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Fuglede-Putnam type commutativity theorems for EP operators

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Abstract

Fuglede-Putnam theorem is not true in general for EP operators on Hilbert spaces. We prove that under some conditions the theorem holds good. If the adjoint operation is replaced by Moore-Penrose inverse in the theorem, we get Fuglede-Putnam type theorem for EP operators – however proofs are totally different. Finally, interesting results on EP operators have been proved using several versions of Fuglede-Putnam type theorems for EP operators on Hilbert spaces.

Keywords

Fuglede-Putnam theorem, Moore-Penrose inverse, EP operator.

AMS Subject Classification

47A05, 15A09, 47B99.

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Contents

1. Introduction

A square matrix A over the complex field $\mathbb{C}$ is said to be an EP matrix if ranges of A and A^* are equal. Although the EP matrix was defined by Schwerdtfeger [19] in 1950, it could not get any greater attention until Pearl [15] characterized it through Moore-Penrose inverse in 1966. The normed space of all bounded linear operators from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$ is denoted by $\mathscr{B}(\mathscr{H},\mathscr{K})$. We write $\mathscr{B}(\mathscr{H},\mathscr{H}) = \mathscr{B}(\mathscr{H})$. If $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$, we denote the kernel of T by $\mathscr{N}(T)$ and the range of T by $\mathscr{R}(T)$. The operator T is said to be invertible if its inverse exists and is bounded. Given $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$, $S \in \mathscr{B}(\mathscr{K},\mathscr{H})$ is the adjoint operator on $\mathscr{H}$ if $\langle Tx,y\rangle = \langle x,Sy\rangle$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$; in this case the operator S is denoted by T^*. If $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ with a closed range, then $T^\dagger$ is the unique linear operator in $B(\mathscr{K},\mathscr{H})$ satisfying

$$TT^\dagger T = T,\ T^\dagger TT^\dagger = T^\dagger,\ TT^\dagger = (TT^\dagger)^* \text{ and } T^\dagger T = (T^\dagger T)^*.$$

The operator $T^\dagger$ is called the Moore-Penrose inverse of T. It is well-known that an operator T has a closed range if and only if its Moore-Penrose inverse $T^\dagger$ exists. The class $\mathscr{B}_c(\mathscr{H})$ denotes the set of all operators in $\mathscr{B}(\mathscr{H})$ having closed ranges. For any nonempty set $\mathscr{M}$ in $\mathscr{H}$, $\mathscr{M}^\perp$ denotes the orthogonal complement of $\mathscr{M}$. Note that if $T \in \mathscr{B}_c(\mathscr{H})$, then $T^* \in \mathscr{B}_c(\mathscr{H})$, $\mathscr{N}(T)^\perp = \mathscr{R}(T^*)$, $\mathscr{N}(T^*)^\perp = \mathscr{R}(T)$ and $\mathscr{R}(T) = \mathscr{R}(TT^*)$. An operator $T \in \mathscr{B}_c(\mathscr{H})$ is said to be an EP operator if $\mathscr{R}(T) = \mathscr{R}(T^*)$. EP matrices and operators have been studied by many authors [1, 2, 5, 10, 11, 13, 14, 20]. It is well-known that if T is normal with a closed range, or an invertible operator, then T is EP. The converse is not true even in a finite dimensional space.

The Fuglede-Putnam theorem (first proved by B. Fuglede [7] and then by C. R. Putnam [16] in a more general version) plays a major role in the theory of bounded (and unbounded) operators. Many authors have worked on it since the papers of Fuglede and Putnam got published [6, 8, 9, 12]. There are various generalizations of the Fuglede-Putnam theorem to non-normal operators, for instance, hyponormal, subnormal, etc. This paper is devoted to the study of Fuglede-Putnam type theorems for EP operators.

In section 2, we give some known characterizations for EP operators and we give a procedure to construct an EP matrix T (preferably non-normal) for the given subspace $\mathscr{W}$ of the unitary space $\mathbb{C}^n$ such that $\mathscr{R}(T) = \mathscr{W}$. This construction has been used in the paper to construct suitable examples of EP matrices. We show in section 3 that the Fuglede theorem [7] is not true in general for EP operators (Example 3.2) and we prove that the commutativity relation in Fuglede-Putnam theorem is true for EP operators if the adjoint operation is replaced by Moore-Penrose inverse. Moreover, several versions of Fuglede-Putnam type theorems are given for EP operators. In the last section, we prove some interesting results using Fuglede-Putnam type theorems for EP operators on Hilbert spaces.

2. Preliminaries

Let $\mathscr{H}$ be a complex Hilbert space. An operator on $\mathscr{H}$ means a linear operator from $\mathscr{H}$ into itself. Given an EP operator T on $\mathscr{H}$, we get a closed subspace $\mathscr{R}(T)$ which is the same as $\mathscr{R}(T^*)$. On the other hand, one may ask whether every closed subspace $\mathscr{M}$ of $\mathscr{H}$ is the range of some EP operator (not necessarily normal) on $\mathscr{H}$. The answer is in the affirmative in a finite dimensional Hilbert space $\mathscr{H}$. We give a procedure to construct such EP matrices and this construction has been used in the sequel to provide suitable examples of EP matrices. We use the letters S,T for EP operators ; M,N for normal operators and A,B for bounded operators.

We start with some known characterizations of EP operators.

Theorem 2.1. *[1, 15] Let $T \in \mathscr{B}_c(\mathscr{H})$. Then the following are equivalent :*

1. *T is EP ;*
2. *$TT^\dagger = T^\dagger T$;*
3. *$\mathscr{N}(T)^\perp = \mathscr{R}(T)$;*
4. *$\mathscr{N}(T) = \mathscr{N}(T^*)$;*
5. *$T^* = PT$, where P is some bijective bounded operator on $\mathscr{H}$.*

Example 2.2. *Let $T : \ell_2 \to \ell_2$ be defined by*

$$T(x_1,x_2,x_3,\dots) = (x_1+x_2, 2x_1+x_2+x_3, -x_1-x_3, x_4, x_5, \dots).$$

Then $T^*(x_1,x_2,x_3,x_4,x_5,\dots) = (x_1+2x_2-x_3, x_1+x_2, x_2-x_3, x_4,\dots)$ and $\mathscr{N}(T) = \mathscr{N}(T^*) = \{(x_1,-x_1,-x_1,0,0,\dots) : x_1 \in \mathbb{C}\}$. But $TT^* \neq T^*T$. Since $\mathscr{N}(T)$ is finite dimensional, $\mathscr{R}(T)$ is closed. Hence T is an EP operator but not normal.

Theorem 2.3. *If $\mathscr{W}$ is a subspace of $\mathbb{C}^n$, then there exists an EP matrix T of order n such that $\mathscr{R}(T) = \mathscr{W}$.*

Proof. If $\mathscr{W}$ is a trivial subspace of $\mathbb{C}^n$, then it holds trivially. Without loss of generality, let $\mathscr{W}$ be a subspace of $\mathbb{C}^n$ with of dimension $n-1$. Then $\mathscr{W}$ can be expressed as $\Big\{(x_1,x_2,\dots,x_{i-1},\sum_{k=1}^{n-1} a_k x_k, x_i,\dots,x_{n-1}) \;:\; x_k \in \mathbb{C}, k = 1,2,\dots,n-1\Big\}$. Let $\Big\{v_j = \big(x_{j1},x_{j2},\dots,x_{j(i-1)},\sum_{k=1}^{n-1} a_k x_{jk}, x_{ji},\dots,x_{j(n-1)}\big), j = 1,2,\dots,n-1\Big\}$ be a basis for $\mathscr{W}$ which can be regarded as column vectors.

Take $T = \begin{bmatrix} v_1 & v_2 & \cdots & v_{i-1} & v' & v_i & \cdots & v_{n-1}\end{bmatrix}$ where

$$v' = \left(\sum_{k=1}^{n-1}\overline{a_k}x_{k1}, \sum_{k=1}^{n-1}\overline{a_k}x_{k2},\dots,\sum_{j=1}^{n-1}\sum_{k=1}^{n-1} a_j\overline{a_k}x_{kj},\dots,\sum_{k=1}^{n-1}\overline{a_k}x_{k(n-1)}\right).$$

Since the columns of T contain a basis of $\mathscr{W}$, $\mathscr{R}(T) = \mathscr{W}$. Now we need to show that T is EP. But the selection of v' ensures that each row of T is in $\mathscr{W}$. Hence $\mathscr{R}(T^*) = \mathscr{W}$. Therefore the result is true when dimension of $\mathscr{W}$ is $n-1$.

For the sake of completeness we also prove the result when the dimension of $\mathscr{W}$ is $n-2$ and continuing the same technique to construct EP matrices for lesser dimension of $\mathscr{W}$. Suppose that $\mathscr{W}$ is of dimension $n-2$. Then $\mathscr{W}$ can be expressed as

$$\Big\{(x_1,x_2,\dots,x_{i-1},\sum_{k=1}^{n-2} a_k x_k, x_i,\dots,x_{\ell-1},\sum_{k=1}^{n-2} b_k x_k, x_\ell,\dots,x_{n-2}) \;:\; x_k \in \mathbb{C}, k = 1,2,\dots,n-2\Big\}.$$

Let

$$\Big\{v_j = \big(x_{j1},\dots,x_{j(i-1)},\sum_{k=1}^{n-2} a_k x_{jk}, x_{ji},\dots,x_{j(\ell-1)},\sum_{k=1}^{n-2} b_k x_{jk}, x_{j\ell},\dots x_{j(n-2)}\big), j = 1,2,\dots,n-2\Big\}$$

be a basis for $\mathscr{W}$ which can be regarded as column vectors. Take $T = \begin{bmatrix} v_1 & \cdots & v_{i-1} & v' & v_i & \cdots & v_{\ell-1} & v'' & v_\ell & \cdots & v_{n-2}\end{bmatrix}$ where

$$v' = \left(\sum_{k=1}^{n-2}\overline{a_k}x_{k1}, \sum_{k=1}^{n-2}\overline{a_k}x_{k2},\dots,\sum_{j=1}^{n-2}\sum_{k=1}^{n-2} a_j\overline{a_k}x_{kj},\dots, \sum_{j=1}^{n-2}\sum_{k=1}^{n-2} b_j\overline{a_k}x_{kj},\dots,\sum_{k=1}^{n-2}\overline{a_k}x_{k(n-2)}\right)$$

and

$$v'' = \left(\sum_{k=1}^{n-2}\overline{b_k}x_{k1}, \sum_{k=1}^{n-2}\overline{b_k}x_{k2},\dots,\sum_{j=1}^{n-2}\sum_{k=1}^{n-2} a_j\overline{b_k}x_{kj},\dots, \sum_{j=1}^{n-2}\sum_{k=1}^{n-2} b_j\overline{b_k}x_{kj},\dots,\sum_{k=1}^{n-2}\overline{b_k}x_{k(n-2)}\right).$$

As in the first case, $\mathscr{R}(T) = \mathscr{R}(T^*) = \mathscr{W}$. □

Remark 2.4. *If T is a complex EP matrix of rank 1, then it must be normal, by the result ([3], Theorem 1.3.3): If T is a complex matrix of rank 1, then its Moore-Penrose inverse is of the form $T^\dagger = \frac{1}{\alpha}T^*$, where $\alpha =$ trace (T^*T).*

Remark 2.5. *If T is a real EP matrix of rank 1, then it must be a symmetric matrix. Indeed, as in Remark 2.4, T is a normal matrix. Hence by spectral theorem $T = UDU^*$, for some unitary matrix U and*

$$D = \begin{bmatrix} d & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$$

where $d =$ trace (T) and $\mathbb{O}$ is the zero matrix of appropriate order. As T is real and $T^* = UD^*U^*$, we have $D = D^*$ and hence T is symmetric.

Example 2.6. *Let $\mathscr{W} = \{(x_1, x_1 + x_2, x_2) : x_1, x_2 \in \mathbb{C}\}$ be a subspace of $\mathbb{C}^3$ with basis $v_1 = (1, 1+i, i), v_2 = (1, 0, -1)$. By the proof of the Theorem 2.3, we have $v' = (2, 1+i, 1)$. Then $T = \begin{bmatrix} 1 & 2 & 1 \\ 1+i & 1+i & 0 \\ i & i-1 & -1 \end{bmatrix}$. Here T is an EP matrix (non-normal) with $\mathscr{R}(T) = \mathscr{W}$.*

Conjecture 2.7. *Let $\mathscr{W}$ be a closed subspace of a Hilbert space $\mathscr{H}$. Then there exists an EP (non-normal) operator T on $\mathscr{H}$ such that $\mathscr{R}(T) = \mathscr{W}$.*

3. Fuglede-Putnam type theorems for EP operators

The well-known Fuglede theorem for a bounded operator is stated as follows.

Theorem 3.1. *[7]. Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator and $A \in \mathscr{B}(\mathscr{H})$. If $AN = NA$, then $AN^* = N^*A$.*

The following example illustrates that Fuglede theorem does not hold good for EP operators. The theorem cannot be extended to the set of EP operators on $\mathscr{H}$ even though every normal operator with a closed range is EP.

Example 3.2. *Consider the EP operator T on ℓ_2 defined by $T(x_1, x_2, x_3, \ldots) = (x_1 - x_2, x_1 + x_3, 2x_1 - x_2 + x_3, x_4, \ldots)$ and $A \in \mathscr{B}(\ell_2)$ defined by $A(x_1, x_2, x_3, \ldots) = (x_2, -x_1 + x_2 - x_3, -2x_1 + x_2, x_4, \ldots)$. Here $AT = TA$ but $AT^* \neq T^*A$.*

We have seen in the above example that Fuglede theorem is not true in general for EP operators. The following theorem is a Fuglede type theorem which proves that if an EP operator and a bounded operator commute, then the EP operator commutes with the Moore-Penrose inverse of the bounded operator. Our result just replaces the "adjoint" operation by the "Moore-Penrose inverse" in the Fuglede theorem stated in Theorem 3.1, however proofs are totally different.

Theorem 3.3. *Let T be an EP operator on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = TA$, then $AT^\dagger = T^\dagger A$.*

Proof. As T is an EP operator, we have $TT^\dagger = T^\dagger T$. From the assumption $AT = TA$, we have
$AT^\dagger = AT^\dagger TT^\dagger = AT(T^\dagger)^2 = TA(T^\dagger)^2 = TT^\dagger TA(T^\dagger)^2 = T^\dagger TAT(T^\dagger)^2 = T^\dagger TAT^\dagger = T^\dagger ATT^\dagger = T^\dagger TT^\dagger AT^\dagger T = (T^\dagger)^2 TAT^\dagger T = (T^\dagger)^2 ATT^\dagger T = (T^\dagger)^2 AT = (T^\dagger)^2 TA = T^\dagger A$. ☐

Example 3.4. *The assumption that T is an EP operator cannot be dropped in Theorem 3.3. For instance, $A = T$ is an bounded operator on ℓ_2 defined by $T(x_1, x_2, x_3, \ldots) = (x_1 + x_2, 2x_1 + 2x_2, x_3, \ldots)$. Then $T^\dagger(x_1, x_2, x_3, \ldots) = (\frac{1}{10}(x_1 + 2x_2), \frac{1}{10}(x_1 + 2x_2), x_3, \ldots)$. Note that T is not an EP operator and $AT = TA$ but $AT^\dagger \neq T^\dagger A$.*

Under some conditions, we prove that Fuglede theorem is true for EP operators and we give examples which embellish that those conditions are necessary.

Theorem 3.5. *Let T be an EP operator on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = TA$ and $AT^*T = T^*TA$, then $AT^* = T^*A$.*

Proof. Suppose $T \in \mathscr{B}(\mathscr{H})$ is an EP operator with $AT = TA$ and $AT^*T = T^*TA$. Then by Theorem 3.3, we have $AT^* = A(TT^\dagger T)^* = AT^*(TT^\dagger)^* = AT^*TT^\dagger = T^*TAT^\dagger = T^*TT^\dagger A = (TT^\dagger T)^*A = T^*A$. ☐

Theorem 3.6. *Let T be an EP operator on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = TA$ and $AT^\dagger T^* = T^\dagger T^*A$, then $AT^* = T^*A$.*

Proof. As $T \in \mathscr{B}(\mathscr{H})$ is an EP operator, we have $TT^\dagger = T^\dagger T$. From the given facts $AT = TA$ and $AT^\dagger T^* = T^\dagger T^*A$, we have $AT^* = A(TT^\dagger T)^* = AT^\dagger TT^* = ATT^\dagger T^* = TAT^\dagger T^* = TT^\dagger T^*A = (TT^\dagger T)^*A = T^*A$. ☐

Example 3.7. *The condition $AT^*T = T^*TA$ is essential in Theorem 3.5. Consider the EP operator T on ℓ_2 defined by $T(x_1, x_2, x_3, \ldots) = (x_1 + x_3, 0, x_3, \ldots)$ and $A \in \mathscr{B}(\ell_2)$ defined by $A(x_1, x_2, x_3, \ldots) = (x_1 + 2x_3, -x_2, x_3, \ldots)$. Then $AT = TA$ and $AT^*T \neq T^*TA$. But $AT^* \neq T^*A$.*

Example 3.8. *The condition $AT^\dagger T^* = T^\dagger T^*A$ cannot be dropped in Theorem 3.6. Let A and T be as in Example 3.2. Then $AT = TA$ and $AT^\dagger T^* \neq T^\dagger T^*A$. But $AT^* \neq T^*A$.*

Fuglede theorem was generalized for two normal operators by Putnam, which is well-known as Fuglede-Putnam theorem and is stated as follows.

Theorem 3.9. *[16] Let N, M be bounded normal operators on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AN = MA$, then $AN^* = M^*A$.*

Fuglede-Putnam theorem is not true in general if we replace bounded normal operators by EP operators, as shown in the following example.

Example 3.10. *Consider the EP operators T and S on ℓ_2 are defined by*

$$T(x_1, x_2, x_3, \ldots) = (x_1 + x_3, 0, x_3, \ldots)$$

and $S(x_1, x_2, x_3, \ldots) = (x_1 + x_2, x_2, , 0, x_4, \ldots)$ and $A \in \mathscr{B}(\ell_2)$ is defined by $A(x_1, x_2, x_3, \ldots) = (x_1 - x_3, x_3, 2x_2, x_4, \ldots)$. Then $AT = SA$. But $AT^* \neq S^*A$.

Theorem 3.11. *Let T, S be EP operators on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = SA$ and $AT^*T = S^*SA$, then $AT^* = S^*A$.*

Proof. Suppose that $T,S \in \mathscr{B}(\mathscr{H})$ are EP operators with $AT = SA$ and $AT^*T = S^*SA$. Then we have $AT^* = A(TT^\dagger T)^* = AT^*TT^\dagger = S^*SAT^\dagger = S^*SS^\dagger A = (SS^\dagger S)^*A = S^*A$. □

Example 3.12. *The condition $AT^*T = S^*SA$ in Theorem 3.11 is essential. Let T,S be EP operators and A be the operator as in Example 3.10. Here $AT^*T \neq S^*SA$ and $AT = SA$ but $AT^* \neq S^*A$.*

Theorem 3.13. *Let T,S be EP operators on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = SA$ and $AT^\dagger T^* = S^\dagger S^*A$, then $AT^* = S^*A$.*

Proof. As T and S are EP operators with $AT = SA$ and $AT^\dagger T^* = S^\dagger S^*A$, we have $AT^* = A(TT^\dagger T)^* = AT^\dagger TT^* = ATT^\dagger T^* = SAT^\dagger T^* = SS^\dagger S^*A = (SS^\dagger S)^*A = S^*A$. □

Example 3.14. *The condition $AT^\dagger T^* = S^\dagger S^*A$ in Theorem 3.13 is essential. Let T,S be EP operators and A be the operator as in Example 3.10. Here $AT^\dagger T^* \neq S^\dagger S^*A$ and $AT = SA$ but $AT^* \neq S^*A$.*

The following Fuglede-Putnam type theorem for EP operators is a generalization of Theorem 3.3 involving two EP operators.

Theorem 3.15. *Let T,S be EP operators on $\mathscr{H}$ and $A \in \mathscr{B}(\mathscr{H})$. If $AT = SA$, then $AT^\dagger = S^\dagger A$.*

Proof. As T and S are EP operators, we have $TT^\dagger = T^\dagger T$ and $SS^\dagger = S^\dagger S$. From the given fact $AT = SA$, we have $AT^\dagger = AT^\dagger TT^\dagger = AT(T^\dagger)^2 = SA(T^\dagger)^2 = SS^\dagger SA(T^\dagger)^2 = S^\dagger S AT(T^\dagger)^2 = S^\dagger SAT^\dagger = S^\dagger ATT^\dagger = S^\dagger SS^\dagger AT^\dagger T = (S^\dagger)^2SAT^\dagger T = (S^\dagger)^2ATT^\dagger T = (S^\dagger)^2AT = (S^\dagger)^2SA = S^\dagger A$. □

Example 3.16. *In the Theorem 3.15, if one of the operators, T or S fails to be EP, then the theorem is not valid. Consider the EP operator T on ℓ_2 defined by $T(x_1,x_2,x_3,\ldots) = (x_1+x_3,0,x_3,\ldots)$ and the non-EP operator S on ℓ_2 defined by $S(x_1,x_2,x_3,\ldots) = (x_1+x_2,0,0,x_4,\ldots)$. Let $A \in \mathscr{B}(\ell_2)$ be defined by $A(x_1,x_2,x_3,\ldots) = (x_2+2x_3,-x_2,-x_2,x_4)$. Then $AT = SA$. But $AT^\dagger \neq S^\dagger A$.*

Theorem 3.17. *Let T,S be EP operators on $\mathscr{H}$. If $A,B \in \mathscr{B}(\mathscr{H})$ with $AT = SB$ and $AT^2 = S^2B$, then $AT^\dagger = S^\dagger B$.*

Proof. Suppose that $A,B,T,S \in \mathscr{B}(\mathscr{H})$ with $AT = SB$ and $AT^2 = S^2B$, where T and S are EP operators. Then $AT^\dagger = A(T^\dagger TT^\dagger) = ATT^\dagger T^\dagger = SBT^\dagger T^\dagger = SS^\dagger SBT^\dagger T^\dagger = S^\dagger S^2BT^\dagger T^\dagger = S^\dagger AT^2T^\dagger T^\dagger = S^\dagger ATT^\dagger = S^\dagger S^\dagger SATT^\dagger = S^\dagger S^\dagger S^2 BT^\dagger = S^\dagger S^\dagger AT^2T^\dagger = S^\dagger S^\dagger AT = S^\dagger S^\dagger SB = S^\dagger B$. □

Example 3.18. *The assumptions that T and S are EP operators in Theorem 3.17 cannot be dropped. For instance, let $A,B,T,S \in \mathscr{B}(\ell_2)$ be defined by*

$$\begin{aligned} A(x_1,x_2,x_3,\ldots) &= (x_2,x_1,x_3,\ldots), \\ B &= I, \\ T(x_1,x_2,x_3,\ldots) &= (x_1+x_2,-x_1-x_2,x_3,\ldots), \\ S(x_1,x_2,x_3,\ldots) &= (-x_1-x_2,x_1+x_2,x_3,\ldots). \end{aligned}$$

Here both T,S are not EP operators with $AT = S = SB$. But $AT^\dagger \neq S^\dagger B$.

Example 3.19. *The condition $AT^2 = S^2B$ in Theorem 3.17 is essential. For instance, let $T,S \in \mathscr{B}(\ell_2)$ be EP operators defined by $T(x_1,x_2,x_3,\ldots) = (x_1-x_2,x_1+x_3,2x_1-x_2+x_3,x_4,\ldots)$ and $S(x_1,x_2,x_3,\ldots) = (x_1+x_2,x_2,x_3,\ldots)$ and let $A,B \in \mathscr{B}(\ell_2)$ be defined by $A(x_1,x_2,x_3,\ldots) = (x_1+2x_2-x_3,-x_1-x_2+x_3,2x_1+2x_2-2x_3,x_4,\ldots)$ and $B(x_1,x_2,\ldots) = (x_1+x_3,0,x_1+x_2,x_4,\ldots)$ be such that $AT = SB$ and $AT^2 \neq S^2B$. But $AT^\dagger \neq S^\dagger B$.*

Theorem 3.20. *Let T be an EP operator on $\mathscr{H}$ and $A,B \in \mathscr{B}(\mathscr{H})$. If $AT = TB$ and $BT = TA$, then $AT^\dagger = T^\dagger B$ and $BT^\dagger = T^\dagger A$.*

Proof. From given hypotheses, $(A+B)T = T(A+B)$. By Theorem 3.3,

$$\begin{aligned} (A+B)T^\dagger &= T^\dagger(A+B) \\ AT^\dagger + BT^\dagger &= T^\dagger A + T^\dagger B \\ AT^\dagger - T^\dagger B &= T^\dagger A - BT^\dagger. \end{aligned} \tag{3.1}$$

Again using given hypotheses, $(A-B)T = -T(A-B)$. By Theorem 3.15,

$$\begin{aligned} (A-B)T^\dagger &= -T^\dagger(A-B) \\ AT^\dagger - BT^\dagger &= -T^\dagger A + T^\dagger B \\ AT^\dagger - T^\dagger B &= -T^\dagger A + BT^\dagger. \end{aligned} \tag{3.2}$$

Adding (3.1) and (3.2), we have $AT^\dagger = T^\dagger B$. Similarly subtracting (3.2) from (3.1), we have $BT^\dagger = T^\dagger A$. □

Theorem 3.21. *Let T,S be EP operators on $\mathscr{H}$ and $A,B \in \mathscr{B}(\mathscr{H})$. If $AT = SB$ and $BT = SA$, then $AT^\dagger = S^\dagger B$ and $BT^\dagger = S^\dagger A$.*

Proof. From given hypotheses, $(A+B)T = S(A+B)$. By Theorem 3.15,

$$\begin{aligned} (A+B)T^\dagger &= S^\dagger(A+B) \\ AT^\dagger + BT^\dagger &= S^\dagger A + S^\dagger B \\ AT^\dagger - S^\dagger B &= S^\dagger A - BT^\dagger. \end{aligned} \tag{3.3}$$

Again using given hypotheses, $(A-B)T = -S(A-B)$. By Theorem 3.15,

$$\begin{aligned} (A-B)T^\dagger &= -S^\dagger(A-B) \\ AT^\dagger - BT^\dagger &= -S^\dagger A + S^\dagger B \\ AT^\dagger - S^\dagger B &= -S^\dagger A + BT^\dagger. \end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), we have $AT^\dagger = S^\dagger B$. Similarly subtracting (3.4) from (3.3), we have $BT^\dagger = S^\dagger A$.

□

4. Consequences of Fuglede-Putnam type theorems for EP operators

The product of EP operators is not an EP operator in general.

Example 4.1. *Let $S,T \in \mathscr{B}(\ell_2)$ be defined by*

$$S(x_1,x_2,x_3,\dots) = (x_1+x_2,x_1+x_2,x_3,\dots)$$

and $T(x_1,x_2,x_3,\dots) = (0,x_2,x_3,\dots)$. Here S and T are EP operators, but the product ST is not an EP operator.

Djordjević has given a necessary and sufficient condition for product of two EP operators to be an EP operator again.

Theorem 4.2. *[4] Let S,T be EP operators on $\mathscr{H}$. Then the following statements are equivalent:*

1. *ST is an EP operator ;*
2. *$\mathscr{R}(ST) = \mathscr{R}(S) \cap \mathscr{R}(T)$ and $\mathscr{N}(ST) = \mathscr{N}(S) + \mathscr{N}(T)$.*

The following example illustrates the fact that there are operators S and T in $\mathscr{B}_c(\mathscr{H})$ such that $ST \in \mathscr{B}_c(\mathscr{H})$ but $TS \notin \mathscr{B}_c(\mathscr{H})$. We have proved that when S and T are EP operators, the closed rangeness of ST implies the closed rangeness of TS and vice-versa.

Example 4.3. *[17] Let S be an operator on ℓ_2 defined by $S(x_1,x_2,x_3,\dots) = (x_1,0,x_2,0,\dots)$ and T be another operator on ℓ_2 defined by $T(x_1,x_2,x_3,\dots) = (\frac{x_1}{1}+x_2, \frac{x_3}{3}+x_4, \frac{x_5}{5}+x_6,\dots)$. One can verify that both S and T are bounded operators and are having closed ranges. Also, $\mathscr{R}(ST)$ is closed but $\mathscr{R}(TS)$ is not closed.*

Theorem 4.4. *[18] Let S and T be EP operators on $\mathscr{H}$. Then $\mathscr{R}(ST)$ is closed if and only if $\mathscr{R}(TS)$ is closed.*

Example 4.5. *Consider the EP operators $S,T \in \mathscr{B}(\ell_2)$ defined by*

$$S(x_1,x_2,x_3,\dots) = (x_1+x_2,x_2,x_3,\dots)$$

and $T(x_1,x_2,x_3,\dots) = (x_1,0,x_3,\dots)$. Here ST is an EP operator, but TS is not EP.

Theorem 4.6. *Let $S,T \in \mathscr{B}(\mathscr{H})$ such that $(ST)^\dagger = T^\dagger S^\dagger$. Then ST and TS are EP if and only if $S^\dagger ST = TSS^\dagger$ and $STT^\dagger = T^\dagger TS$.*

Proof. Suppose ST and TS are EP. Then $(ST)^\dagger$ and $(TS)^\dagger$ are also EP. Hence we have $S^\dagger(ST)^\dagger = S^\dagger T^\dagger S^\dagger = (TS)^\dagger S^\dagger$. Therefore by Theorem 3.15, we have $S^\dagger ST = TSS^\dagger$. In a similar way we have $(ST)^\dagger T^\dagger = T^\dagger S^\dagger T^\dagger = T^\dagger (TS)^\dagger$. Now we use Theorem 3.15, we get $STT^\dagger = T^\dagger TS$. Conversely, suppose we have

$$S^\dagger ST = TSS^\dagger \quad (4.1)$$
$$STT^\dagger = T^\dagger TS. \quad (4.2)$$

From the equation (4.1), we get $T^\dagger S^\dagger ST = T^\dagger TSS^\dagger$ and from the equation (4.2), we get $STT^\dagger S^\dagger = T^\dagger TSS^\dagger$. Since the right side of these two equations are same, we have $T^\dagger S^\dagger ST = STT^\dagger S^\dagger$. Hence $(ST)^\dagger ST = ST(ST)^\dagger$. Therefore ST is EP. Similarly from the equation (4.1), we get $S^\dagger STT^\dagger = TSS^\dagger T^\dagger$ and from the equation (4.2), we get $S^\dagger STT^\dagger = S^\dagger T^\dagger TS$. Therefore $TSS^\dagger T^\dagger = S^\dagger T^\dagger TS$. Hence $TS(TS)^\dagger = (TS)^\dagger TS$. Thus TS is EP. □

Corollary 4.7. *Let $S = UP \in \mathbb{C}^{n\times n}$ be a polar decomposition of S where $U \in \mathbb{C}^{n\times n}$ is unitary and $P \in \mathbb{C}^{n\times n}$ is positive semidefinite Hermitian and let $T \in \mathbb{C}^{n\times n}$ with $(ST)^\dagger = T^\dagger S^\dagger$. If TU is EP and $PTU = TUP$, then ST and TS are EP.*

Proof. Suppose TU is EP and $PTU = TUP$, then $TSS^\dagger = T(UP)(UP)^\dagger = TUPP^\dagger U^* = PTUP^\dagger U^* = PP^\dagger TUU^* = PP^\dagger T = P^\dagger PT = P^\dagger U^* UPT = (UP)^\dagger UPT = S^\dagger ST$. Since TU is EP and $PTU = TUP$, we have $P(TU)^\dagger = (TU)^\dagger P$. Therefore $STT^\dagger = UPTUU^* T^\dagger = UPTU(TU)^\dagger = UTUP(TU)^\dagger = UTU(TU)^\dagger P = U(TU)^\dagger TUP = UU^* T^\dagger TUP = T^\dagger TS$. Thus by Theorem 4.6, ST and TS are EP. □

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