} k(x,t)g(t)dt \approx h \sum_{j=0}^i w_{ij}k(x_i,t_i)g(t_i) = h \sum_{j=0}^i w_{ij}k_{ij}g(t_j) + E_{i,t}(k(x_i,t),g(t))\dots \quad (2.8)$$

If $x = x_i = a + ih$, $h = \frac{(x-a)}{n}$, $x_i = y_i$, $i = 1, 2, 3, \dots, n$. Where $E_{i,t}(k(x_i,t),g(t))$ error is function and w_{ij} represents a weight function. If Simpson’s 1/3 rule are aided them

$$g_n = f_n + \frac{3h}{10} \sum_{i=0}^n w_{ij}k[nh, ih]g(i), \quad n = 2, 4, 6, \dots \quad (2.9)$$

Numerical Results:

Example 1:

Suppose a Volterra integral equation of second kinds.

$$g(x) = x^2 - \frac{x^5}{4} + \int_0^x xtg(t)dt$$

Exact solutions

$$g(x) = x^2$$

Solution:

In an analytical successive approximation method

$$g_0(x) = f_0(x) = 0$$

$$g_1(x) = f(x) + \int_0^x xyg_0(t)dt = x^2 - \frac{x^5}{4}$$

$$g_2(x) = f(x) + \int_0^x xyg_1(t)dt = x^2 - \frac{x^8}{28}$$

$$g_3(x) = f(x) + \int_0^x xyg_2(t)dt = x^2 - \frac{x^{11}}{280}$$

.....

$$g_n(x) = f(x) + \int_0^x xyg_{n-1}(t)dt = x^2 - \frac{x^{3n+2}}{4.7.10\dots(3n+1)}$$

The relation between the exact and approximate solutions Trapezoidal Rule (1), Simpson’s Rule (2) and Weddle’s Rule (3).

example 2:

Suppose a Volterra integral equation of second kinds

$$g(x) = x - \frac{1}{6}x^3 + \int_0^x (x-t)g(t)dt \quad (2.10)$$

Exact Solutions $g(x = x)$.

Solution:

By using successive approximation method

$$g_0(x) = 0$$

$$g_1(x) = x - \frac{1}{6}x^3$$

$$g_2(x) = x - \frac{1}{120}x^5$$

$$g_3(x) = x - \frac{1}{5040}x^7$$

$$g_4(x) = x - \frac{1}{362880}x^9$$

The above results is obtained n^{th} terms

$$g_0(x) = 0$$

$$g_1(x) = x - \frac{1}{3!}x^3$$

$$g_2(x) = x - \frac{1}{5!}x^5$$

$$g_3(x) = x - \frac{1}{7!}x^7$$

$$g_4(x) = x - \frac{1}{9!}x^9$$

.....

.....

$$g_n(x) = x - \frac{1}{(2n+1)!}x^{2n+1}$$

The relation between the exact and approximate solutions Trapezoidal Rule(1), Simpson’s Rule(2) and Weddle’s Rule(3).



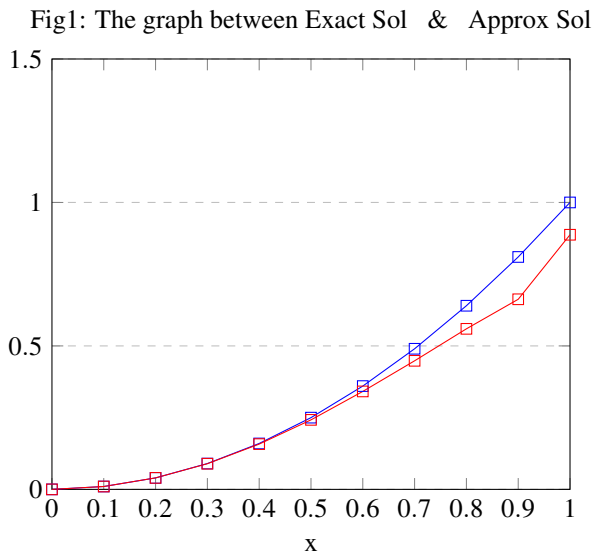
x	Exact Solution	Trapezoidal Rule	Simpson's Rule	Weddle Rule
0.0	0.0000000000	0.0000000000000	0.0000000000000	0.0000000000000
0.1	0.0100000000	0.0100010002432	0.0100001987638	0.0100001998652
0.2	0.0400000000	0.0400080130586	0.0399999992758	0.0399999999825
0.3	0.0900000000	0.0894118525161	0.0894154325789	0.0894154387543
0.4	0.1600000000	0.1577803842349	0.1577483942528	0.1577483957824
0.5	0.2500000000	0.2422437892675	0.2422537459227	0.2422537450852
0.6	0.3600000000	0.3413831339542	0.3413136662475	0.3413136662150
0.7	0.4900000000	0.4480902067359	0.4481088633943	0.4481088654032
0.8	0.6400000000	0.5595191145237	0.5540507125839	0.5540507107685
0.9	0.8100000000	0.6625406282574	0.6625684723574	0.6625684723582
1.0	1.0000000000	0.8874258974357	0.8874257469841	0.8874254895204

Table 1. Numerical results we obtain Trapezoidal Rule, Simpson's Rule, Weddle Rule in n=60.

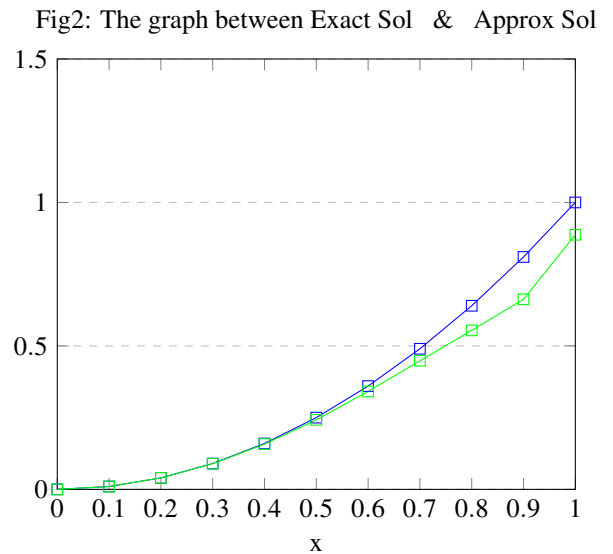
ErrorTrapezoidal	Error Simpson's	Error Weddle
0.0000000000000	0.0000000000000	0.0000000000000
1.000243200E-06	1.987638000E-07	1.998652000E-07
8.013058600E-06	7.242000000E-10	1.750000000E-11
5.881474839E-04	5.845674211E-04	5.845612457E-04
2.219615765E-03	2.251605747E-03	2.251604218E-03
7.756210733E-03	7.756210733E-03	7.746254915E-03
1.861686605E-02	1.861686605E-02	1.868633379E-02
4.190979326E-02	4.190979326E-02	4.189113460E-02
8.048088548E-02	8.048088548E-02	8.594928923E-02
1.474593372E-01	1.474593717E-01	1.474315276E-01
1.125757411E-01	1.125741026E-01	1.125745105E-01

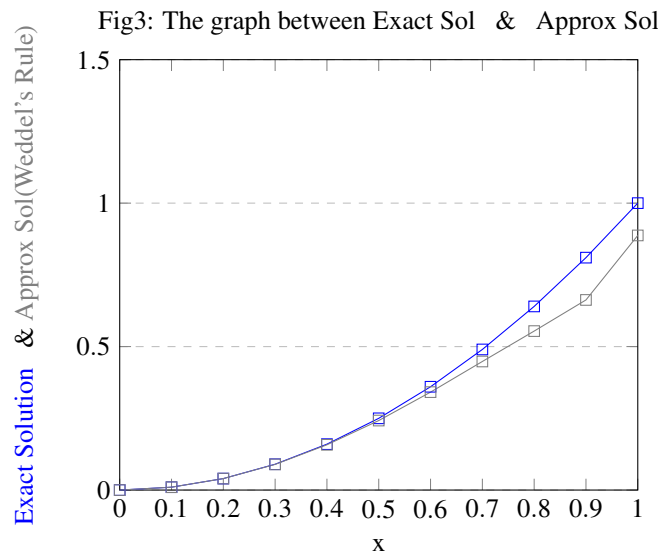
Table 2. Numerical Error we obtain Trapezoidal Rule, Simpson's Rule and Weddle's Rule.

Exact Solution & Approx Sol(Trapezoidal Rule)



Exact Solution & Approx Sol(Simpson's Rule)





x	Exact Solution	Trapezoidal Rule	Simpson's Rule	Weddle Rule
0.0	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.100000000000	0.09999332266	0.09999332622	0.09999332836
0.2	0.200000000000	0.19998657852	0.1999993953	0.1999998765
0.3	0.300000000000	0.29627637648	0.29628150295	0.29628150951
0.4	0.400000000000	0.39380531475	0.39379036649	0.39379036789
0.5	0.500000000000	0.48054407037	0.48054878064	0.48054878546
0.6	0.600000000000	0.57147882274	0.57145087168	0.57145087768
0.7	0.800000000000	0.72484097438	0.72480115963	0.72480011612
0.8	0.800000000000	0.72484097438	0.72480115963	0.72480011612
0.9	0.900000000000	0.78088288220	0.78088610835	0.78088610991
1.0	1.000000000000	0.84367894254	0.84367895432	0.84367895854

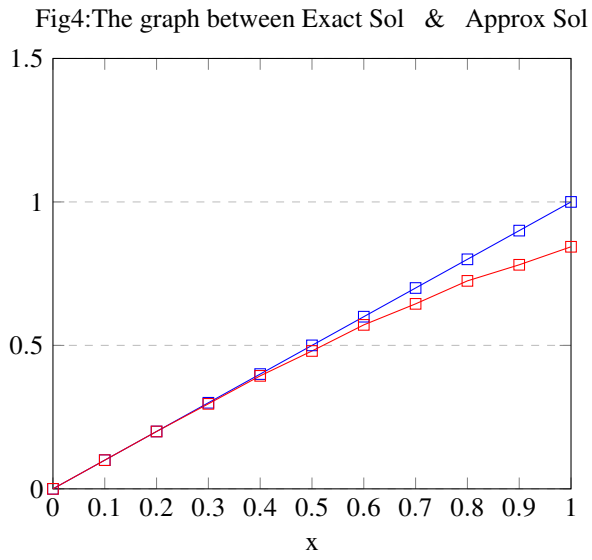
Table 3. Numerical results we obtain Trapezoidal Rule, Simpson's Rule, Weddle Rule in n=60.

Error Trapezoidal	Error Simpson's	Error Weddle
0.0000000000	0.0000000000	0.0000000000
6.677334000E-06	6.673780000E-06	6.671640000E-06
1.342148000E-05	6.047000000E-08	1.235000000E-08
3.723623520E-03	3.718497050E-03	3.718490490E-03
6.194685250E-03	6.209633510E-03	6.209632110E-03
1.945592963E-02	1.949121936E-02	1.945121454E-02
2.852117726E-02	2.854912832E-02	2.854912322E-02
5.524522359E-02	5.524114703E-02	5.524114428E-02
7.515902562E-02	7.519884037E-02	7.519988388E-02
1.191171178E-01	1.191138917E-01	1.191138901E-01
1.563210575E-01	1.563210468E-01	1.563210415E-01

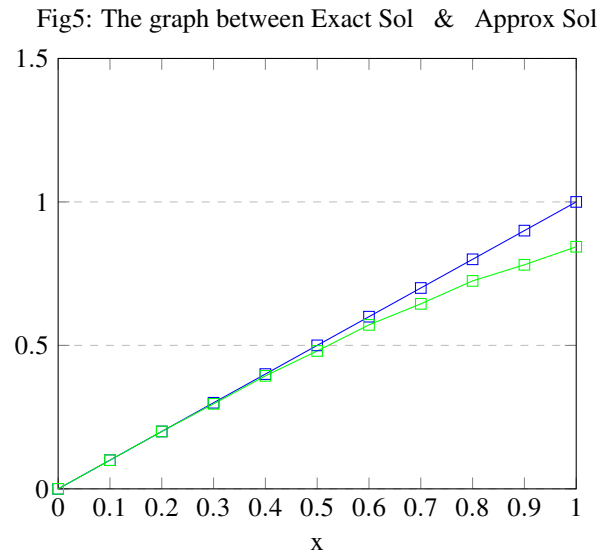
Table 4. Numerical Error we obtain Trapezoidal Rule, Simpson's Rule and Weddle's Rule.



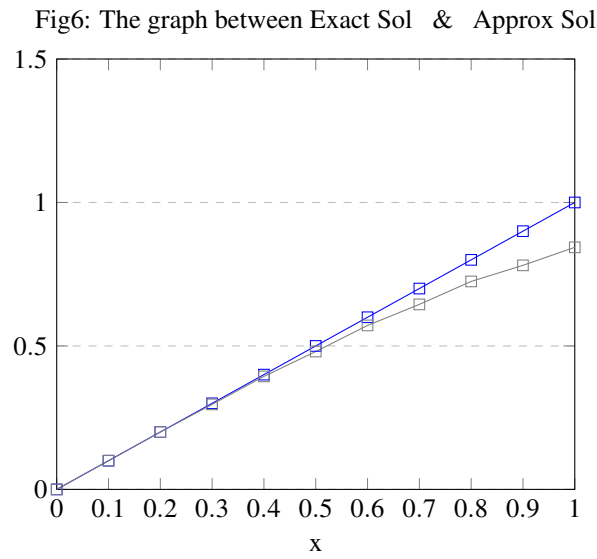
Exact Solution & Approx Sol(Trapezoidal Rule)



Exact Solution & Approx Sol(Simpson's Rule)



Exact Solution & Approx Sol(Weddle's Rule)



Conclusion:

In this paper describe new approaches of numerical solution and accuracy of Volterra integral equation of second kinds by Quadrature method. The estimated solutions obtained by the Trapezoidal rule, Simpson rule and Weddle rule. All compared results more accurate & effective in form of numerical accuracy. This research gives new ideas in further research in the field of numerical solution of integral equations.

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