Malaya Journal of Matematik, Vol. 9, No. 1, 701-708, 2021

https://doi.org/10.26637/MJM0901/0123

A study of variational iteration method for solving various types of problems

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Abstract

This paper presents the application of He's Variational Iteration Method for various types of problems. This method is applied to find successive approximate solutions of first order differential equation with single condition, second order differential equations with two conditions, an isoperimetric problem and Volterra integral equations of second kind. It was shown that they are converging to their exact solutions. The successive approximations and exact solution are shown graphically.

Keywords

Successive Approximate Solution, Variational Iteration Method

AMS Subject Classification

65F10.

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Article History: Received **14** January **2021**; Accepted **27** February **2021**

Contents

1. Introduction

J.H. He [1–3] has initiated the Variational Iteration Method. Usually modeling problems yield either differential or partial differential equations. For their physical interpretation we need their solutions. The methods given in the theory of differential equations may fail to give solutions of certain differential and partial differential equations. There are several methods to get the approximate solutions of such equations. Adomain decomposition method, pertubation method, numerical methods can be applied to get approximate solution of differential and partial differential equation. Numerous researchers [4–7] are using variation iteration method to find the solutions of such problems. The variational iteration method is easy to compute the successive approximations. In a fewer iterations we get exact solutions of problems if solution of those problems exist. In this paper, we find the solutions of first order differential equation with one given condition, second order differential equation with two given conditions, isoperimetric problem and Volterra integral equation using the variational iteration method. Further it is observed that the successive approximations converges to the exact solution in two iterations in some problems.

2. Variational Iterative Method

We briefly explain the concept of the Variational Iterative Method to find the solution of the equations of the type

$$L[y(x)] + N[y(x)] = g(x) \tag{2.1}$$

where L is linear operator, N is a non-linear operator and g(x) is known continuous function. This method is introduced by

J.H.He [8] and the solutions of the problems are approximated by a set of functions that may contain unknown constants which are to be determined either from initial conditions or boundary values. He introduced two terms namely restricted variation and correction functional. The correction functional of (2.1) is given by

$$y_{n+1}(x) = y_n(x) + \int_{x_0}^{x} \lambda(t,x)[L[y_n(t)] + N[y_n(t)] - g(t)]dt \tag{2.2}$$

where $\lambda(t,x)$ is a general Lagrange Multiplier which can be computed using variational theory. Let $\lambda^f(t,x)$ be the computed Lagrange multiplier. Further $y_n(x)$ is the n^{th} approximate solution and $\tilde{y_n}(x)$ denotes restricted variation, that it $\delta\tilde{y}_n(x) = 0$. The successive approximations $y_n(x)$ are computed by the variation iteration formula

$$y_{n+1}(x) = y_n(x) + \int_{x_0}^{x} \lambda^f(t,x)\{L[y_n(t)] + N[y_n(t)] - g(t)\}dt \tag{2.3}$$

and properly chosen initial approximation $y_0(x)$.

The solution of (2.1) is given by

$$y(x) = \lim_{n\to\infty} y_n(x)$$

Ji-Huan He and XuHong Wu [9] given that the Lagrange multiplier is

$$\lambda^f(t,x) = \frac{(-1)^k}{(k-1)!}(t-x)^{k-1} \tag{2.4}$$

for the differential equation

$$y^{(k)} + f(y, y', y'', ..., y^{(k-1)}) = 0. \tag{2.5}$$

3. Variational Iterative method for obtaining the solution of first order differential equation with single given condition

Example 3.1. *Suppose* $y'(x) + y = 2e^x$ *and the given condition is* $y(0) = 1$.

Here $Ly = y', Ny = y$ and $g(x) = 2e^x$. Order of linear operator is one and given equation is in the form of equation (2.1). For the problem which is under consideration, the $(n+1)^{th}$ approximation is given by

$$y_{n+1}(x) = y_n(x) - \int_0^x \{y_n'(t) + y_n(t) - 2e^t\}dt \tag{3.1}$$

Choose $y_0(x) = y(0)$. Then $y_0(x) = 1$ since $y(0) = 1$. Substituting $n = 0$ in equation (3.1) we get

$$y_1(x) = y_0(x) - \int_0^x \{y_0'(t) + y_0(t) - 2e^t\}dt$$

Simplifying we get

$$y_1(x) = 1 - \int_0^x \{0 + 1 - 2e^t\}dt = 2e^x - 1 - x.$$

Substituting $n = 1$ in equation (3.1), we get

$$y_2(x) = y_1(x) - \int_0^x \{y_1'(t) + y_1(t) - 2e^t\}dt$$
$$= y_1(x) - \int_0^x \{2e^t - 1 + 2e^t - 1 - t - 2e^t\}dt$$
$$= 1 + x + \frac{x^2}{2}.$$

Similarly, we get

$$y_3 = 2e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}), y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!},$$
$$y_5 = 2e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}),$$
$$y_6 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

and so on.

The exact solution and approximate solutions of IVP are shown in the Fig.1. It is observed that the even numbered approximations are below the exact solution curve and odd numbered approximations are above of it. Further it is observed that the subsequence $y_2, y_4, y_6, y_8, ...$of $\{y_n\}$ converge to exact solution e^x and another subsequence $y_1, y_3, y_5, ...$is also converge to exact solution $y = e^x$.

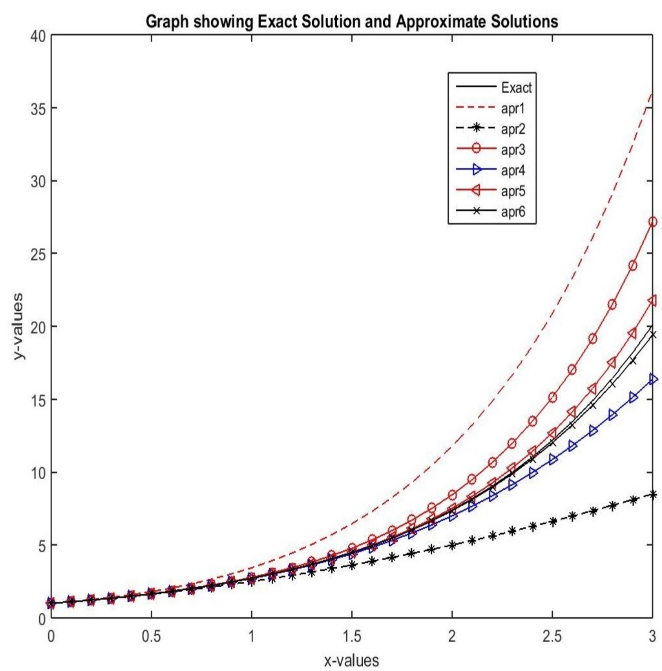

Fig.1

Example 3.2. *Suppose $y'(x) - 2x = \cos x$ and given condition is $y(0) = 0$.*

The exact solution of this first order differential equation is $y = x^2 + \sin x$. The variation iteration formula for the problem considered is

$$y_{n+1}(x) = y_n(x) - \int_0^x \{y_n'(t) - 2t - \cos t\}dt \quad (3.2)$$

Substituting $n = 0$ in (3.2) we obtain

$$y_1(x) = y_0(x) - \int_0^x \{y_0'(t) - 2t - \cos t\}dt \quad (3.3)$$

Choose $y_0(x) = 0$. Then the equation (3.3) reduces to

$$y_1(x) = -\int_0^x \{-2t - \cos t\}dt = x^2 + \sin x \quad (3.4)$$

Substituting $n = 1$ in equation (3.2) we get

$$y_2(x) = y_1(x) - \int_0^x \{y_1'(t) - 2t - \cos t\}dt$$
$$= y_1(x) - \int_0^x \{2t + \cos t - 2t - \cos t\}dt = y_1(x)$$

Therefore, $y_k(x) = x^2 + \sin x$ for $k = 1, 2, 3, ...$
We can observe that in the first iteration itself we got the exact solution.

4. Variational Iterative method for obtaining the solution of second order differential equations with two given conditions

Example 4.1. *Suppose the second order differential equation is $y'' - y + x = 0$ and the given two conditions are $y(0) = 0$ and $y'(0) = 0$.*

The exact solution of it is $y(x) = \frac{e^{-x}}{2} - \frac{e^x}{2} + x$. The variation iteration formula for the problem under consideration is

$$y_{n+1}(x) = y_n(x) + \int_0^x (t - x)\{y_n''(t) - y_n(t) + t\}dt \quad (4.1)$$

Choose $y_0(x) = 0$. Substituting $n = 0$ in the equation (4.1) we get

$$y_1(x) = y_0(x) + \int_0^x (t - x)\{y_0''(t) - y_0(t) + t\}dt = -\frac{x^3}{3!}$$

Similarly substituting $n = 1, 2, \ldots$ in (4.1) we get

$$y_2(x) = -\frac{x^3}{3!} - \frac{x^5}{5!}, y_3(x) = -\frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!},$$
$$y_4(x) = -\frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!}, \ldots.$$

and n^{th} approximation is given by $y_n(x) = -\sum_{k=1}^{n} \frac{x^{2k+1}}{(2k+1)!}$
Further $y(x) = \lim_{n\to\infty} y_n(x) = \frac{e^{-x}}{2} - \frac{e^x}{2} + x$ which is exact solution of the problem considered. Curves representing exact solution, first, second, third and fourth approximations are shown graphically in Fig.2 to Fig.5 . It is observed that higher numbered approximations are very very close to the curve representing the exact solution of the problem.

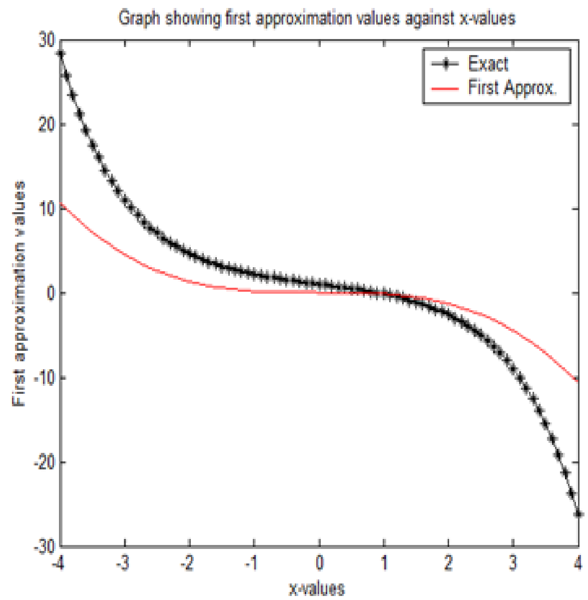

Fig.2

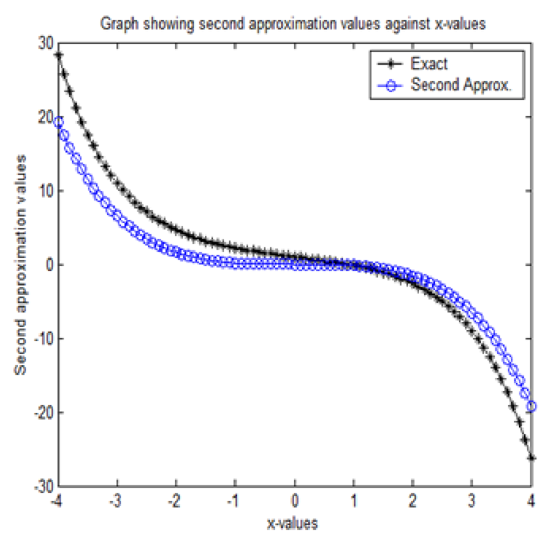

Fig.3

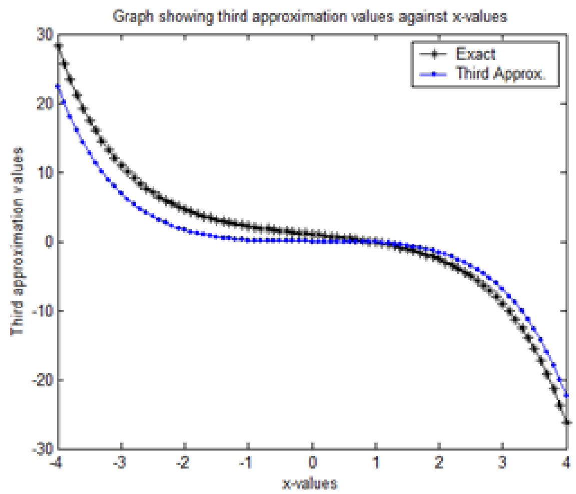

Fig.4

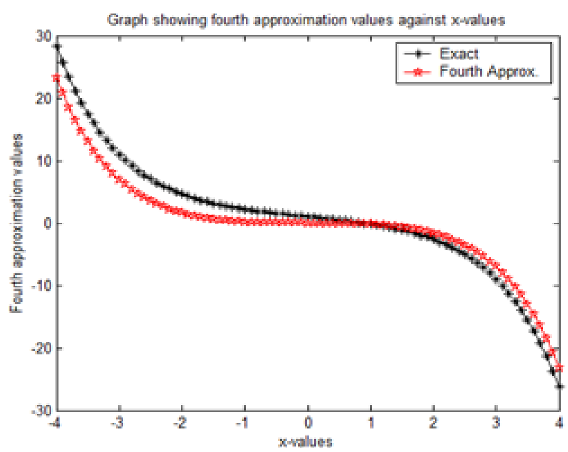

Fig.5

Example 4.2. *We know that $y''(x)+16y(x)=0$ with the conditions $y(0)=1$ and $y'(0)=0$ represent simple harmonic motion. The exact solution of this problem is $y=\cos(4x)$.*

The variation iteration formula for $y''(x)+16y(x)=0$ is

$$y_{n+1}(x)=y_n(x)+\int_0^x (t-x)\{y_n''(t)+16y_n(t)\}dt \quad (4.2)$$

Choose $y_0(x)=1$. Substitute $n=0$ in the equation (4.2) we get

$$y_1(x)=y_0(x)+\int_0^x (t-x)\{y_0''(t)+16y_0(t)\}dt$$
$$=1+\int_0^x (t-x)\{16\}dt=1-8x^2$$

$$y_1(x)=1-\frac{(4x)^2}{2!} \quad (4.3)$$

Substituting $n=1$ in the equation (4.2) we get

$$y_2(x)=y_1(x)+\int_0^x (t-x)\{y_1''(t)+16y_1(t)\}dt$$
$$=1-8x^2+\int_0^x (t-x)\{-16+16(1-8t^2)\}dt$$
$$y_2(x)=y_1(x)+\frac{32}{3}x^4=1-\frac{(4x)^2}{2!}+\frac{(4x)^4}{4!}$$

Continuing this process we get

$$y_n(x)=1-\frac{(4x)^2}{2!}+\frac{(4x)^4}{4!}-\frac{(4x)^6}{6!}+.......+(-1)^n\frac{(4x)^{2n}}{(2n)!}$$

Consequently the solution of the differential equation is

$$y(x)=\lim_{n\to\infty} y_n(x)=\cos(4x)$$

In Fig.6 to Fig.9 we have drawn the curves representing the exact solution and first, second , third, fourth approximations separately to know how the approximations are approaching to the exact values of function which is the solution of considered problem. It is observed that higher numbered approximations are close to the exact solution.

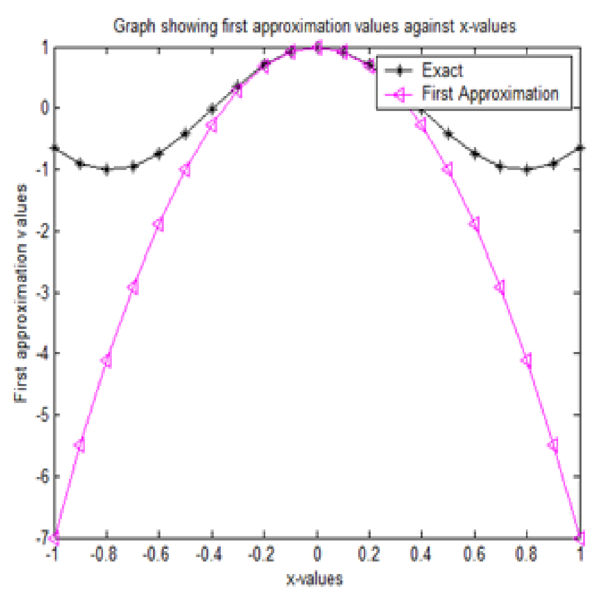

Fig.6

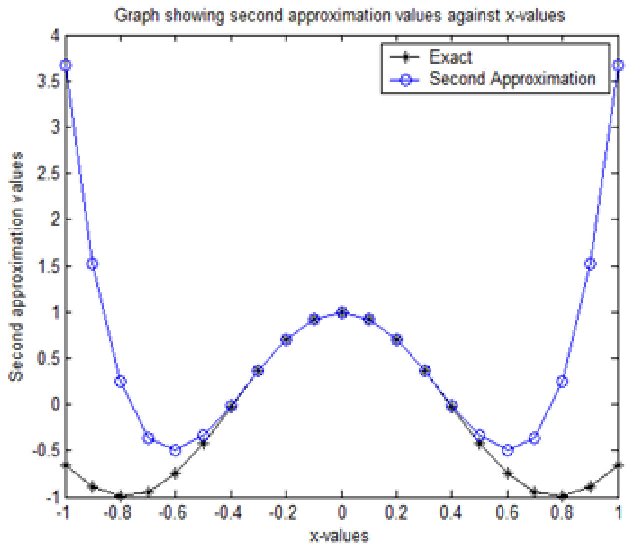

Fig.7

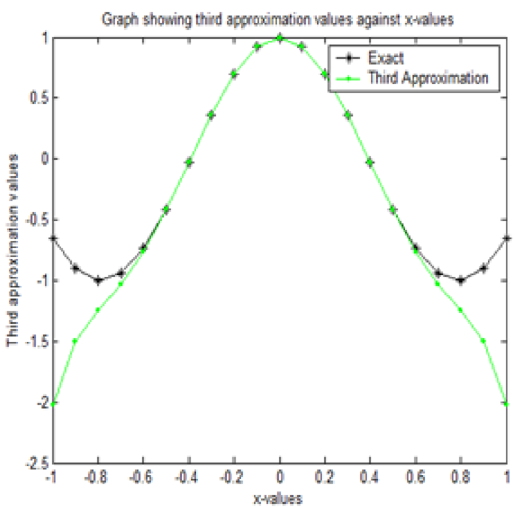

Fig.8

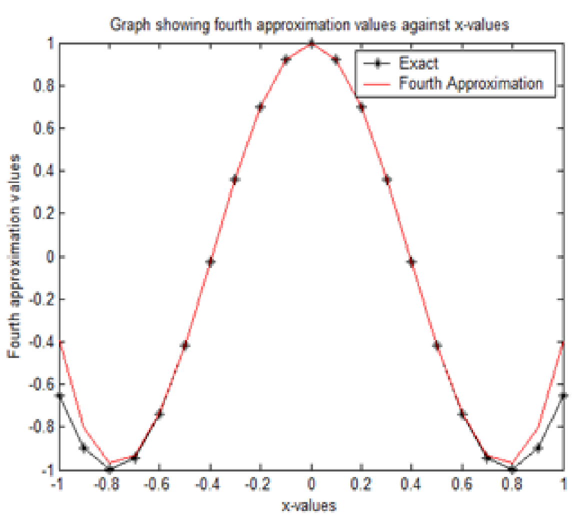

Fig.9

5. Variational Iterative method for the solution of Volterra Integral equation

Example 5.1. *Let the Volterra integral equation be*

$$y(x) = x - \int_0^x (x-t)y(t)dt \tag{5.1}$$

Using Leibnitz rule, differentiating (5.1) with 'x' we obtain

$$y'(x) = 1 - \int_0^x \frac{\partial}{\partial x}\{(x-t)y(t)\}dt \Rightarrow y'(x) = 1 - \int_0^x y(t)dt \tag{5.2}$$

Simplifying (5.2) we get

$$y'(x) + \int_0^x y(t)dt - 1 = 0 \tag{5.3}$$

Hence the iteration formula is given by

$$y_{n+1}(x) = y_n(x) - \int_0^x [y_n'(t) - 1 + \int_0^t y_n(s)ds]dt \tag{5.4}$$

From equation (5.1) we have $y(0) = 0$. Choose $y_0(x) = 0$.
Substituting $n = 0$ in (5.4) we get $y_1(x) = x$.
Now, substituting $n = 1$ in the equation (5.4) we get

$$\begin{aligned} y_2(x) &= y_1(x) - \int_0^x [y_1'(t) - 1 + \int_0^t y_1(s)ds]dt \\ &= x - \int_0^x [1 - 1 + \int_0^t sds]dt = x - \frac{x^3}{3!} \end{aligned}$$

$$y_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, y_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, ...$$

$$y(x) = \lim_{n\to\infty} y_n(x) = \sin x$$

In this example it is observed that the fourth approximation is more accurate than preceeding approximations to the exact solution and it is shown in the Fig.10 to Fig.13 .

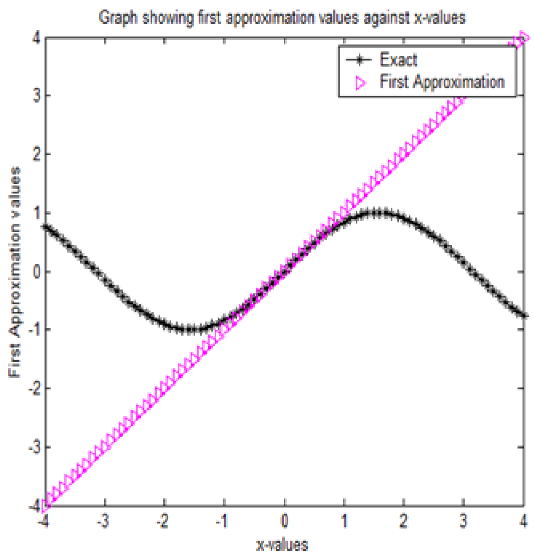

Fig.10

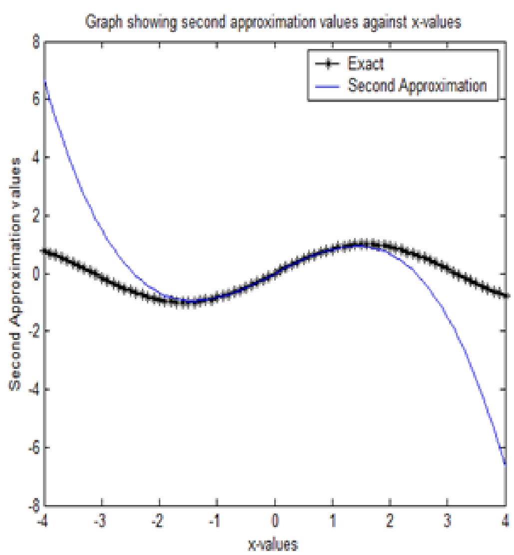

Fig.11

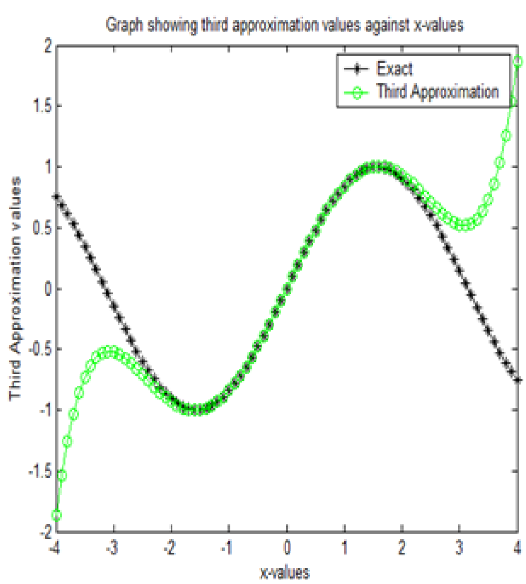

Fig.12

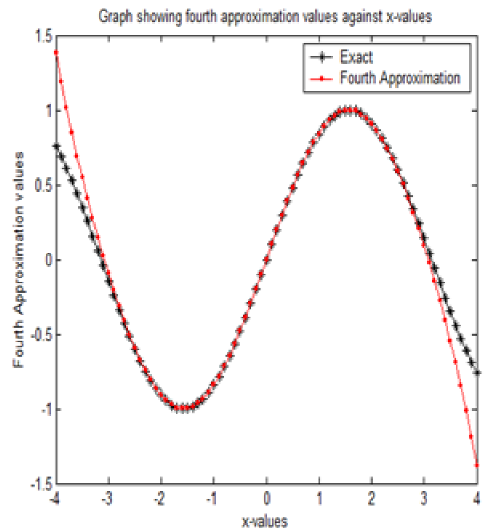

Fig.13

Example 5.2. *The equation given below is a Volterra integral equation*

$$y(x) = 1 + x + \int_0^x (x-t)y(t)dt \tag{5.5}$$

Differentiating the above equation we get

$$y'(x) = 1 + \int_0^x \frac{\partial}{\partial x}[(x-t)y(t)]dt$$

i.e.,

$$y'(x) - \int_0^x y(t)dt - 1 = 0 \tag{5.6}$$

The variational iteration formula for (5.6) is given by

$$y_{n+1}(x) = y_n(x) - \int_0^x [y_n'(t) - 1 - \int_0^t y_n(s)ds]dt \tag{5.7}$$

From (5.5) we get $y(0) = 1$. Choose$y_0(x) = 1$. Substituting $n = 0$ in equation (5.7) we obtain

$$y_1(x) = y_0(x) - \int_0^x [y_0'(t) - 1 - \int_0^t y_0(s)ds]dt$$
$$= 1 - \int_0^x [0 - 1 - \int_0^t 1ds]dt = 1 + x + \frac{x^2}{2!}$$

Thus $y_1(x) = 1 + x + \frac{x^2}{2!}$. Substituting $n = 1, 2, 3, \dots.$ in equation (5.7) we obtain

$$y_2(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!},$$

$$y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!},$$

$$y_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}$$

,...... and so on

The solution of the Volterra integral equation is

$$y(x) = \lim_{n\to\infty} y_n(x) = e^x$$

The approximations and exact solution of the problem are shown in the Fig.14 to Fig.17 . It is observed that the curve representing the fourth approximation and the curve representing the exact solutions coincide for small values of x.

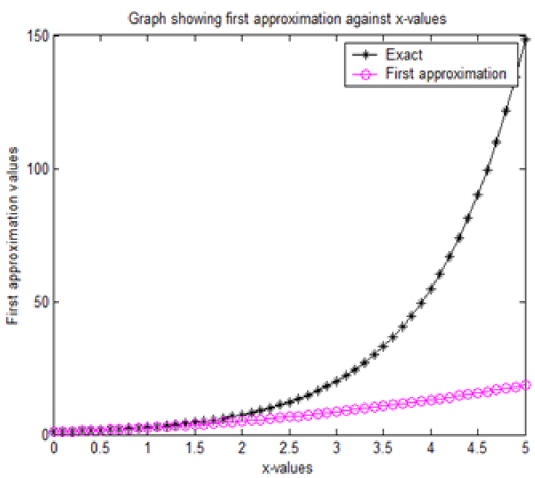

Fig.14

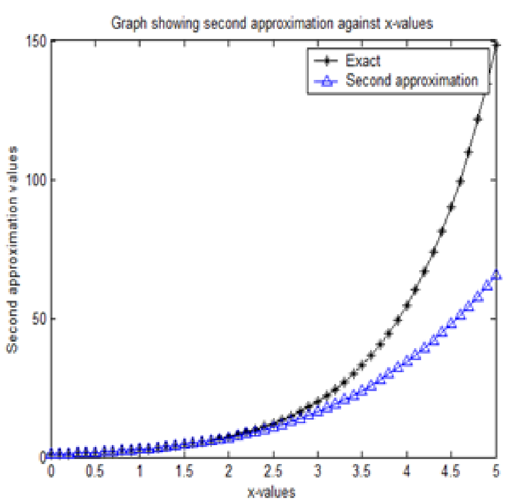

Fig.15

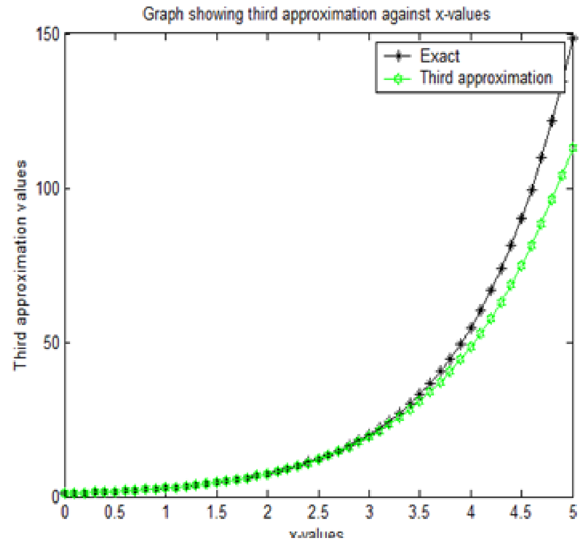

Fig.16

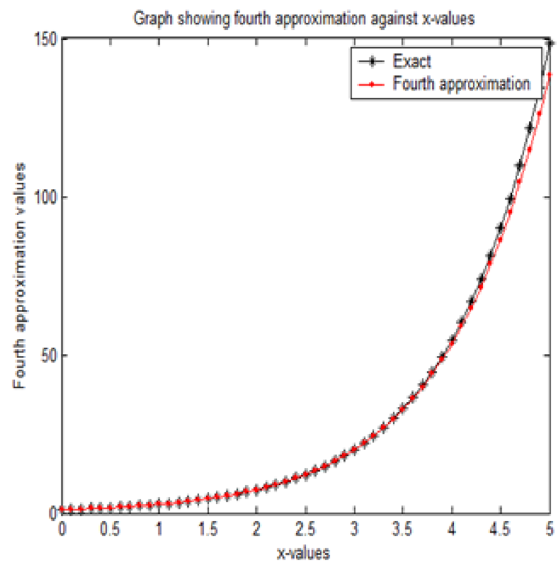

Fig.17

6. Variational Iteration Method for solving isoperimetric problem

Example 6.1. *We obtain the extremal of the functional* $\int_0^1 y'^2(x)dx$ *subject to the conditions* $y(0) = 1$ *,*$y(1) = 1$ *and* $\int_0^1 y(x)dx = 2$*. The exact solution of this problem is* $y(x) = 1+6x-6x^2$.

Let F^* be the auxiliary function and λ_{lmi} be the Lagrange multiplier of isoperimetric problem.
Then $F^* = y'(x)^2 + \lambda_{lmi} y(x)$
Then, Euler Lagrange equation $\frac{\partial F^*}{\partial y} - \frac{d}{dx}\frac{\partial F^*}{\partial y'} = 0$ for this problem gives the following differential equation.

$$2y''(x) - \lambda_{lmi} = 0 \tag{6.1}$$

The variational iteration formula for the differential equation

(6.1) we obtain as

$$y_{n+1}(x) = y_n(x) + \int_0^x (t-x)\{y_n''(t) - \lambda_{lmi}\}dt \tag{6.2}$$

we choose

$$y_0(x) = 1 + ax - ax^2 \tag{6.3}$$

So that it satisfies $y(0) = 1$, $y(1) = 1$. From the isoperimetric condition

$$\int_0^1 y_0(x)dx = 2 \Rightarrow \int_0^1 (1+ax-ax^2)dx = 2 \Rightarrow a = 6 \tag{6.4}$$

Hence

$$y_0(x) = 1 + 6x - 6x^2 \tag{6.5}$$

Substituting $n = 0$ in (6.2) we get

$$y_1(x) = y_0(x) + \int_0^x (t-x)\{y_0''(t) - \lambda_{lmi}\}dt$$

In view of (6.4) and (6.5) we have

$$y_1(x) = 1 + 6x - 6x^2 + \int_0^x (t-x)\{-24 - \lambda_{lmi}\}dt$$

$$y_1(x) = 1 + 6x - 6x^2 + (24 + \lambda_{lmi})\frac{x^2}{2} \tag{6.6}$$

Now from the isoperimetric condition we have

$$\int_0^1 y_1(x)dx = 2 \Rightarrow \int_0^1 \{1 + 6x - 6x^2 + (24+\lambda_{lmi})\frac{x^2}{2} = 2 \tag{6.7}$$

$$\Rightarrow \lambda_{lmi} = -24 \tag{6.8}$$

Hence, $y_1(x) = 1 + 6x - 6x^2$ which is equal to the exact solution. It can be shown that $y_k(x) = y_1(x)(k = 2,3,4,\ldots.)$. In the first iteration itself we got the exact solution of the considered isoperimetric problem.

7. Conclusion

This paper is aimed to obtain the approximate solution of initial value problems of first and second orders, Volterra integral equation and isoperimetric problem. By observing the solutions of problems that we considered one can say that Variational Iterative Method developed by J.H.He can be applied to various types of problems. By observing the graphs of exact and approximate solutions we can say that solution is obtained in fewer iterations.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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