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Inclusion properties for subclasses of multivalent regular functions defined on the unit disk

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Abstract

For subclasses of p-valent regular functions defined on the open unit disc, we prove certain inclusion theorems using multiplier and integral transform operator.

Keywords

Multivalent regular functions, subordination, multiplier transform operator, Integral transform operator, Inclusion theorems.

AMS Subject Classification

30C45, 30C50.

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Contents

1. Introduction

Let it denote by A, the class of all regular functions defined on the open unit disc $\mathbb{U} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of univalent functions. If US^*, UC, UK respectively denotes the subclasses of S, consisting of functions $\eta(z)$ of the form,

$$\eta(z) = z + \sum_{n=2}^{\infty} \eta_n z^n.$$

satisfying the inequality

$$Re\left(\frac{z\eta'(z)}{\eta(z)}\right) > 0, |z| < 1$$

$$Re\left(1 + \frac{z\eta''(z)}{\eta'(z)}\right) > 0, |z| < 1.$$

and

$$Re\left(\frac{z\eta'(z)}{\xi(z)}\right) > 0, |z| < 1$$

where $\xi(z) = z + \sum_{n=2}^{\infty} \xi_n z^n \in US^*$. Let A_p denote the class of all p-valent regular functions of the form

$$\eta_p(z) = z^p + \sum_{i=k}^{\infty} \eta_i \zeta^{i+p} \tag{1.1}$$

Note that when $p = 1$, $\eta_1(z) = \eta(z)$. If US^*_p, UC_p, UK_p respectively denote the subclass of A_p, consisting of p-valently starlike, p-valently convex and p-valently close-to-convex functions, then

$$US^*_p = \left\{\eta_p \in A_p : \frac{1}{p} Re\left(\frac{z\eta'_p(z)}{\eta_p(z)}\right) > 0\right\},$$

$$UC_p = \left\{\eta_p \in A_p : \frac{1}{p} Re\left(1 + \frac{z\eta''_p(z)}{\eta'_p(z)}\right) > 0\right\},$$

and

$$UK_p = \left\{\eta_p \in A_p : \frac{1}{p} Re\left(\frac{z\eta'_p(z)}{\xi_p(z)}\right) > 0\right\}$$

where

$$\xi_p = z^p + \sum_{n=p+1}^{\infty} \xi_n z^n \in US^*_p. \tag{1.2}$$

Equivalently,

$$\eta_p \in US_p^* \ if\ and\ only\ if\ \left|\frac{z\eta_p'(z)}{\eta_p(z)} - p\right| < p,$$

$$\eta_p \in UC_p \ if\ and\ only\ if\ \left|\left(1+\frac{z\eta_p''(z)}{\eta_p'(z)}\right) - p\right| < p$$

and

$$\eta_p \in UK_p \ if\ and\ only\ if\ \left|\frac{z\eta_p'(z)}{\xi_p(z)} - p\right| < p$$

where ξ_p satisfies

$$\left|\frac{z\eta_p'(z)}{\eta_p(z)} - p\right| < p.$$

Similarly, the class of all p-valently starlike functions of order γ, p-valently convex functions of order γ and p-valently close to convex functions of order γ, namely $US_p^*(\gamma)$, $UC_p(\gamma)$, $UK_p(\gamma)$ respectively satisfies the equivalent conditions, $0 \leq \gamma < p$

$$\left|\frac{z\eta_p'(z)}{\eta_p(z)} - p\right| < p-\gamma, \tag{1.3}$$

$$\left|\left(1+\frac{z\eta_p''(z)}{\eta_p'(z)}\right) - p\right| < p-\gamma, \tag{1.4}$$

$$\left|\frac{z\eta_p'(z)}{\xi_p(z)} - p\right| < p-\gamma. \tag{1.5}$$

where $\xi_p(z)$ satisfies

$$\left|\frac{z\eta_p'(z)}{\eta_p(z)} - p\right| < p-\beta, \quad 0 \leq \beta < p.$$

The Convolution(Or Hadamard product) of two p-valent regular functions η_p and ξ_p, denoted by $(\eta_p * \xi_p)(z)$ and is defined by

$$(\eta_p * \xi_p)(z) = z^p + \sum_{n=p+1}^{\infty} \eta_n \xi_n z^n.$$

Given two functions η and ξ, which are analytic in $\mathbb{U}$, the function η is said to be subordinate to ξ in $\mathbb{U}$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with

$$w(0) = 0, \quad |w(z)| < 1, \quad (z \in \mathbb{U})$$

such that

$$\eta(z) = \xi(w(z)), \quad (z \in \mathbb{U}).$$

Furthermore, if the function ξ is univalent in $\mathbb{U}$, then we have the following equivalence relation

$$\eta \prec \xi \Leftrightarrow \eta(0) = \xi(0)\ and\ \ \eta(U) \subset \xi(U).$$

2. Preliminaries

In this section, using the principle of subordination between regular functions, we denote by $US_p^*(\gamma,\xi)$, $UC_p(\gamma,\xi)$, $UK_p(\gamma,\xi)$, the classes of p-valently starlike of order γ, p-valently convex of order γ, p-valently close to convex of order γ respectively. Interms of a subordinating function $\chi(z)$, these classes are defined as follows:

$$US_p^*(\gamma,\chi) = \left\{\eta_p \in A_p \ : \frac{1}{p-\gamma}\left(\frac{z\eta_p'(z)}{\eta_p(z)} - \gamma\right) \prec \chi(z)\right\}, \tag{2.1}$$

$$UC_p(\gamma,\chi) = \left\{\eta_p \in A_p \ : \frac{1}{p-\gamma}\left(1+\frac{z\eta_p''(z)}{\eta_p'(z)} - \gamma\right) \prec \chi(z)\right\} \tag{2.2}$$

and

$$UK_p(\gamma,\chi) = \left\{\eta_p \in A_p \ : \frac{1}{p-\gamma}\left(\frac{z\eta_p'(z)}{\mu_p(z)} - \gamma\right) \prec \chi(z)\right\}. \tag{2.3}$$

where $\mu_p(z) \in US^*(l)$, $0 \leq l < 1$.
We know that the set of all p-valently analytic functions A_p defined on the unit disc $\mathbb{U}$ is a linear space over the complex field. In [13], the authors defined a generalized multiplier operator $U^m_{p,\lambda,\xi}$ as follows:
For a fixed $\xi_p \in A_p$ of the form (1.2), Define $U^m_{p,\lambda,\xi} : A_p \to A_p$ by

$$U^0_{p,\lambda,\xi}\ \eta_p(z) = \eta(z),$$

$$U^1_{p,\lambda,\xi}\ \eta_p(z) = \ (1-\lambda)(\eta_p * \xi_p)(z) + \frac{\lambda}{p}\, z\ ((\eta_p * \xi_p)(z))',$$

$$U^m_{p,\lambda,\xi}\ \eta_p(z) = \ U'_{p,\lambda,\xi}\ (U^{m-1}_{p,\lambda,\xi}\eta_p(z)). \tag{2.4}$$

It can be easily verified that

$$U^m_{p,\lambda,\xi}\ \eta_p(z) = z^p + \sum_{n=1}^{\infty}\left[1+\frac{n\lambda}{p}\right]^m \eta_n\ \xi_n\ z^{n+p}. \tag{2.5}$$

and

$$U^{m+1}_{p,\lambda,\xi}\ \eta_p(z) = (1-\lambda)U^m_{p,\lambda,\xi}\ (\eta_p * \xi_p)(z) + \frac{\lambda}{p}\, z\, U^m_{p,\lambda,\xi}\ ((\eta_p * \xi_p)(z))'. \tag{2.6}$$

such that

$$z\left(U^m_{p,\lambda,\xi}\ \eta_p(z)\right)' = \frac{p}{\lambda}\left[U^{m+1}_{p,\lambda,\xi}\ \eta_p(z) - (1-\lambda)\, U^m_{p,\lambda,\xi}\ \eta_p(z)\right]$$

(2.7)

Now, by using the operator $U^m_{p,\lambda,\xi}$, we introduce the following subclasses of p-valently starlike,p-valently convex and p-valently close-to-convex functions for the functions $\xi,\chi \in US^{*\,m,\gamma}_{p,\lambda}(\xi,\chi)$, $0 \le \lambda < p$, $p \in \mathbb{N}$.

$$US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi) = \left\{\eta_p \in A_p : U^m_{p,\lambda,\xi}\ \eta_p(z)\ \in US^*_p(\gamma,\chi)\right\},$$

$$UC^{m,\gamma}_{p,\lambda}(\xi;\chi) = \left\{\eta_p \in A_p : U^m_{p,\lambda,\xi}\ \eta_p(z)\ \in UC_p(\gamma,\chi)\right\},$$

$$UK^{m,\gamma}_{p,\lambda}(\xi;\chi) = \left\{\eta_p \in A_p : U^m_{p,\lambda,\xi}\ \eta_p(z)\ \in UK_p(\gamma,\chi)\right\}.$$

Remark 1.1: If we let $m = 0$, $U^m_{p,\lambda,\xi}\ \eta_p(z)$ reduces to $\eta(z)$. This class was introduced and investigated in [13]. In this paper we prove certain theorems involving multiplier and integral operators.

Remark 1.2: For the choices of a subordinating functions $\chi(z) = \frac{1+(2\alpha-1)z^p}{1-z^p}$ and $\chi(z) = \frac{1+(2\alpha-p)\beta z^p}{1-\beta z^p}$, we obtain various subclasses of A_p. For example,

1. $US^*_p\left(\gamma, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = S^*(0)$, starlike function of order 0, when $p = 1$, $\alpha = 0$, $US^*_p\left(\gamma, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = S^*(\gamma)$, $0 \le \gamma < 1$ when $p = 1$, the classes of starlike functions of order γ.
2. $UC_p\left(\gamma, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = C(0)$, when $\alpha = 0$, $p = 1$ and $UC_p\left(\gamma, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = C(\gamma)$, when $p = 1, 0 \le \gamma < 1$.
3. $UK_p\left(1, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = K(0)$, when $\alpha = 0, p = 1$ and $UK_p\left(1, \frac{1+(2\alpha-1)z^p}{1-z^p}\right) = K(\beta)$,when $0 \le \beta < 1, p = 1$.

To prove our results, we need the following lemmas [9].
We denote by $\prod$, the set of all regular functions of the form $\chi(z)$ such that $\chi(0) = 1$ and $Re\ \chi(z) \ge\ 0$.

Lemma 2.1. *Let ϕ be convex, univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $Re\ (k\phi(z)+\gamma) > 0$, $k,\gamma \in \mathbb{C}$.If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{kp(z+\gamma)} \prec \phi(z)\ \ implies\ \ p(z) \prec\ \phi(z), (z\ \in\ \mathbb{U}).$$

Lemma 2.2. *Let ϕ be convex, univalent in $\mathbb{U}$ and w be analytic in $\mathbb{U}$ with $Re\ (w(z)) \ge\ 0$.If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$,then*

$$p(z) + w(z)\ zp'(z)\ \prec\ \phi(z)\ \ implies\ \ p(z) \prec\ \phi(z), (z\ \in \mathbb{U}).$$

3. Inclusion theorems involving the operator $U^m_{p,\lambda,\xi}$

Theorem 3.1. *Let $\eta_p \in A_p$ and let $\chi \in \Pi$ with $Re\left\{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}\right\} > 0$.
Then $US^{m+1}_{p,\lambda,\xi}(\gamma,\chi) \subset\ US^m_{p,\lambda,\xi}(\gamma,\chi)$.*

Proof. Let

$$\eta_p(z) \in\ US^{m+1}_{p,\lambda,\xi}(\gamma,\chi).$$

Then

$$Re\ \frac{1}{p-\gamma}\left[\frac{z(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m+1}_{p,\lambda,\xi}\eta_p(z)} - \gamma\right] > 0$$

Take

$$q(z) = \frac{1}{p-\gamma}\left[\frac{z\left(U^m_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^m_{p,\lambda,\xi}\ \eta_p(z)} - \gamma\right] \tag{3.1}$$

where $q(z)$ is analytic in U with $p(0) = 1$.Using (2.7) in (3.1),we get

$$\left(\frac{p}{\lambda}\right)\frac{U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)}{U^m_{p,\lambda,\xi}\ \eta_p(z)} = (p-\gamma)\ q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}. \tag{3.2}$$

Differentiate (3.2) logarithmically with respect to z, we obtain

$$\frac{\left(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^m_{p,\lambda,\xi}\ \eta_p(z)} - \frac{\left(U^m_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^m_{p,\lambda,\xi}\ \eta_p(z)} = \frac{(p-\gamma)q'(z)}{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}}. \tag{3.3}$$

This implies that

$$\frac{z\left(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)} = (p-\gamma)q(z)+\gamma+\frac{(p-\gamma)q'(z)}{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}}$$

By simplifying the above equation, we get

$$\frac{1}{p-\gamma}\left[\frac{z(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)} - \gamma\right] = q(z)+\frac{zq'(z)}{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}}. \tag{3.4}$$

Applying lemma(2.1) to (3.4), it follows that $q \prec \chi$, that is $\eta_p\ \in US^m_{p,\lambda,\xi}\ (\gamma,\chi)$. □

Theorem 3.2. *Let $\eta_p \in A_p$ and let $\chi \in \Pi$ with $Re\left\{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}\right\} > 0$.
Then $UC^{m+1}_{p,\lambda,\xi}(\gamma,\chi) \subset\ UC^m_{p,\lambda,\xi}(\gamma,\chi)$.*

Proof. Applying the result,

$$\eta_p(z) \in UC^m_{p,\lambda,\xi}\ (\gamma,\chi) \iff \frac{z\eta_p'(z)}{p}\ \in US^m_{p,\lambda,\xi}\ (\gamma,\chi).$$

We conclude that

$$\eta_p(z) \in UC^{m+1}_{p,\lambda,\xi}(\gamma,\chi) \Longleftrightarrow U^{m,\gamma}_{p,\lambda,\xi}\ \eta_p(z) \in UC^{m+1}_{p,\lambda,\xi}(\gamma,\chi).$$

$$\Longleftrightarrow \frac{z\left(U^{m,\gamma}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{p} \in US^{m+1}_{p,\lambda,\xi}(\gamma,\chi) \subset US^{m}_{p,\lambda,\xi}(\gamma,\chi).$$

$$\Longleftrightarrow \frac{z\left(U^{m,\gamma}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{p} \in US^{m}_{p,\lambda,\xi}(\gamma,\chi).$$

$$\Longleftrightarrow U^{m,\gamma}_{p,\lambda,\xi}\ \eta_p(z) \in UC^{m}_{p,\lambda,\xi}(\gamma,\chi).$$

$$\Longleftrightarrow \eta_p(z) \in UC^{m}_{p,\lambda,\xi}(\gamma,\chi).$$

□

Theorem 3.3. *Let $\eta_p \in A_p$ and let $\chi \in \Pi$ with $Re\left\{(p-\gamma)q(z)+\gamma+\frac{(1-\lambda)p}{\lambda}\right\} > 0$. Then $UK^{n+1}_{p,\lambda,\xi}(\gamma,\chi) \subset UK^{n}_{p,\lambda,\xi}(\gamma,\chi)$.*

Proof. Let $\eta_p \in UK^{m+1}_{p,\lambda,\xi}$, Then by definition there exists a function $\mu_p \in US^{m+1}_{p,\lambda,\xi}$ such that

$$\frac{1}{p-\gamma}\left[\frac{z(U^{m+1}_{p,\lambda,\xi}\eta_p(z))'}{U^{m+1}_{p,\lambda,\xi}\mu_p(z)} - \gamma\right] \prec \chi(z).$$

Take

$$q(z) = \frac{1}{p-\gamma}\left[\frac{z\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right]. \tag{3.5}$$

$$z\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)' = [(p-\gamma)q(z)+\gamma]\,U^{m}_{p,\lambda,\xi}\mu_p(z)$$

where $q(z)$ is analytic in U with $q(0)=1$. Using (2.7) and (3.5), we have

$$\left(\frac{p}{\lambda}\right)U^{m+1}_{p,\lambda,\xi}\ \eta_p(z) = \frac{p(1-\lambda)}{\lambda}\,U^{m}_{p,\lambda,\xi}\ \eta_p(z) + z\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)'. \tag{3.6}$$

Differentiating (3.6) with respect to z, and multiplying by z, we get

$$\left(\frac{p}{\lambda}\right)z\left(U^{m+1}_{p,\lambda,\xi}\eta_p(z)\right)' = \frac{p(1-\lambda)}{\lambda}z\left(U^{m}_{p,\lambda,\xi}\eta_p(z)\right)' + [(p-\gamma)z\,q'(z)]U^{m}_{p,\lambda,\xi}\mu_p(z)$$
$$[(p-\gamma)q(z)+\gamma]\,z\left(U^{m}_{p,\lambda,\xi}\mu_p(z)\right)'. \tag{3.7}$$

Since $\mu_p(z) \in US^{m+1}_{p,\lambda,\xi}(\gamma,\chi) \Rightarrow \mu_p(z) \in US^{m}_{p,\lambda,\xi}(\gamma,\chi)$, we can take

$$h(z) = \frac{1}{p-\gamma}\left[\frac{z(U^{m}_{p,\lambda,\xi}\ \mu_p(z))'}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right]. \tag{3.8}$$

Using(2.7) and(3.8), we get

$$\frac{p}{\lambda}\left[\frac{U^{m+1}_{p,\lambda,\xi}\ \mu_p(z)}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)}\right] = \frac{p(1-\lambda)}{\lambda} + (p-\gamma)h(z) + \gamma. \tag{3.9}$$

From (3.7), we have

$$\left(\frac{p}{\lambda}\right)\frac{z(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)} = [(p-\gamma)q(z)+\gamma]\left[(p-\gamma)h(z)+\gamma+\frac{p(1-\lambda)}{\lambda}\right] + (p-\gamma)\,zq'(z). \tag{3.10}$$

From(3.9) and (3.10), we have

$$\frac{1}{p-\gamma}\left[\frac{z\left(U^{m+1}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^{m+1}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right] = q(z) + \frac{zq'(z)}{\frac{p(1-\lambda)}{\lambda}+(p-\gamma)h(z)+\gamma}. \tag{3.11}$$

Since $0 \le \lambda < p$ and $h(z) \prec \chi(z)$ in U, then $Re(\frac{p(1-\lambda)}{\lambda}+(p-\gamma)h(z)+\gamma) > 0$, so by taking $w(z) = \frac{1}{\frac{p(1-\lambda)}{\lambda}+(p-\gamma)h(z)+\gamma}$ and applying lemma(2.1) we can show that $p \prec \chi$. □

4. Inclusion theorems involving the integral operator $F_{p,c}$

In this section we consider the generalized Libera integral operator $F_{p,c}$ defined by

$$F_{p,c}\ \eta_p(z) = \frac{p+c}{z^c}\int_0^1 t^{c-1}\ \eta(t)dt, \quad (c > -p; \eta_p \in A_p).$$

Now

$$F_{p,c}\ U^{m}_{p,\lambda,\xi}\eta_p(z) = \frac{p+c}{z^c}\int_0^1 t^{c-1}\ U^{m}_{p,\lambda,\xi}\eta(t)dt, \quad (c > -p; \eta_p \in A_p). \tag{4.1}$$

From (4.1), we have

$$z(F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z))' = (p+c)U^{m}_{p,\lambda,\xi}\ \eta_p(z) - c\,F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z). \tag{4.2}$$

Theorem 4.1. *Let $c > -p$ and $\chi \in \Pi$ with $Re\{(p-\gamma)q(z)+\gamma+c\} > 0$. If $U^{m}_{p,\lambda,\xi}\ \eta_p(z) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi)$, then $F_{p,c}(U^{m}_{p,\lambda,\xi}\ \eta_p(z)) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi)$.*

Proof. Let

$$U^{m}_{p,\lambda,\xi}\ \eta_p(z) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi).$$

Then

$$Re\left(\frac{1}{p-\gamma}\right)\left(\frac{z\left(F_{p,c}\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)\right)'}{F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z)}\right) > 0.$$

Take

$$q(z) = \left(\frac{1}{p-\gamma}\right)\left(\frac{z\left(F_{p,c}\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)\right)'}{F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z)}\right) \quad (4.3)$$

where q is analytic in $\mathbb{U}$ with $q(0) = 1$.By using(4.2) and (4.3), we have

$$(p+c)\frac{U^{m}_{p,\lambda,\xi}\ \eta_p(z)}{F_{p,c}\ U^{m}_{p,\lambda,\xi}\eta_p(z)} = (p-\gamma)q(z)+\gamma+c. \quad (4.4)$$

Differentiating(4.4) logarithmically with respect to z, we get

$$\frac{z(U^{m}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m}_{p,\lambda,\xi}\ \eta_p(z)} = \frac{z(F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z))'}{F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \eta_p(z)} + \frac{(p-\gamma)zq'(z)}{(p-\gamma)q(z)+\gamma+c}. \quad (4.5)$$

Simplifying the above equation, we get

$$\frac{1}{p-\gamma}\left[\frac{z(U^{m}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m}_{p,\lambda,\xi}\ \eta_p(z)} - \gamma\right] = q(z) + \frac{zq'(z)}{(p-\gamma)q(z)+\gamma+c}. \quad (4.6)$$

By using the lemma (2.1),we get $F_{p,c}(U^{m}_{p,\lambda,\xi}\ \eta_p(z)) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi)$. □

Theorem 4.2. *Let $c > -p$ and $\chi \in \Pi$ with $Re\{(p-\gamma)q(z)+\gamma+c\} > 0$. If $U^{m}_{p,\lambda,\xi}\ \eta_p(z) \in UK^{m,\gamma}_{p,\lambda}(\xi;\chi)$, then $F_{p,c}(U^{m}_{p,\lambda,\xi}\ \eta_p(z)) \in UK^{m,\gamma}_{p,\lambda}(\xi;\chi)$.*

Proof. Let $U^{m}_{p,\lambda,\xi}\ \eta_p(z) \in UK^{m,\gamma}_{p,\lambda}(\xi,\chi)$ then there exist a function $U^{m}_{p,\lambda,\xi}\ \mu_p(z) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi)$. such that

$$\frac{1}{p-\gamma}\left[\frac{z\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)'}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right] \prec \chi(z).$$

Now let

$$q(z) = \frac{1}{p-\gamma}\left[\frac{z\left(F_{p,c}\left(U^{m}_{p,\lambda,\xi}\ \eta_p(z)\right)\right)'}{F_{p,c}(U^{m}_{p,\lambda,\xi}\ \mu_p(z))} - \gamma\right]$$

where q is analytic in $\mathbb{U}$ with $q(0) = 1$, then using (4.2) in above expression, we obtain

$$[(p-\gamma)q(z)+\gamma]\ F_{p,c}\ U^{m}_{p,\lambda,\xi}\ \mu_p(z) + c\ F_{p,c}U^{m}_{p,\lambda,\xi}\ \eta_p(z) = (p+c)\ U^{m}_{p,\lambda,\xi}\ \eta_p(z). \quad (4.7)$$

Then differentiate (4.7) with respect to z we get,

$$(p+c)\frac{z(U^{m}_{p,\lambda,\xi}\ \eta_p(z))'}{F_{p,c}U^{m}_{p,\lambda,\xi}\ \mu_p(z)} = [(p-\gamma)q(z)+\gamma][(p-\gamma)h(z)+\gamma+c] + zq'(z)(p-\gamma). \quad (4.8)$$

Since $\mu_p(z) \in US^{*\,m,\gamma}_{p,\lambda,\xi}(\xi;\chi)$, then by theorem(4.1), we have $F_{p,c}U^{m}_{p,\lambda,\xi}\mu_p(z) \in US^{*\,m,\gamma}_{p,\lambda}(\xi;\chi)$. Let

$$h(z) = \frac{1}{p-\gamma}\left[\frac{z(F_{p,c}U^{m}_{p,\lambda,\xi}\ \mu_p(z))'}{F_{p,c}U^{m}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right]. \quad (4.9)$$

From (4.9), we get

$$(p+c)U^{m}_{p,\lambda,\xi}\mu_p(z) = c\ F_{p,c}U^{m}_{p,\lambda,\xi}\ \mu_p(z) + [(p-\gamma)h(z)+\gamma]F_{p,c}U^{m}_{p,\lambda,\xi}\ \mu_p(z). \quad (4.10)$$

This gives

$$(p+c)\frac{U^{m}_{p,\lambda,\xi}\mu_p(z)}{F_{p,c}U^{m}_{p,\lambda,\xi}\mu_p(z)} = c+(p-\gamma)h(z)+\gamma. \quad (4.11)$$

From (4.8), we have

$$(p+c)\frac{z\left(U^{m}_{p,\lambda,\xi}\eta_p(z)\right)'}{F_{p,c}U^{m}_{p,\lambda,\xi}\mu_p(z)} = [(p-\gamma)q(z)+\gamma][(p-\gamma)h(z)+\gamma+c] + (p-\gamma)\ zq'(z). \quad (4.12)$$

From equations(4.11) and(4.12), we have

$$\frac{1}{p-\gamma}\left[\frac{z(U^{m}_{p,\lambda,\xi}\ \eta_p(z))'}{U^{m}_{p,\lambda,\xi}\ \mu_p(z)} - \gamma\right] = q(z) + \frac{zq'(z)}{(p-\gamma)h(z)+\gamma+c}. \quad (4.13)$$

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