Malaya Journal of Matematik, Vol. 9, No. 1, 731-734, 2021
https://doi.org/10.26637/MJM0901/0128

On total dominator coloring of middle graph, total graph and shadow graph

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Abstract
A total dominator coloring of a graph G is a proper coloring of G in which every vertex of the graph is adjacent to every vertex of some color class. The minimum number of colors required for a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we discuss some results on strict strong chromatic number of middle graph, total graph, and shadow graph.

Keywords
Proper coloring, total dominator coloring, total dominator chromatic number, middle graph, total graph, shadow graph.

AMS Subject Classification
05C15, 05C69.

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Article History: Received **18** January **2020**; Accepted **26** February **2021**

Contents

1. Introduction

All graphs considered here are simple. For graph theoretic terminology we refer to [3]. Let $G=(V,E)$ be a graph. The *degree* of a vertex $v \in V$ in a graph G is defined to be the number of edges incident with v and is denoted by deg (v). A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendent vertex* or a *leaf*. Any vertex which is adjacent to a pendant vertex is called a *support vertex*. The *open neighborhood* and *closed neighborhood* of v is $N(v)=\{u \in V : uv \in E\}$ and $N[v]=N(v) \cup \{v\}$ respectively. A subset S of V is called a *dominating set (total dominating set)* of G if every vertex in $V-S$ (every vertex in V) is adjacent to a vertex in S. The *domination number* γ (*total domination number* γ_t) is the minimum cardinality of a dominating set (total dominating set) in G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set.

A *proper vertex coloring* of G is an assignment of colors to the vertices of G in such a way that adjacent vertices are assigned distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required for a proper coloring of G. The concept of total dominator coloring was introduced by M.A. Henning [7]. A *total dominator coloring* of G is a proper coloring of G in which every vertex of the graph is adjacent to every vertex of some color class, that is, each vertex totally dominates every vertex of some other color class. The minimum number of colors required for a total dominator coloring of G is called the *total dominator chromatic number* of G and is denoted by $\chi_{td}(G)$. Some basic results on total dominator coloring are given in [1, 2, 5, 8–13]. Given a graph G, we subdivide each edge of G exactly once and two vertices, say x and y are adjacent if (i) $x,y \in E(G)$ are adjacent in G or (ii) $x \in V(G)$ and $y \in E(G)$ are incident in G. The graph obtained by this process is called *middle graph* of G denoted by $M(G)$. Given a graph G, we subdivide each edge of G exactly once and two vertices, say x and y are adjacent if (i) $x,y \in E(G)$ are adjacent in G or (ii) $x \in V(G)$ and $y \in E(G)$ are incident in G or (iii) $x,y \in V(G)$ are adjacent in G. The graph obtained by this process is called *total graph* of G denoted by $T(G)$. The *shadow graph*, denoted by $S(G)$ is obtained from G by taking for each vertex $v \in V(G)$ we have a vertex $v' \in V'$ such that $N(v)=N(v')$. In this paper, we prove the results on total dominator chromatic number of middle graphs and total graphs on path, cycle, and complete wheel graph.

2. Results

2.1 Middle Graph

In this section, we prove the exact values for middle graph on path, cycle, and wheel graph.

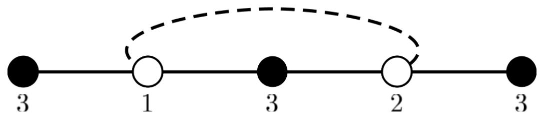

Figure 1. Total dominator chromatic number of middle graph on path P_3 is 3.

Theorem 2.1. *For a path $P_n, \chi_{td}(M(P_n)) = n$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path P_n and let $m_{i,j}$ be the vertex which divides an edge v_iv_j. Consider a coloring $\mathscr{C} = \{\{m_{i,j}\} : 1 \leq i \leq n-1 \, and \, j = i+1\} \bigcup \{v_1 \cup v_2 \cup \cdots \cup v_n\}$ of $M(P_n)$ of cardinality n. Clearly, each vertex v_i and v_j totally dominates the color class $\{\{m_{i,j}\} : 1 \leq i \leq n-1 \, and \, j = i+1\}$. Also the middle vertices $m_{i,j}$ totally dominates either the color class $m_{i-1,j}$ or $m_{i+1,j}$, where $1 \leq i \leq n-2 \, and \, j = i+1$. Hence $\chi_{td}(M(P_n)) = n$ □

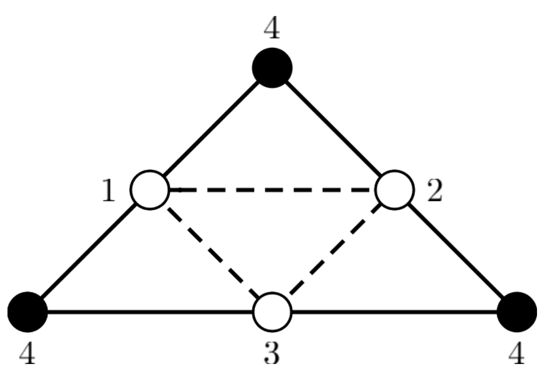

Figure 2. Total dominator chromatic number of middle graph on cycle C_3 is 4.

Theorem 2.2. *For a cycle C_n,*

$$\chi_{td}(M(C_n)) = \begin{cases} n+1, & n \equiv 0 (mod\, 3) \\ n, & otherwise \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle C_n and let $m_{i,j}$ be the vertex which divides an edge v_iv_j. From figure 2, we observe that $\chi_{td}(M(C_3)) = 4$ and from figure 3, we observe that $\chi_{td}(M(C_4)) = 4$. For $n \geq 5$, consider a coloring $\mathscr{C} = \{m_{1,2} \cup m_{3,4}\} \bigcup \{\{m_{i,j}\} : 3 \leq i \leq n-1 \, and \, j = i+1\} \bigcup \{m_{2,3}\} \bigcup \{m_{m_{n,1}}\} \bigcup \{v_1 \cup v_2 \cup \cdots \cup v_n\}$ of $M(C_n)$ of cardinality n. Clearly each vertex $v \in M(C_n)$ totally dominates some

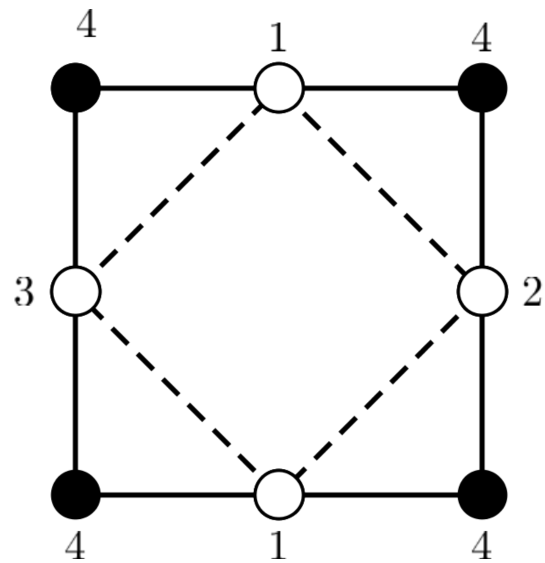

Figure 3. Total dominator chromatic number of middle graph on cycle C_4 is 4.

color class of $\{\{m_{i,j}\} : 3 \leq i \leq n-1 \, and \, j = i+1\}$. Hence
$\chi_{td}(M(C_n)) = \begin{cases} n+1, & n \equiv 0 (mod\, 3) \\ n, & \text{otherwise} \end{cases}$ □

Theorem 2.3. *For a wheel graph $W_{1,n-1}$,*

$$\chi_{td}(M(W_{1,n-1})) = \begin{cases} n+2, & n \text{ is odd} \\ n+3, & n \text{ is even} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a wheel graph $W_{1,n-1}$ in which v_1 is the central vertex and let $m_{i,j}$ be the vertex which divides an edge v_iv_j.

When n is odd,

consider a coloring $\mathscr{C} = \{\{m_{i,j}\} : 1 = 1 \, and \, 2 \leq j \leq n\} \bigcup \{v_1 \cup v_2 \cup \cdots \cup v_n\} \bigcup \{m_{2,3} \cup m_{4,5} \cup \cdots \cup m_{n-1,n}\} \bigcup \{m_{3,4} \cup m_{5,6} \cup \cdots \cup m_{n,2}\}$ of $M(W_{1,n-1})$ of cardinality $n+2$. Clearly all the vertices $v \in M(W_{1,n-1})$ totally dominates some color class of $\{\{m_{i,j}\} : 1 = 1 \, and \, 2 \leq j \leq n\}$. Hence $\chi_{td}(M(W_{1,n-1})) = n+2$.

When n is even,

consider a coloring $\mathscr{C} = \{\{m_{i,j}\} : 1 = 1 \, and \, 2 \leq j \leq n\} \bigcup \{m_{n,1}\} \bigcup \{v_1 \cup v_2 \cup \cdots \cup v_n\} \bigcup \{m_{2,3} \cup m_{4,5} \cup \cdots \cup m_{n-2,n-1}\} \bigcup \{m_{3,4} \cup m_{5,6} \cup \cdots \cup m_{n-1,n}\}$ of $M(W_{1,n-1})$ of cardinality $n+3$. Clearly all the vertices $v \in M(W_{1,n-1})$ totally dominates some color class of $\{\{m_{i,j}\} : 1 = 1 \, and 2 \leq j \leq n\}$. Hence $\chi_{td}(M(W_{1,n-1})) = n+3$. □

2.2 Total Graph

In this section, we prove the exact values for total graph on path, cycle, and wheel graph.

Theorem 2.4. *For a path P_n,*

$$\chi_{td}(T(P_n)) = \begin{cases} n+1, & for\, n = 2 \\ n, & otherwise \end{cases}$$

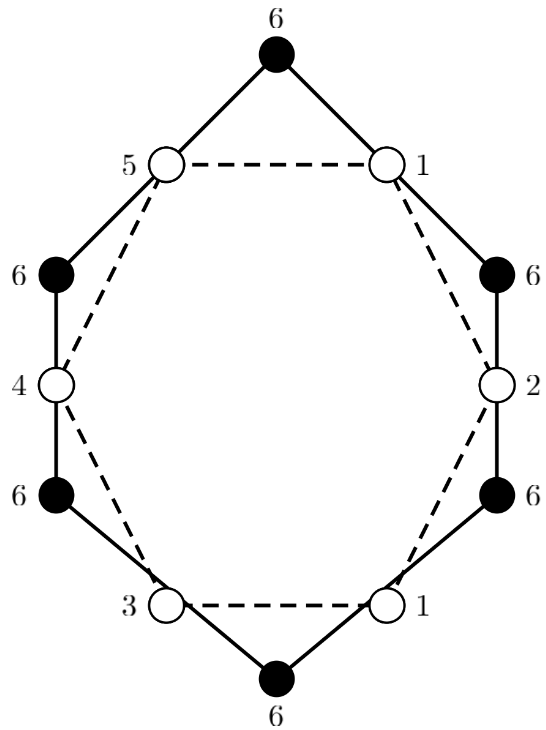

Figure 4. Total dominator chromatic number of middle graph on cycle C_6 is 6.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a path P_n and let $m_{i,j}$ be the vertex which divides an edge v_iv_j. For $n = 2$, it is easy to observe that $\chi_{td}(T(P_2)) = 3 = n+1$. For $n \geq 3$, consider a coloring $\mathscr{C} = \{\{v_i\} : 2 \leq i \leq n-1\} \bigcup \{V_1 \cup V_2\}$ of $T(P_n)$ of cardinality n. Where $\{V_1 \cup V_2\}$ is used to color the vertices other than $v_i, 2 \leq i \leq n-1$. Clearly, each vertex $v \in T(P_n)$ totally dominates some color class of $\{\{v_i\} : 2 \leq i \leq n-1\}$. Hence $\chi_{td}(T(P_n)) = n$. □

Theorem 2.5. *For a cycle C_n,*

$$\chi_{td}(T(C_n)) = \begin{cases} n, & \text{for } n = 3 \\ n+1, & \text{otherwise} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle C_n and let $m_{i,j}$ be the vertex which divides an edge v_iv_j. For $n = 3$, it is easy to observe that $\chi_{td}(T(C_3)) = 3 = n$. For $n \geq 4$.

When n is odd,

consider a coloring $\mathscr{C} = \{\{v_i\} : 2 \leq i \leq n-1\} \bigcup \{m_{n,1}\} \bigcup \{V_1 \cup V_2\}$ of $T(C_n)$ of cardinality $n+1$. Where $\{V_1 \cup V_2\}$ is used to color the vertices other than $v_i, 2 \leq i \leq n-1$ and $m_{n,1}$. Clearly, each vertex $v \in T(C_n)$ totally dominates some color class of $\{\{v_i\} : 2 \leq i \leq n-1\}$. Hence $\chi_{td}(T(C_n)) = n+1$.

When n is even,

consider a coloring $\mathscr{C} = \{\{v_i\} : 2 \leq i \leq n-1 \, and \, n \neq 4\} \bigcup \{v_4 \cup m_{n,1}\} \bigcup \{V_1 \cup V_2\}$ of $T(C_n)$ of cardinality $n+1$. Where $\{V_1 \cup V_2\}$ is used to color the vertices other than $v_i, 2 \leq i \leq n-1$ and $m_{n,1}$. Clearly, each vertex $v \in T(C_n)$ totally dominates some color class of $\{\{v_i\} : 2 \leq i \leq n-1\}$. Hence $\chi_{td}(T(C_n)) = n+1$. □

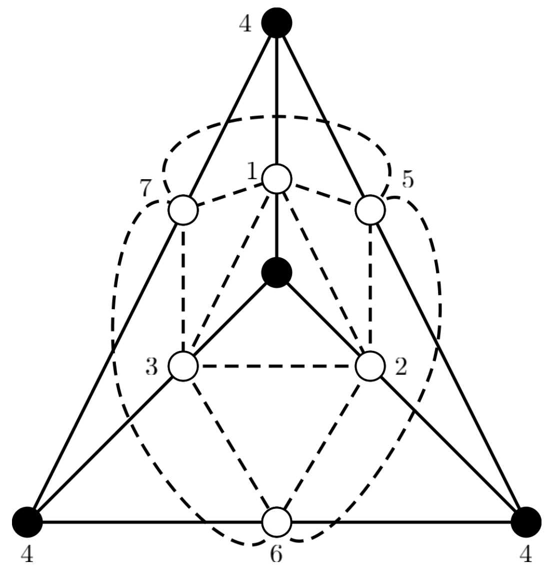

Figure 5. Total dominator chromatic number of middle graph on wheel graph $W_{1,3}$ is 7.

Theorem 2.6. *For a wheel graph $W_{1,n-1}$, $\chi_{td}(T(W_{1,n-1})) = n+3$.*

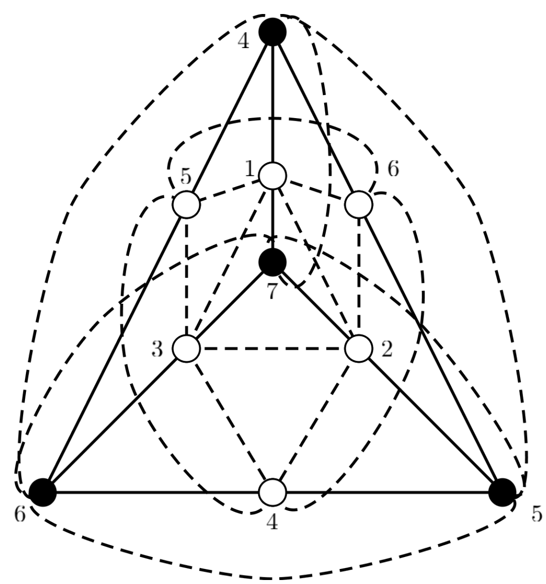

Figure 6. Total dominator chromatic number of total graph on wheel graph $W_{1,3}$ is 7.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of a wheel graph $W_{1,n-1}$ in which v_1 is the central vertex and let $m_{i,j}$ be the vertex which divides an edge v_iv_j.

When $n = 4$, it is easy to observe from figure 7 that $\chi_{td}(T(W_{1,n-1})) = 7$. For $n \geq 5$.

When $n-1$ is even,

consider a coloring $\mathscr{C} = \big\{\{m_{i,j}\} : 1 = 1 \, and \, 2 \leq j \leq n\big\} \bigcup \{v_2 \cup v_4 \cup \cdots \cup v_{n-1}\} \bigcup \{v_3 \cup v_5 \cup \cdots \cup v_n\} \bigcup \{m_{2,3} \cup m_{4,5} \cup$

$\cdots \cup m_{n-1,n}\} \bigcup \{m_{3,4} \cup m_{5,6} \cup \cdots \cup m_{n-2,n-1} \cup m_{n,2}\}$ of $T(W_{1,n-1})$ of cardinality $n+3$. Clearly all the vertices $v \in T(W_{1,n-1})$ totally dominates some color class of $\{\{m_{i,j}\} : 1 = 1\, and 2 \leq j \leq n\}$. Hence $\chi_{td}(T(W_{1,n-1})) = n+3$.

When $n-1$ is odd,

consider a coloring $\mathscr{C} = \{\{m_{i,j}\} : 1 = 1\, and\, 2 \leq j \leq n\}$ $\bigcup \{v_2 \cup v_4 \cup \cdots \cup v_{n-2}\} \bigcup \{v_3 \cup v_5 \cup \cdots \cup v_{n-1} \cup m_{n,2}\} \bigcup \{v_n \cup m_{2,3} \cup m_{4,5} \cup \cdots \cup m_{n-2,n-1}\} \bigcup \{m_{3,4} \cup m_{5,6} \cup \cdots \cup m_{n-1,n}\}$ of $T(W_{1,n-1})$ of cardinality $n+3$. Clearly all the vertices $v \in T(W_{1,n-1})$ totally dominates some color class of $\{\{m_{i,j}\} : 1 = 1\, and 2 \leq j \leq n\}$. Hence $\chi_{td}(T(W_{1,n-1})) = n+3$. □

2.3 Shadow Graph

In this section, we give exact value for shadow graph.

Theorem 2.7. *Let G be a graph. Then* $\chi_{td}(S(G)) = \chi_{td}(G) + 1$.

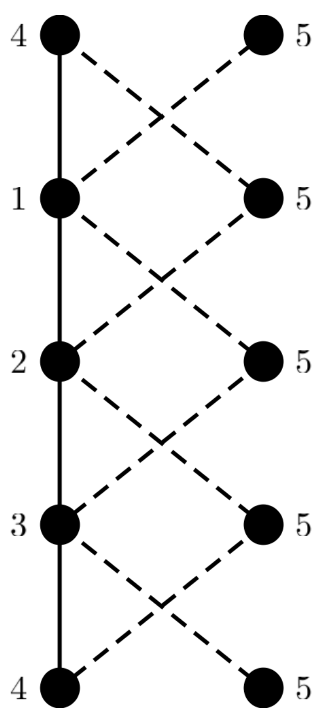

Figure 7. Total dominator chromatic number of shadow graph on path P_5 is 5=4+1=$\chi_{td}(P_5)+1$.

Proof. Let $\mathscr{C}$ be a χ_{td}-coloring of G. Now consider a coloring $\mathscr{C}_1 = \mathscr{C} \cup \{V'\}$ of $S(G)$. Clearly $\mathscr{C}_1$ is a proper coloring of $S(G)$ of cardinality $\chi_{td}(G)+1$. Each vertex $v \in V(G)$ totally dominates some color class in $\mathscr{C}_1$ of G. The vertices $v \in V'$ totally dominates the color class which its twin vertex totally dominates in $\mathscr{C}$ of G. Hence $\chi_{td}(S(G)) = \chi_{td}(G)+1$. □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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