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The area of the Bézier polygonal region of the BézierCurve and derivatives in E^3

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Abstract. In the paper, we have first defined the area of the Bézier polygonal region which contains the n^{th} $order$ Bézier Curve and its first, second and third derivatives based on the control points of n^{th} $order$ Bézier curve in E^3. Further, the area of the Bézier polygonal region containing the 5^{th} $order$ Bézier curve and the corresponding derivatives is examined based on the control points of $5^{th}order$ Bézier Curve in E^3.

AMS Subject Classifications: 53A04, 53A05.

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Contents

1. Introduction

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion. To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users design the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D paths for key frame interpolation. We have been motivated by the following studies. In [2, 6], the use of Bézier curves on object modeling purposes has been given for Computer-Aided Geometric designs. Moreover, Bézier curves with curvature and torsion continuity has been examined in [8]. In [13], Frenet apparatus of the cubic Bézier curves have been examined in E^3. The matrix representations for a given Bézier curve and its derivatives have been contented in [7, 10–12, 17].In addition, the use and the generation method of Bézier curves have other possible applications as given in [1, 3–5, 9]. Recently, the examination of a Bézier curve by means of curve pairs such as involute, Bertrand or Mannheim partner curves has been given in [14–16].

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2. Preliminaries

A Bézier curve is defined by a set of control points P_0 through P_n, where n is called its order. If $n = 1$ for linear, if $n = 2$ for quadratic, if $n = 3$ for cubic Bézier curve, etc. The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve. Generaly Bézier curve can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ and has the following form, the points P_i are called control points for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with P_0 and finishing with P_n, is called the Bézier polygon (or control polygon). Bézier curve with $n+1$ control points $P_0, P_1, ..., P_n$ has the following equation [2, 6]

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} (t) [P_i] \ , \quad t \in [0, 1]$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ are the binomial coefficients.

Theorem 2.1. *The derivatives of a given Bézier curve* $\mathbf{B}(t)$ *is*

$$\mathbf{B}'(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} Q_i$$

where $Q_i = n(P_{i+1} - P_i)$ *[2, 6].*

Given points P_0 and P_1, a linear Bézier curve is simply a straight line between those two points. Linear Bézier curve is given by $\alpha(t) = (1-t)P_0 + tP_1$ and also the matrix form of a linear Bézier curve is

$$\alpha(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}.$$

A quadratic Bézier curve is the path traced by the function $\alpha(t)$, given points P_0, P_1 and P_2 which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from P_0 to P_1 and from P_1 to P_2 respectively.and also a quadratic Bézier curve has the matrix form with control points P_0 , P_1 and P_2

$$\alpha(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}.$$

Four points in the plane or in higher-dimensional space define a cubic Bézier curve with the following equation $\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$ with the matrix form of a cubic Béziercurve with control points P_0, P_1, P_2, and P_3, is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

We have already examined the cubic Bézier curves and involutes in [13] and [14], respectively.

3. The area of the Bézier polygonal regions

Definition 3.1. *The* P_i *polygon formed by connecting the Bézier control points with lines, starting with* P_0 *and finishing with* P_n*, is called the Bézier polygon (or control polygon). The convex hull of the Bézier polygon contains the Bézier curve.*

Definition 3.2. *The area of the Bézier polygonal region containing the n^{th} $order$ Bézier Curve which is given as*

$$\alpha(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} (t) [P_i] \ , \quad t \in [0,1] .$$

with control points P_0, P_1, ..., P_n is defined as the sum of the area of the each area of triangles $\Delta(P_0, P_1, P_2)$, $\Delta(P_0, P_2, P_3)$, $\Delta(P_0, P_3, P_4)$, ...,$\Delta(P_0, P_{n-1}, P_n)$ as in the following way

$$A(P_0, P_1, ..., \ P_n) = A(P_0, P_1, P_2) + A(P_0, P_2, P_3) + ... + A(P_0, P_{n-1}, P_n) .$$

Theorem 3.3. *The area of the Bézier polygonal region containing the 5^{th} $order$ BézierCurve and derivatives in E^3 is*

$$A(P_0, P_1, P_2, P_3, P_4, \ P_5) = \frac{1}{2} \sum_{i=1}^{4} \| P_0 \wedge (P_i + P_{i+1}) \|$$

Proof. From the definition the area of the Bézier polygonal region containing the 5^{th} $order$ Bézier Curve

$$\alpha(t) = \sum_{i=0}^{5} \binom{5}{i} t^i (1-t)^{5-i} (t) [P_i] \ , \quad t \in [0,1] .$$

with control points P_0, P_1, P_2, P_3, P_4, and P_5 is defined as the sum of the area of the each area of triangles $\Delta(P_0, P_1, P_2)$, $\Delta(P_0, P_2, P_3)$, $\Delta(P_0, P_3, P_4)$, and $\Delta(P_0, P_4, P_5)$ as in the following way

$$A(P_0, P_1, P_2, P_3, P_4, \ P_5) = A(P_0, P_1, P_2) + A(P_0, P_2, P_3) + A(P_0, P_3, P_4) + A(P_0, P_4, P_5) .$$

The matrix representation of 5^{th} $order$ Bézier curve with control points P_0, P_1, P_2, P_3, P_4, and P_5 is

$$\alpha(t) = \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

The area of the Bézier polygonal region that contains the 5^{th} $order$ BézierCurve with control points P_0, P_1, P_2, P_3, P_4, and P_5 is defined as the sum of the area of the

$$\begin{aligned} A(P_0, P_1, P_2, P_3, P_4, \ P_5) &= A(P_0, P_1, P_2) + A(P_0, P_2, P_3) + A(P_0, P_3, P_4) + A(P_0, P_4, P_5) \\ &= \frac{1}{2} \begin{pmatrix} \|P_0P_1 \wedge P_0P_2\| + \|P_0P_2 \wedge P_0P_3\| + \|P_0P_3 \wedge P_0P_4\| \\ + \|P_0P_4 \wedge P_0P_5\| \end{pmatrix} \\ &= \frac{1}{2} \sum_{i=1}^{4} \| P_0 \wedge (P_i + P_{i+1}) \| . \end{aligned}$$

■

We can generalize the above theorem to the n^{th} $order$ of a Bézier curve, hence we get the following theorem;

Theorem 3.4. *The area of the Bézier polygonal region having the n^{th} order Bézier Curve and derivatives in E^3 is*

$$A(P_0, P_1, P_2, P_3, ..., \ P_n) = \frac{1}{2} \sum_{i=1}^{n-1} \| P_0 \wedge (P_i + P_{i+1}) \| .$$

Theorem 3.5. *The area of the Bézier polygonal region having the first derivative of* 5^{th} *order of a Bézier curve as a* $4th$ *order Bézier curve with control points*P_0, P_1, P_2, P_3, P_4, *and* P_5 *of* 5^{th} *order BézierCurve*

$$A\left(Q_0,Q_1,Q_2,Q_3,Q_4\right)=\frac{25}{2}\sum_{i=1}^{3}\left\|\left(P_0-P_1\right)\wedge\left(P_i-P_{i+2}\right)\right\|$$

Proof. The matrix representation of the first derivative of 5^{th} $order$ of a Bézier curve as a $4th$ order Béziercurve with control points Q_0,Q_1,Q_2,Q_3,Q_4

$$\alpha'\left(t\right)=\begin{bmatrix}t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}1&-4&6&-4&1\\-4&12&-12&4&0\\6&-12&6&0&0\\-4&4&0&0&0\\1&0&0&0&0\end{bmatrix}\begin{bmatrix}Q_0\\Q_1\\Q_2\\Q_3\\Q_4\end{bmatrix}$$

where the control points, $Q_0=5\left(P_1-P_0\right),\,Q_1=5\left(P_2-P_1\right),\,Q_2=5\left(P_3-P_2\right),Q_3=5\left(P_4-P_3\right)$, and $Q_4=5\left(P_5-P_4\right)$ respectively. The area of the Bézier polygonal region contains the first derivative of 5^{th} $order$ of a Bézier curve as a $4th$ order Bézier curve with control points Q_0,Q_1,Q_2,Q_3,Q_4 is

$$\begin{aligned}A\left(Q_0,Q_1,Q_2,Q_3,Q_4\right)&=\frac{1}{2}\sum_{i=1}^{3}\left\|Q_0\wedge\left(Q_i+Q_{i+1}\right)\right\|\\&=A\left(Q_0,Q_1,Q_2\right)+A\left(Q_0,Q_2,Q_3\right)+A\left(Q_0,Q_3,Q_4\right)\\&=\frac{1}{2}\left(\left\|Q_0Q_1\wedge Q_0Q_2\right\|+\left\|Q_0Q_2\wedge Q_0Q_3\right\|+\left\|Q_0Q_3\wedge Q_0Q_4\right\|+\left\|Q_0Q_4\wedge Q_0Q_5\right\|\right)\\&=\frac{1}{2}\left(\left\|\left(Q_1+Q_2\right)\wedge\left(-Q_0\right)\right\|+\left\|\left(Q_2+Q_3\right)\wedge\left(-Q_0\right)\right\|+\left\|\left(Q_3+Q_4\right)\wedge\left(-Q_0\right)\right\|\right)\\&=\frac{1}{2}\left(\left\|Q_0\wedge\left(Q_1+Q_2\right)\right\|+\left\|Q_0\wedge\left(Q_2+Q_3\right)\right\|+\left\|Q_0\wedge\left(Q_3+Q_4\right)\right\|\right)\\&=\frac{1}{2}\sum_{i=1}^{3}\left\|Q_0\wedge\left(Q_i+Q_{i+1}\right)\right\|\end{aligned}$$

Also using the control points P_0, P_1, P_2, P_3, P_4, and P_5 of 5^{th} $order$ BézierCurve

$$\begin{aligned}2A\left(Q_0,Q_1,Q_2,Q_3,Q_4\right)=&\left\|Q_0\wedge\left(Q_1+Q_2\right)\right\|+\left\|Q_0\wedge\left(Q_2+Q_3\right)\right\|+\left\|Q_0\wedge\left(Q_3+Q_4\right)\right\|\\=&\left\|5\left(P_1-P_0\right)\wedge\left(5\left(P_2-P_1\right)+5\left(P_3-P_2\right)\right)\right\|\\&+\left\|5\left(P_1-P_0\right)\wedge\left(5\left(P_3-P_2\right)+5\left(P_4-P_3\right)\right)\right\|\\&+\left\|5\left(P_1-P_0\right)\wedge\left(5\left(P_4-P_3\right)+5\left(P_5-P_4\right)\right)\right\|\\=&25\left\|\left(P_1-P_0\right)\wedge\left(\left(P_2-P_1\right)+\left(P_3-P_2\right)\right)\right\|\\&+25\left\|\left(P_1-P_0\right)\wedge\left(\left(P_3-P_2\right)+\left(P_4-P_3\right)\right)\right\|\\&+25\left\|\left(P_1-P_0\right)\wedge\left(\left(P_4-P_3\right)+\left(P_5-P_4\right)\right)\right\|\\=&25\left\|\left(P_0-P_1\right)\wedge\left(P_1-P_3\right)\right\|\\&+25\left\|\left(P_0-P_1\right)\wedge\left(P_2-P_4\right)\right\|\\&+25\left\|\left(P_0-P_1\right)\wedge\left(P_3-P_5\right)\right\|.\end{aligned}$$

This complete the proof. ■

If we generalize the above theorem to the n^{th} $order$ of a Bézier curve we get the following theorem;

Theorem 3.6. *The area of the Bézier polygonal region containing the first derivative of n^{th} $order$ of a Bézier curve as a $(n-1)^{th}$ order Béziercurve with control points $Q_0, Q_1, Q_2, ..., Q_{n-1}$ is*

$$A\left(Q_0, Q_1, Q_2, ..., Q_{n-1}\right) = \frac{1}{2}\sum_{i=1}^{n-2} \left\| Q_0 \wedge \left(Q_i + Q_{i+1}\right)\right\|.$$

Also using the control points P_0, P_1, ..., P_n of n^{th} $order$ BézierCurve

$$A\left(Q_0, Q_1, Q_2, ..., Q_{n-1}\right) = \frac{1}{2}n^2\sum_{i=1}^{n-2} \left\| \left(P_0 - P_1\right) \wedge \left(P_i - P_{i+2}\right)\right\|.$$

Theorem 3.7. *The area of the Bézier polygonal region containing the second derivative of 5^{th} $order$ of a Bézier curve as a $3rd$ order Béziercurve with control points P_0, P_1, P_2, P_3, P_4, and P_5 of 5^{th} $order$ BézierCurve is*

$$A\left(R_0, R_1, ...,\ R_{n-2}\right) = \frac{20^2}{2}\sum_{i=1}^{n-3} \left\| \left(P_0 - 2P_1 + P_2\right) \wedge \left(P_i - P_{i+1} - P_{i+2} + P_{i+3}\right)\right\|.$$

Proof. The matrix representation of the second derivative of 5^{th} $order$ of a Bézier curve with control points R_0, R_1, R_2, R_3 is

$$\alpha^{''}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

where R_0, R_1, R_2, R_3 are control points. The area of the Bézier polygonal region having the second derivative of 5^{th} $order$ of a Bézier curve as a $3rd$ order Béziercurve with control points R_0, R_1, R_2, and R_3 is

$$A\left(R_0, R_1, ...,\ R_{n-2}\right) = \frac{1}{2}\sum_{i=1}^{2} \left\| R_0 \wedge \left(R_i + R_{i+1}\right)\right\|$$

Also using the control points P_0, P_1, P_2, P_3, P_4, and P_5 of 5^{th} $order$ BézierCurve, and

$$R_0 = 20\left(P_0 - 2P_1 + P_2\right), R_1 = 20\left(P_1 - 2P_2 + P_3\right),$$
$$R_2 = 20\left(P_2 - 2P_3 + P_4\right), R_3 = 20\left(P_3 - 2P_4 + P_5\right)$$

and

$$R_1 + R_2 = 20\left(P_1 - P_2 - P_3 + P_4\right), R_2 + R_3 = 20\left(P_2 - P_3 - P_4 + P_5\right)$$

we get the proof as in the following way

$$\begin{aligned} A\left(R_0, R_1, R_2,\ R_3\right) &= \frac{1}{2}\left(\left\|R_0 \wedge \left(R_1 + R_2\right)\right\| + \left\|R_0 \wedge \left(R_2 + R_3\right)\right\|\right) \\ &= \frac{1}{2}\left(\left\|20\left(P_0 - 2P_1 + P_2\right) \wedge \left(R_1 + R_2\right)\right\| + \left\|20\left(P_0 - 2P_1 + P_2\right) \wedge \left(R_2 + R_3\right)\right\|\right) \\ A\left(R_0, R_1, R_2,\ R_3\right) &= \frac{20^2}{2}\sum_{i=1}^{2} \left\| \left(P_0 - 2P_1 + P_2\right) \wedge \left(P_i - P_{i+1} - P_{i+2} + P_{i+3}\right)\right\|. \end{aligned}$$

■

If we generalize the above theorem to the n^{th} $order$ of a Bézier curve we get the following theorem;

Theorem 3.8. *The area of the Bézier polygonal region contains the second derivative of n^{th} $order$ of a Bézier curve as a $(n-2)^{th}$ order Béziercurve with control points is $R_0, R_1, ..., R_{n-2}$*

$$A\left(R_0, R_1, R_2,\ R_3\right) = \frac{1}{2}\sum_{i=1}^{n-3} \left\| R_0 \wedge \left(R_i + R_{i+1}\right)\right\| .$$

Also using the control points $P_0, P_1, ...,\ P_n$ of n^{th} $order$ BézierCurve

$$A\left(R_0, R_1, R_2,\ R_3\right) = \frac{1}{2}\left(n(n-1)\right)^2 \sum_{i=1}^{2} \left\| \left(P_0 - 2P_1 + P_2\right) \wedge \left(P_i - P_{i+1} - P_{i+2} + P_{i+3}\right)\right\| .$$

Theorem 3.9. *The area of the Bézier polygonal region containing the third derivative of 5^{th} $order$ of a Bézier curve as a $2nd$ order Béziercurve with control points S_0, S_1, S_2 is*

$$A\left(S_0, S_1, S_2\right) = \frac{1}{2}\left\| S_0 \wedge \left(S_1 + S_2\right)\right\|$$

Also using the control points $P_0, P_1, P_2, P_3, P_4,$ and P_5 of 5^{th} $order$ BézierCurve

$$A\left(S_0, S_1, S_2\right) = 2.60^2 \left\| \left(-P_0 + 3P_1 - 3P_2 + P_3\right) \wedge \left(-2P_0 + 5P_1 + 2P_3 + 5P_4 + P_5\right)\right\|$$

Proof. The matrix representation of the third derivative of 5^{th} $order$ of a Bézier curve with control points S_0, S_1, S_2 is

$$\alpha^{'''}(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}$$

where

$S_0 = 60\left(6P_1 - 2P_0 - 6P_2 + 2P_3\right)$, $S_1 = 60\left(2P_1 - P_0 - 2P_3 + P_4\right)$, and
$S_2 = 60\left(3P_1 - P_0 - 4P_2 + 4P_3 - 3P_4 + P_5\right)$
hence

$$S_1 + S_2 = 60\left(5P_1 - 2P_0 + 2P_3 + 5P_4 + P_5\right)$$

The area of the Bézier polygonal region for the third derivative of 5^{th} $order$ of a Bézier curve as a $2nd$ order Béziercurve with control points S_0, S_1, S_2 is

$$A\left(S_0, S_1, S_2\right) = \frac{1}{2}\left\| S_0 \wedge \left(S_1 + S_2\right)\right\| .$$

Hence

$$\begin{aligned} A\left(S_0, S_1, S_2\right) &= \frac{1}{2}\left\| \left(S_1 - S_0\right) \wedge \left(S_2 - S_0\right)\right\| \\ &= \frac{1}{2}\left\| S_0 \wedge \left(S_1 + S_2\right)\right\| \\ &= \frac{60^2}{2}\left\| \left(-2P_0 + 6P_1 - 6P_2 + 2P_3\right) \wedge \left(5P_1 - 2P_0 + 2P_3 + 5P_4 + P_5\right)\right\| \\ A\left(S_0, S_1, S_2\right) &= \frac{60^2}{2}\left\| \left(-P_0 + 3P_1 - 3P_2 + P_3\right) \wedge \left(-2P_0 + 5P_1 + 2P_3 + 5P_4 + P_5\right)\right\| . \end{aligned}$$

We have the proof. ■

If we generalize the above theorem to the n^{th} $order$ of a Bézier curve we get the following theorem;

Theorem 3.10. *The area of the Bézier polygonal region for the third derivative of n^{th} order of a Bézier curve as a $(n-3)^{nd}$ order Béziercurve with control points $S_0, S_1, ..., S_{n-3}$ is*

$$A\left(S_0, S_1, ..., S_{n-3}\right) = \frac{1}{2}\sum_{i=1}^{2} \left\| S_0 \wedge \left(S_i + S_{i+1}\right)\right\|$$

Also using the control points P_0, P_1, P_2, P_3, P_4, and P_5 of 5^{th} order BézierCurve

$$A\left(S_0, S_1, ..., S_{n-3}\right) = \frac{\left({}^{n}\boldsymbol{P}_3\right)^2}{2}\sum_{i=1}^{2} \left\|\left(-P_0 + 3P_1 - 3P_2 + P_3\right) \wedge \left(-2P_{i-1} + 5P_i + 2P_{i+2} + 5P_{i+3} + P_{i+4}\right)\right\|,$$

where ${}^{n}\boldsymbol{P}_3 = n(n-1)(n-2)$ is permutation.

Theorem 3.11. *The length of the T_0T_1, of the fourth derivative of 5^{th} order of a Bézier curve is a linear Béziercurve, with control points T_0, and T_1 is*

$$\left\|T_0T_1\right\| = 5.4.3.2.1\left\|-P_0 + 5P_1 - 10P_2 + 10P_3 - 5P_4 + P_5\right\|$$

Proof. The fourth derivative of 5^{th} *order* of a Bézier curve has the following representation.

$$\alpha'^{v}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}$$

where

$$T_0 = 120P_0 - 480P_1 + 720P_2 - 480P_3 + 120P_4$$
$$T_1 = 120P_1 - 480P_2 + 720P_3 - 480P_4 + 120P_5$$

are the control points of the fourth derivative of $5^{th}order$ of a Bézier curve based on the P_0, P_1, P_2, ..., and P_5.

$$\begin{aligned} \left\|T_0T_1\right\| &= \left\| \begin{matrix} \left(120P_1 - 480P_2 + 720P_3 - 480P_4 + 120P_5\right) \\ -\left(120P_0 - 480P_1 + 720P_2 - 480P_3 + 120P_4\right) \end{matrix} \right\| \\ &= \left\|600P_1 - 120P_0 - 1200P_2 + 1200P_3 - 600P_4 + 120P_5\right\| \\ &= 5.4.3.2.1\left\|-P_0 + 5P_1 - 10P_2 + 10P_3 - 5P_4 + P_5\right\| \end{aligned}$$

■

Example 3.12. *Let $\alpha(t)$ be a $5th$ order Bézier curve given by the following parametrization:*

$$\alpha(t) = \begin{matrix} \big(74t^5 - 210t^4 + 180t^3 - 50t^2 + 5t + 1, \\ -79t^5 + 185t^4 - 130t^3 + 10t^2 + 10t + 1, \\ -63t^5 + 95t^4 - 30t^3 - 5t + 2\big) \end{matrix}$$

with control points, $P_0 = (1,1,2), P_1 = (2,3,1), P_2 = (-2,6,0), P_3 = (7,-3,-4), P_4 = (5,0,5)$, $P_5 = (0,-3,-1)$.

The area of the Bézier polygonal region containing the 5^{th} $order$ *Bézier curve is*

$$\begin{aligned}
&A\left(P_0,P_1,P_2,P_3,P_4,\ P_5\right)\\
&=\frac{1}{2}\sum_{i=1}^{4}\left\|P_0\wedge\left(P_i+P_{i+1}\right)\right\|\\
&=\frac{1}{2}\left(\left\|P_0\wedge(0,9,1)\right\|+\left\|P_0\wedge(5,3,-4)\right\|+\left\|P_0\wedge(12,-3,1)\right\|+\left\|P_0\wedge(5,-3,4)\right\|\right)\\
&=\frac{1}{2}\left\|\begin{vmatrix} i & j & k\\ 1 & 1 & 2\\ 0 & 9 & 1\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ 1 & 1 & 2\\ 5 & 3 & -4\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ 1 & 1 & 2\\ 12 & -3 & 1\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ 1 & 1 & 2\\ 5 & -3 & 4\end{vmatrix}\right\|\\
&=39.\,531\ unit\ square.
\end{aligned}$$

The area of the Bézier polygonal region containing the first derivative of 5^{th} $order$ *of a Bézier curve is*

$$\begin{aligned}
&A\left(Q_0,Q_1,Q_2,Q_3,Q_4\right)\\
&=\frac{1}{2}5^2\sum_{i=1}^{3}\left\|\left(P_0-P_1\right)\wedge\left(P_i-P_{i+2}\right)\right\|\\
&=\frac{1}{2}5^2\left(\left\|\left(P_0-P_1\right)\wedge\left(P_1-P_3\right)\right\|+\left\|\left(P_0-P_1\right)\wedge\left(P_2-P_4\right)\right\|+\left\|\left(P_0-P_1\right)\wedge\left(P_3-P_5\right)\right\|\right)\\
&=\frac{1}{2}5^2\left(\left\|\left(-1\ \ -2\ \ \ 1\right)\wedge\left(P_1-P_3\right)\right\|+\left\|\left(P_0-P_1\right)\wedge\left(P_2-P_4\right)\right\|\right.\\
&\qquad\left.+\left\|\left(P_0-P_1\right)\wedge\left(P_3-P_5\right)\right\|\right)\\
&=\frac{1}{2}25\left(\left\|\begin{vmatrix} i & j & k\\ -1 & -2 & 1\\ -5 & 6 & 5\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ -1 & -2 & 1\\ -7 & 6 & -5\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ -1 & -2 & 1\\ 7 & 0 & -3\end{vmatrix}\right\|\right)\\
&=\frac{1551.\,0}{2}\\
&=775.\,5\ unit\ square
\end{aligned}$$

The area of the Bézier polygon that contains the second derivative of 5^{th} $order$ *of a Bézier curve as a* $3rd$ *order Béziercurve with control points* R_0, R_1, R_2, R_3 *is*

$$\begin{aligned}
A\left(R_0,R_1,R_2,\ R_3\right)&=\frac{1}{2}20^2\sum_{i=1}^{2}\left\|\left(P_0-2P_1+P_2\right)\wedge\left(P_i-P_{i+1}-P_{i+2}+P_{i+3}\right)\right\|\\
&=\frac{1}{2}20^2\left(\left\|\begin{matrix}-5 & 1 & 0\end{matrix}\wedge\left(P_1-P_2-P_3+P_4\right)\right\|+\left\|\begin{matrix}-5 & 1 & 0\end{matrix}\wedge\left(P_2-P_3-P_4+P_5\right)\right\|\right)\\
&=\frac{1}{2}20^2\left(\left\|\begin{matrix}-5 & 1 & 0\end{matrix}\wedge\left(\begin{matrix}2 & 0 & 10\end{matrix}\right)\right\|+\left\|\begin{matrix}-5 & 1 & 0\end{matrix}\wedge\left(\begin{matrix}-14 & 6 & -2\end{matrix}\right)\right\|\right)\\
&=\frac{1}{2}20^2\left\|\begin{vmatrix} i & j & k\\ -5 & 1 & 0\\ 2 & 0 & 10\end{vmatrix}\right\|+\left\|\begin{vmatrix} i & j & k\\ -5 & 1 & 0\\ -14 & 6 & -2\end{vmatrix}\right\|\\
&=\frac{20431}{2}\\
&=10.216\ unit\ square.
\end{aligned}$$

The area of the Bézier polygonal region containing the third derivative of 5^{th} $order$ *of a Bézier curve using the* *control points* P_0, P_1, P_2, P_3, P_4, *and* P_5 *of* 5^{th} *order Bézier Curve is*

$$
\begin{aligned}
A(S_0,S_1,S_2) &= \frac{1}{2}60^2 \left\|(-P_0+3P_1-3P_2+P_3)\wedge(5P_1-2P_0+2P_3+5P_4+P_5)\right\| \\
&= \frac{1}{2}.60^2 \left\| \begin{matrix} \begin{pmatrix} -\begin{pmatrix}1 & 1 & 2\end{pmatrix}+3\begin{pmatrix}2 & 3 & 1\end{pmatrix} \\ -3\begin{pmatrix}-2 & 6 & 0\end{pmatrix}+\begin{pmatrix}7 & -3 & -4\end{pmatrix} \end{pmatrix} \\ \wedge \begin{pmatrix} 5\begin{pmatrix}2 & 3 & 1\end{pmatrix}-2\begin{pmatrix}1 & 1 & 2\end{pmatrix}+2\begin{pmatrix}7 & -3 & -4\end{pmatrix} \\ +5\begin{pmatrix}5 & 0 & 5\end{pmatrix}+\begin{pmatrix}0 & -3 & -1\end{pmatrix} \end{pmatrix} \end{matrix} \right\| \\
&= \frac{1}{2}60^2 \left\|\begin{pmatrix}18 & -13 & -3\end{pmatrix}\wedge\begin{pmatrix}47 & 4 & 17\end{pmatrix}\right\| \\
&= \frac{1}{2}.60^2 \left\| \begin{vmatrix} i & j & k \\ 18 & -13 & -3 \\ 47 & 4 & 17 \end{vmatrix} \right\| \\
&= 5696.0/2 \\
&= 2848.0\ unit\ square
\end{aligned}
$$

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