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On strict strong coloring of central graphs

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Abstract

A strict strong coloring of a graph G is a proper coloring of G in which every vertex of the graph is adjacent to every vertex of some color class. The minimum number of colors required for a strict strong coloring of G is called the strict strong chromatic number of G and is denoted by $\chi_{ss}(G)$. In this paper we discuss some results on strict strong chromatic number of central graphs.

Keywords

Proper coloring, strict strong coloring, strict strong chromatic number, central graphs.

AMS Subject Classification

05C15, 05C69.

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Contents

1. Introduction

All graphs considered here are simple. For graph theoretic terminology we refer to [4]. Let $G=(V,E)$ be a graph. The *degree* of a vertex $v \in V$ in a graph G is defined to be the number of edges incident with v and is denoted by deg (v). A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendent vertex* or a *leaf*. Any vertex which is adjacent to a pendant vertex is called a *support vertex*. The *open neighborhood* and *closed neighborhood* of v is $N(v)=\{u \in V : uv \in E\}$ and $N[v]=N(v)\cup\{v\}$ respectively. A subset S of V is called a *dominating set (total dominating set)* of G if every vertex in $V-S$ (every vertex in V) is adjacent to a vertex in S. The *domination number* γ (*total domination number* γ_t) is the minimum cardinality of a dominating set (total dominating set) in G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set.

A *proper vertex coloring* of G is an assignment of colors to the vertices of G in such a way that adjacent vertices are assigned distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required for a proper coloring of G. The concept of strong coloring was introduced by I.E. Zverovich [19]. It defines a dominance relation between graph vertices and color classes. A *strict strong coloring* of G is a proper coloring of G in which every vertex of the graph is adjacent to every vertex of some color class, that is, each vertex totally dominates every vertex of some other color class. The minimum number of colors required for a strict strong coloring of G is called the *strict strong chromatic number* of G and is denoted by $\chi_{ss}(G)$. Some basic results on strict strong colorings are given in [2, 10, 13, 17]. Given a graph G, we subdivide each edge of G exactly once and join all the non-adjacent vertices of G. The graph obtained by this process is called central graph of G denoted by $C(G)$. In this paper, we prove the results on strict strong chromatic number of central graphs on path, cycle, complete graph and complete bipartite graph.

2. Results

In this section, we prove the exact values for path, cycle, complete graph and complete bipartite graph.

Theorem 2.1. *For a path P_n,*

$$\chi_{ss}(C(P_n)) = \begin{cases} \lfloor \frac{2n-1}{3} \rfloor + 1, & n \equiv 2 (mod\, 3) \\ \lfloor \frac{2n-1}{3} \rfloor + 2, & otherwise \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path P_n and let $c_{i,j}$ be the vertex which divides an edge v_iv_j.

Case 2.2. $n \equiv 0 (mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0 (mod\, 3)\} \bigcup \{v_{n-1}\} \bigcup \{v_n\} \bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-2\} \bigcup \{(c_{i,j} : 1 \le i \le n-1 \, and \, j = i+1)\}$ of $C(P_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-4$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n-1}{3} \rfloor + 2$.

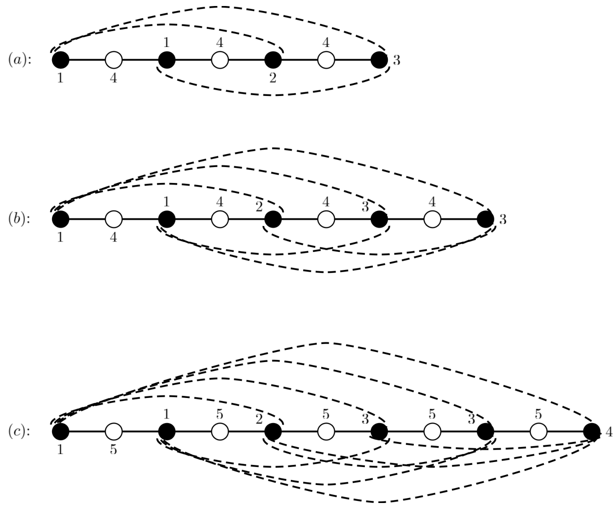

Figure 1. (a): Strict strong chromatic number of central graph on path P_4 is 4. (b): Strict strong chromatic number of central graph on path P_5 is 4. (c): Strict strong chromatic number of central graph on path P_6 is 5.

Case 2.3. $n \equiv 1(mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0(mod\, 3)\} \bigcup \{v_n\}$ $\bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-3\}$ $\bigcup \big\{(c_{i,j} : 1 \le i \le n-1\, and\; j = i+1)\big\}$ of $C(P_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-3$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n-1}{3} \rfloor + 2$.

Case 2.4. $n \equiv 2(mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0(mod\, 3)\}$ $\bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-1\}$ $\bigcup \big\{(c_{i,j} : 1 \le i \le n-1\, and\; j = i+1) \cup c_{n,1}\big\}$ of $C(C_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-4$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n-1}{3} \rfloor + 1$. □

Theorem 2.5. *For a cycle C_n,*

$$\chi_{ss}(C(C_n)) = \begin{cases} \lfloor \frac{2n}{3} \rfloor + 1, & n \equiv 0(mod\, 3) \\ \lfloor \frac{2n}{3} \rfloor + 2, & otherwise \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle C_n and let $c_{i,j}$ be the vertex which divides an edge $v_i v_j$.

Case 2.6. $n \equiv 0(mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0(mod\, 3)\} \bigcup \{v_{n-1}\}$ $\bigcup \{v_n\} \bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-2\}$ $\bigcup \big\{(c_{i,j} : 1 \le i \le n-1\, and\; j = i+1) \cup c_{n,1}\big\}$ of $C(C_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-4$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n}{3} \rfloor + 1$.

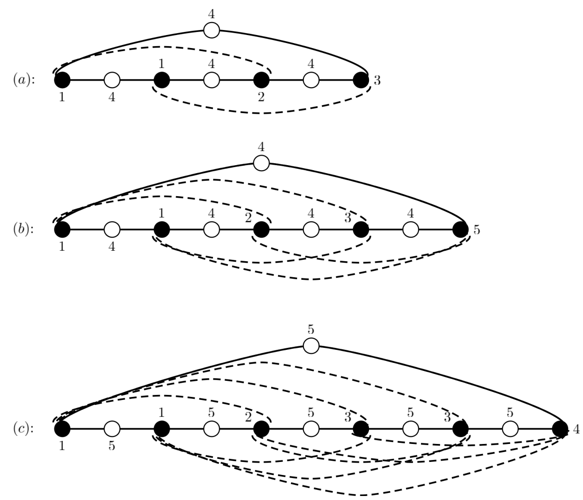

Figure 2. (a): Strict strong chromatic number of central graph on cycle C_4 is 4. (b): Strict strong chromatic number of central graph on cycle C_5 is 5. (c): Strict strong chromatic number of central graph on cycle C_6 is 5.

Case 2.7. $n \equiv 1(mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0(mod\, 3)\} \bigcup \{v_n\}$ $\bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-3\}$ $\bigcup \big\{(c_{i,j} : 1 \le i \le n-1\, and\; j = i+1) \cup c_{n,1}\big\}$ of $C(C_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-3$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n}{3} \rfloor + 2$.

Case 2.8. $n \equiv 2(mod\, 3)$

Consider a coloring $\mathscr{C} = \{\{v_i\} : i \equiv 0(mod\, 3)\} \bigcup \{v_{n-1}\}$ $\bigcup \{v_n\} \bigcup \{\{v_i \cup v_{i+1}\} : i = 1,4,7,\ldots,n-4\}$ $\bigcup \big\{(c_{i,j} : 1 \le i \le n-1\, and\; j = i+1) \cup c_{n,1}\big\}$ of $C(C_n)$. Clearly the vertices $c_{i,j}$, where $i = 1,4,7,\ldots,n-4$ and $j = i+1$, totally dominates the color class $\{\{v_i \cup v_{i+1}\}$ and rest of the vertices totally dominates some color class $\{\{v_i\} : i \equiv 0(mod\, 3)\}$. Hence $\chi_{ss}(C(C_n)) = \lfloor \frac{2n}{3} \rfloor + 2$. □

Theorem 2.9. *For a complete graph K_n, $\chi_{ss}(C(K_n)) = 2(n-1)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle C_n and let $c_{i,j}$ be the vertex which divides an edge $v_i v_j$. Consider a coloring $\mathscr{C} = \{\{v_i\} : 1 \le i \le n-2\} \bigcup \{v_{n-1} \cup v_n\}$ $\bigcup \big\{c_{i,j} : 1 \le i \le n-2\, and\; j = i+1\big\} \bigcup \{c_{n-1,n}\}$ of $C(K_n)$. Each vertex $v_i, 1 \le i \le n-2$, totally dominates the color class $\big\{c_{i,j} : 1 \le i \le n-2\, and\; j = i+1\big\}$ and vice versa. Further the vertices v_{n-1} and v_n totally dominates the color class $\{c_{n-1,n}\}$ and the vertex $c_{n-1,n}$ totally dominates the color class $\{v_{n-1} \cup v_n\}$. Hence $\chi_{ss}(C(K_n)) = 2(n-1)$. □

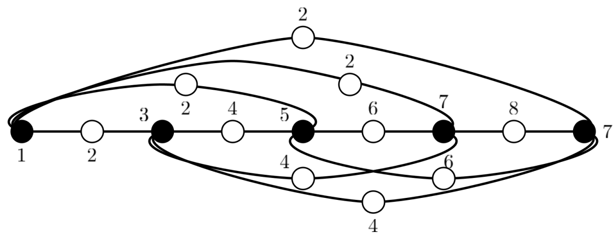

Figure 3. Strict strong chromatic number of central graph on complete graph K_5 is 8.

Theorem 2.10. *For a complete bipartite graph $K_{m,n}$,*

$$\chi_{ss}(C(K_{m,n})) = \begin{cases} m+n, & \text{if } m \neq 2 \text{ or } n \neq 2 \\ m+n+1, & \text{otherwise} \end{cases}$$

Proof. Let $v_{m1}, v_{m2}, \dots, v_{mk}, v_{n1}, v_{n2}, \dots, v_{nk}, k \geq 1$ be the vertices of $K_{m,n}$ and let $c_{i,j}$ be the vertex which divides the edge $v_{mi}v_{nj}, 1 \leq i, j \leq k$.

Case 2.11. $m = n = 1$.

In this case it is easy to observe from figure 5 that $\chi_{ss}(C(K_{1,1})) = 2 = m+n$.

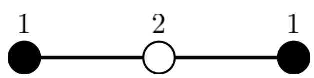

Figure 4. Strict strong chromatic number of central graph on complete bipartite graph $K_{1,1}$ is 2.

Case 2.12. $m = 2$ *or* $n = 2$.

Consider a coloring $\mathscr{C} = \{v_{m1} \cup v_{n1}\} \bigcup \{v_{m2}\} \bigcup \{\{v_{ni}\} : 2 \leq i \leq k\} \bigcup \{c_{i,j} : i = \{m1, m2\}, n1 \leq j \leq n(k-1)\} \bigcup \{c_{m2,nk}\}$. Clearly each vertex v_{mi}, where $i = \{1,2\}$, v_{nj}, where $1 \leq j \leq k$ and $c_{i,j}, i \neq m1$ and $j \neq n1$, totally dominates some color class $\{\{v_{ni}\} : 2 \leq i \leq k\}$. Further the vertex $c_{m1,n1}$ totally dominates the color class $\{v_{m1} \cup v_{n1}\}$ and the vertex v_{m2} totally dominate the color class $\{c_{m2,nk}\}$. Hence $\chi_{ss}(C(K_{2,n})) = m+n+1$. Similarly, we can prove that $\chi_{ss}(C(K_{m,2})) = m+n+1$.

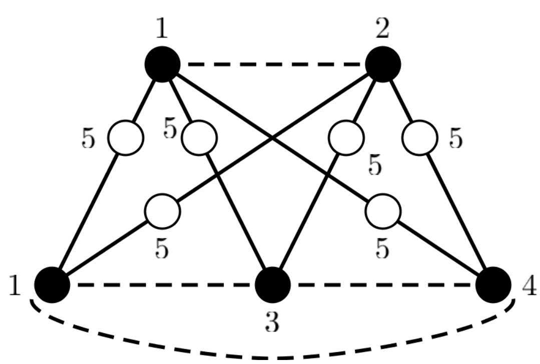

Figure 5. Strict strong chromatic number of central graph on complete bipartite graph $K_{2,3}$ is 5.

Case 2.13. $m \geq 3$ *and* $n \geq 3$.

Consider a coloring $\mathscr{C} = \{v_{m1} \cup v_{n1}\} \bigcup \{\{v_{mi}\} : 2 \leq i \leq k\} \bigcup \{\{v_{nj}\} : 2 \leq j \leq k\} \bigcup \{c_{i,j} : m1 \leq i \leq mk, n1 \leq j \leq nk\}$. Clearly each vertex v_{mi}, v_{nj} and $c_{i,j}$, where $i \neq m1$ and $j \neq n1$ totally dominates some color class $\{\{v_{mi}\} : 2 \leq i \leq k\}$ or some color class $\{\{v_{nj}\} : 2 \leq j \leq k\}$. Further the vertex $c_{m1,n1}$ totally dominate the color class $\{v_{m1} \cup v_{n1}\}$. Hence $\chi_{ss}(C(K_{m,n})) = m+n$. □

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