Malaya Journal of Matematik, Vol. 7, No. 4, 625-631, 2019
https://doi.org/10.26637/MJM0704/0004

Nonexistence of global solution to system of semi-linear wave models with fractional damping

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Abstract
In this article we are interested to Cauchy problem for system of semi-linear fractional evolution equations. Some authors were concerned with studying of global existence of solutions for the hyperbolic nonlinear equations with a damping term.
Our goal is to extend some results obtained by the authors, by studying the system of semi-linear wave models with fractional damping term and fractional Laplacian.
We use the test functions method to prove the nonexistence of the sought solutions in the weak formulation.

Keywords
Derivatives in the sense of Caputo, fractional Laplacian, test function, weak solution.

AMS Subject Classification
35A01, 35D30, 47J35, 93C20.

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Article History: Received **18** April **2019**; Accepted **16** August **2019**

Contents

1. Introduction

In this paper, we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt}-\Delta u+D_{0|t}^{\alpha_1+1}u+(-\Delta)^{\frac{\beta_1}{2}}u_t=h(t,x)\,|v|^q\,,\ t>0,\ x\in\mathbb{R}^N \\ v_{tt}-\Delta v+D_{0|t}^{\alpha_2+1}v+(-\Delta)^{\frac{\beta_2}{2}}v_t=k(t,x)\,|u|^p\,,\ t>0,\ x\in\mathbb{R}^N \\ u(0,x)=u_0(x)>0,\quad u_t(0,x)=u_1(x)>0,\quad x\in\mathbb{R}^N \\ v(0,x)=v_0(x)>0,\quad v_t(0,x)=v_1(x)>0,\quad x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p>1, q>1, 0<\alpha_i<1, 0<\beta_i\leq 2, i=1,2$ are constants.

$D_{0/t}^{\alpha_i}$ denotes the derivatives of order α_i in the sense of Caputo and $(-\Delta)^{\frac{\beta_i}{2}}$ is $\frac{\beta_i}{2}-$ fractional power of the $(-\Delta)$.
The functions h and k are non-negatives and assumed to satisfy the conditions:

$$h(tR^{\frac{2}{\alpha_1+1}},xR)=R^{\mu}h(t,x),\ \ k(tR^{\frac{2}{\alpha_2+1}},xR)=R^{\nu}k(t,x),\quad \text{where } \nu\geq 0,\ \mu\geq 0,\ \text{and } R>0. \tag{1.2}$$

In the beginning of this work we note that Chen and Holm [2] studied the equation

$$\nabla^2 p=\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}+\frac{2\alpha_0}{c_0^{1-y}}\frac{\partial}{\partial t}(-\nabla^2)^{\frac{y}{2}}p,$$

where $0\leq y\leq 2$ and $(-\nabla^2)^{\frac{y}{2}}$ is $\frac{y}{2}$-fractional Laplacian which generalize the two cases:

- when $y=2$ the equation

$$\nabla^2 p=\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}+\frac{\mu}{c_0^2}\frac{\partial}{\partial t}(-\nabla^2)p,$$

  which governs the propagation of sound through a viscous fluid, where c_0 is the small signal sound speed, and $\mu=2\alpha_0c_0^3$ the collective thermoviscous coefficient.

- when $y=0$ the standard damped wave equation

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0}\frac{\partial p}{\partial t},$$

which describes the frequency-independent attenuation.

The problem of global existence of solutions for nonlinear hyperbolic equations with damping term have been studied by many researchers in several contexts (see [8], [11], [12] , [13], [20], [21]), for example, the following Cauchy problem:

$$\begin{cases} u_{tt}-\Delta u+u_t=|u|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N \\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.3}$$

F. Sun and M. Wang [19] extended the works of authors above to the case of a system :

$$\begin{cases} u_{tt}-\Delta u+u_t=|v|^p, & (t,x)\in(0,+\infty)\times\mathbb{R}^N \\ v_{tt}-\Delta v+v_t=|u|^q, & (t,x)\in(0,+\infty)\times\mathbb{R}^N \\ u(0,x)=u_0(x), \ u_t(0,x)=u_1(x) \\ v(0,x)=v_0(x), \ v_t(0,x)=v_1(x), \end{cases} \tag{1.4}$$

they showed that if $\max\left\{\frac{1+p}{pq-1},\frac{1+q}{pq-1}\right\}\geq\frac{N}{2}$ for $N\geq 1$ where $p,q\geq 1$ and satisfy $pq>1$, then every solution with initial data having positive average value does not exist globally.
A. Hakem [11] treated the same type of (1.3), then he extended this result to the case of a system :

$$\begin{cases} u_{tt}-\Delta u+g(t)u_t=|v|^p, & (t,x)\in(0,+\infty)\times\mathbb{R}^N \\ v_{tt}-\Delta v+f(t)v_t=|u|^q, & (t,x)\in(0,+\infty)\times\mathbb{R}^N \\ u(0,x)=u_0(x), \ u_t(0,x)=u_1(x) \\ v(0,x)=v_0(x), \ v_t(0,x)=v_1(x), \end{cases} \tag{1.5}$$

$g(t)$ and $f(t)$ are functions behaving like t^β and t^α, respectively, where
$0\leq\beta,\alpha<1$.
Hakem [11] showed that, if

$$\frac{N}{2}\leq\frac{1}{pq-1}\max\Big\{1-\beta+p(1-\alpha),1-\alpha+q(1-\beta)\Big\} - \max\big(\alpha,\beta\big),$$

then the problem (1.5) has only the trivial solution.
Our purpose of this work is to generalize some of the above results, so with the suitable choice of the test function, we were able to prove a nonexistence result to (1.1) in the weak formulation.

2. Preliminaries

Let us start by introducing the definitions concerning fractional derivatives in the sense of Caputo and the weak local solution to problem (1.1).

Definition 2.1. *Let $0<\alpha<1$ and $\zeta'\in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^\alpha_{0|t}\zeta(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\zeta'(s)}{(t-s)^\alpha}ds,$$

and

$$D^\alpha_{t|T}\zeta(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^T\frac{\zeta'(s)}{(s-t)^\alpha}ds,$$

where Γ denotes the gamma function (see [14] p 79).

In general

$$D^\alpha_{0|t}\zeta(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\zeta^n(s)}{(t-s)^{-n+\alpha+1}}ds,$$

where $n=[\alpha]+1, \alpha>0$.
By using the property

$$D^\alpha_{0|t}(D^m\zeta(t))=D^{\alpha+m}_{0|t}\zeta(t)$$

where $m\in\mathbb{N}$ and $n-1<\alpha<n$.
We have, in particular

$$D^{\alpha+1}_{0|t}\zeta(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\zeta_{tt}(s)}{(t-s)^\alpha}ds, \quad (0<\alpha<1).$$

Definition 2.2. *Let $Q_T=(0,T)\times\mathbb{R}^N$, $0<T<+\infty$.*
We say that $(u,v)\in\big(L^1_{loc}(Q_T)\big)^2$ is a local weak solution to problem (1.1) on Q_T,
if $(hv^q,ku^p)\in\big(L^1_{loc}(Q_T)\big)^2$, and it satisfies

$$\begin{aligned}&\int_{Q_T}h|v|^q\zeta_1\,dx\,dt+\int_{\mathbb{R}^N}u_1(x)\zeta_1(0,x)\,dx+\int_{\mathbb{R}^N}u_0(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx\\&+\int_{\mathbb{R}^N}u_1(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx+\int_{\mathbb{R}^N}\zeta_1(0,x)(-\Delta)^{\frac{\beta_1}{2}}u_0(x)\,dx\\&=\int_{Q_T}u\zeta_{1_{tt}}\,dx\,dt-\int_{Q_T}u\Delta\zeta_1\,dx\,dt\\&-\int_{Q_T}uD^{\alpha_1+1}_{t|T}\zeta_1\,dx\,dt-\int_{Q_T}u(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1_t}\,dx\,dt.\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&\int_{Q_T}k|u|^p\zeta_1\,dx\,dt+\int_{\mathbb{R}^N}v_1(x)\zeta_2(0,x)\,dx+\int_{\mathbb{R}^N}v_0(x)D^{\alpha_2}_{t|T}\zeta_2(0,x)\,dx\\&+\int_{\mathbb{R}^N}v_1(x)D^{\alpha_2}_{t|T}\zeta_2(0,x)\,dx+\int_{\mathbb{R}^N}\zeta_2(0,x)(-\Delta)^{\frac{\beta_2}{2}}v_0(x)\,dx\\&=\int_{Q_T}u\zeta_{2_{tt}}\,dx\,dt-\int_{Q_T}u\Delta\zeta_2\,dx\,dt\\&-\int_{Q_T}vD^{\alpha_2+1}_{t|T}\zeta_2\,dx\,dt-\int_{Q_T}v(-\Delta)^{\frac{\beta_2}{2}}\zeta_{2_t}\,dx\,dt.\end{aligned} \tag{2.2}$$

for all test function $\zeta_i\in C^{2,2}_{t,x}(Q_T)$ such as $\zeta_i\geq 0$ and $\zeta_i(T,x)=\zeta_{i_t}(T,x)=0,\ i=1,2$

(see [12]).

Remark 2.3. *To get the definition 2.2, we multiplying the first equation in (1.1) by ζ_1 and the second equation by ζ_2, integrating by parts on $Q_T = (0,T)\times\mathbb{R}^N$ and using the definition 2.1*

The integrals in the above definition are supposed to be convergent.
If in the definition $T = +\infty$, the solution (u,v) is called global.
Now, we recall the following integration by parts formula:

$$\int_0^T \phi(t)(D^{\alpha}_{0|t}\psi)(t)dt = \int_0^T (D^{\alpha}_{t|T}\phi)(t)\psi(t)dt,$$

(see [18], p 46).

3. Main results

We now in position to announce our result.

Theorem 3.1. *Let $p>1, q>1, 0<\alpha_i<1, 0\le\beta_i\le 2, i=1,2,$ and*

$$N_1 := \frac{-\frac{2}{\alpha_1+1}(pq-q) - \frac{2}{\alpha_2+1}(q-1) + pq\rho + q\sigma + q\nu + \mu}{pq-1}$$

and

$$N_2 := \frac{-\frac{2}{\alpha_2+1}(pq-p) - \frac{2}{\alpha_1+1}(p-1) + pq\sigma + p\rho + p\mu + \nu}{pq-1}$$

and the conditions (1.2) are fulfilled.
If the initial data satisfies

$$\int_{\mathbb{R}^N} u_i(x)\,dx > 0, \int_{\mathbb{R}^N} v_i(x)\,dx > 0, \int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta_1}{2}} u_0(x)\,dx > 0,$$
$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{\beta_2}{2}} v_0(x)\,dx > 0, \quad i = 0,1 \tag{3.1}$$

$$N \le \max\{N_1; N_2\},$$

then, every weak solution of the problem (1.1) does not exist globally in time .

Proof. We notice that, in all steps of proof , $C>0$ is a real positive number which may change from line to line.
Set $\zeta_i(t,x) = \Phi^{\ell}\left(\frac{t^{2(\alpha_i+1)}}{R^4}\right)\Phi^{\ell}\left(\frac{|x|^2}{R^2}\right)$, $i=1,2$ such as Φ is a decreasing function $C_0^2(\mathbb{R}^+)$, satisfies

$$0\le\Phi\le 1 \text{ and } \Phi(r) = \begin{cases} 1 & \text{if } 0\le r\le 1, \\ 0 & \text{if } r\ge 2. \end{cases}$$

$R>0$, and $\ell \ge 2\max\{\tilde{p},\tilde{q}\}$, where $p\tilde{p} = p+\tilde{p}$ and $q\tilde{q} = q+\tilde{q}$.
Multiplying the first equation of (1.1) by ζ_1 and integrating by parts on
$Q_T = (0,T)\times\mathbb{R}^N$, we get

$$\begin{aligned}\int_{Q_T} h|v|^q\zeta_1\,dxdt &+ \int_{\mathbb{R}^N} u_1(x)\zeta_1(0,x)\,dx \\ &+ \int_{\mathbb{R}^N} u_0(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx \\ &+ \int_{\mathbb{R}^N} u_1(x)D^{\alpha_1}_{t|T}\zeta_1(0,x)\,dx \\ &+ \int_{\mathbb{R}^N} \zeta_1(0,x)(-\Delta)^{\frac{\beta_1}{2}}u_0(x)\,dx \\ &= \int_{Q_T} u\zeta_{1tt}\,dxdt - \int_{Q_T} u\Delta\zeta_1\,dxdt \\ &- \int_{Q_T} uD^{\alpha_1+1}_{t|T}\zeta_1\,dxdt - \int_{Q_T} u(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1t}\,dxdt.\end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned}\int_{Q_T} h|v|^q\zeta_1\,dxdt &\le \int_{Q_T}|u||\zeta_{1tt}|\,dxdt + \int_{Q_T}|u||\Delta\zeta_1|\,dxdt \\ &+ \int_{Q_T}|u|\left|D^{\alpha_1+1}_{t|T}\zeta_1\right|dxdt + \int_{Q_T}|u|\left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_{1t}\right|dxdt.\end{aligned} \tag{3.3}$$

We have also

$$\begin{aligned}\int_{Q_T} k|u|^p\zeta_2\,dxdt &\le \int_{Q_T}|v||\zeta_{2tt}|\,dxdt + \int_{Q_T}|v||\Delta\zeta_2|\,dxdt \\ &+ \int_{Q_T}|v|\left|D^{\alpha_2+1}_{t|T}\zeta_2\right|dxdt + \int_{Q_T}|v|\left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_{2t}\right|dxdt.\end{aligned} \tag{3.4}$$

To estimate $\int_{Q_T}|u||\zeta_{1tt}|\,dxdt$, we observe that it can be rewritten as

$$\int_{Q_T}|u||\zeta_{1tt}|\,dxdt = \int_{Q_T}|u|(k\zeta_2)^{\frac{1}{p}}|\zeta_{1tt}|(k\zeta_2)^{\frac{-1}{p}}\,dxdt.$$

Using Hölder's inequality, we obtain

$$\int_{Q_T}|u||\zeta_{1tt}|\,dxdt \le \left(\int_{Q_T}|u|^p(k\zeta_2)\,dxdt\right)^{\frac{1}{p}} \times\left(\int_{Q_T}|\zeta_{1tt}|^{\frac{p}{p-1}}(k\zeta_2)^{\frac{-1}{p-1}}\,dxdt\right)^{\frac{p-1}{p}}.$$

Proceeding as above, we have

$$\int_{Q_T}|u||\Delta\zeta_1|\,dxdt \le \left(\int_{Q_T}|u|^p(k\zeta_2)\,dxdt\right)^{\frac{1}{p}} \times\left(\int_{Q_T}|\Delta\zeta_1|^{\frac{p}{p-1}}(k\zeta_2)^{\frac{-1}{p-1}}\,dxdt\right)^{\frac{p-1}{p}}.$$

and

$$\int_{Q_T} |u| \left| D_{t|T}^{\alpha_1+1}\zeta_1 \right| dx\,dt \le \left(\int_{Q_T} |u|^p (k\zeta_2)\, dx\,dt \right)^{\frac{1}{p}} \times \left(\int_{Q_T} \left| D_{t|T}^{\alpha_1+1}\zeta_1 \right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}},$$

and

$$\int_{Q_T} |u| \left| (-\Delta)^{\frac{\beta_1}{2}}\zeta_1 \right| dx\,dt \le \left(\int_{Q_T} |u|^p (k\zeta_2)\, dx\,dt \right)^{\frac{1}{p}} \times \left(\int_{Q_T} \left| (-\Delta)^{\frac{\beta_1}{2}}\zeta_1 \right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}}.$$

Finally, we infer

$$\int_{Q_T} h|v|^q \zeta_1\, dx\,dt \le \left(\int_{Q_T} |u|^p (k\zeta_2)\, dx\,dt \right)^{\frac{1}{p}} \mathscr{A}, \quad (3.5)$$

where

$$\begin{aligned}\mathscr{A} = &\left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}} + \left(\int_{Q_T} |\Delta\zeta_1|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}} \\ &+ \left(\int_{Q_T} \left| D_{t|T}^{\alpha_1+1}\zeta_1 \right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}} + \left(\int_{Q_T} \left| (-\Delta)^{\frac{\beta_1}{2}}\zeta_1 \right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt \right)^{\frac{p-1}{p}}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} k|u|^p \zeta_2\, dx\,dt \le \left(\int_{Q_T} |v|^q (h\zeta_1)\, dx\,dt \right)^{\frac{1}{q}} \mathscr{B}, \quad (3.6)$$

where

$$\begin{aligned}\mathscr{B} = &\left(\int_{Q_T} |\zeta_{2tt}|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt \right)^{\frac{q-1}{q}} + \left(\int_{Q_T} |\Delta\zeta_2|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt \right)^{\frac{q-1}{q}} \\ &+ \left(\int_{Q_T} \left| D_{t|T}^{\alpha_2+1}\zeta_2 \right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt \right)^{\frac{q-1}{q}} + \left(\int_{Q_T} \left| (-\Delta)^{\frac{\beta_2}{2}}\zeta_2 \right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt \right)^{\frac{q-1}{q}}.\end{aligned}$$

By the choice of ζ_i, it is easy to show that $\mathscr{A}$ and $\mathscr{B}$ are finite. By combining inequalities (3.5) and (3.6) together , it yield

$$\left(\int_{Q_T} h|v|^q \zeta_1\, dx\,dt \right)^{\frac{pq-1}{pq}} \le \mathscr{A}\mathscr{B}^{\frac{1}{p}}. \quad (3.7)$$

Similarly, we get

$$\left(\int_{Q_T} k|u|^p \zeta_2\, dx\,dt \right)^{\frac{pq-1}{pq}} \le \mathscr{B}\mathscr{A}^{\frac{1}{q}}. \quad (3.8)$$

Now, in $\mathscr{A}$ we consider the scale of variables:

$$t = \tau R^{\frac{2}{\alpha_1+1}}, \quad x = yR,$$

while in $\mathscr{B}$ we use:

$$t = \tau R^{\frac{2}{\alpha_2+1}}, \quad x = yR.$$

We define the set Ω and the functions ϕ_i by

$$\Omega := \left\{ (\tau, y) \in \mathbb{R}_+ \times \mathbb{R}^N; \tau^{2(\alpha_i+1)} \le 2, |y|^2 \le 2 \right\}$$

and

$$\zeta_i(t,x) = \zeta_i\left(\tau R^{\frac{2}{\alpha_i+1}}, Ry\right) := \phi_i(\tau, y)$$

and use the fact that

$$dx\,dt = R^{\left(N+\frac{2}{\alpha_i+1}\right)} dy\,d\tau,\ \zeta_{i_{tt}} = R^{\frac{-4}{\alpha_i+1}} \phi_{i_{\tau\tau}},\ \Delta_x\zeta_i = R^{-2}\Delta_y\phi_i$$

$D^{\alpha_i+1}_{t|TR^{\frac{2}{\alpha_i+1}}}\zeta_i = R^{-2} D^{\alpha_i+1}_{\tau|T}\phi_\tau$, $(-\Delta)_x^{\frac{\beta_i}{2}}\zeta_{i_t} = R^{-\left(\beta_i+\frac{2}{\alpha_i+1}\right)}(-\Delta)_y^{\frac{\beta_i}{2}}\phi_{i_\tau}$, $i = 1, 2$.

Thus,

$$\left(\int_{Q_T} |\zeta_{1tt}|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_1}\left(\int_{\Omega} |\phi_{1tt}|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\, dy\,d\tau\right)^{\frac{p-1}{p}},$$

$$\left(\int_{Q_T} |\Delta\zeta_1|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_2}\left(\int_{\Omega} |\Delta\phi_1|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\, dy\,d\tau\right)^{\frac{p-1}{p}},$$

$$\left(\int_{Q_T} \left|D_{t|T}^{\alpha_1+1}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_3}\left(\int_{\Omega} \left|D_{t|T}^{\alpha_1+1}\phi_1\right|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\, dy\,d\tau\right)^{\frac{p-1}{p}},$$

$$\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_1}{2}}\zeta_1\right|^{\frac{p}{p-1}} (k\zeta_2)^{\frac{-1}{p-1}}\, dx\,dt\right)^{\frac{p-1}{p}} = R^{\gamma_4}\left(\int_{\Omega} \left|(-\Delta)^{\frac{\beta_1}{2}}\phi_1\right|^{\frac{p}{p-1}} (k\phi_2)^{\frac{-1}{p-1}}\, dy\,d\tau\right)^{\frac{p-1}{p}},$$

and

$$\left(\int_{Q_T} |\zeta_{2tt}|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt\right)^{\frac{q-1}{q}} = R^{\lambda_1}\left(\int_{\Omega} |\phi_{2tt}|^{\frac{q}{q-1}} (h\phi_1)^{\frac{-1}{q-1}}\, dy\,d\tau\right)^{\frac{q-1}{q}},$$

$$\left(\int_{Q_T} |\Delta\zeta_2|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt\right)^{\frac{q-1}{q}} = R^{\lambda_2}\left(\int_{\Omega} |\Delta\phi_2|^{\frac{q}{q-1}} (h\phi_1)^{\frac{-1}{q-1}}\, dy\,d\tau\right)^{\frac{q-1}{q}},$$

$$\left(\int_{Q_T} \left|D_{t|T}^{\alpha_2+1}\zeta_2\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt\right)^{\frac{q-1}{q}} = R^{\lambda_3}\left(\int_{\Omega} \left|D_{t|T}^{\alpha_2+1}\phi_2\right|^{\frac{q}{q-1}} (h\phi_1)^{\frac{-1}{q-1}}\, dy\,d\tau\right)^{\frac{q-1}{q}},$$

$$\left(\int_{Q_T} \left|(-\Delta)^{\frac{\beta_2}{2}}\zeta_2\right|^{\frac{q}{q-1}} (h\zeta_1)^{\frac{-1}{q-1}}\, dx\,dt\right)^{\frac{q-1}{q}} = R^{\lambda_4}\left(\int_{\Omega} \left|(-\Delta)^{\frac{\beta_2}{2}}\phi_2\right|^{\frac{q}{q-1}} (h\phi_1)^{\frac{-1}{q-1}}\, dy\,d\tau\right)^{\frac{q-1}{q}},$$

where

$$\begin{cases} \gamma_1 = \left(N+\dfrac{2}{\alpha_1+1}\right)\dfrac{1}{\tilde{p}} - \dfrac{4}{\alpha_1+1} - \dfrac{\nu}{p} \\ \gamma_2 = \left(N+\dfrac{2}{\alpha_1+1}\right)\dfrac{1}{\tilde{p}} - 2 - \dfrac{\nu}{p} \\ \gamma_3 = \left(N+\dfrac{2}{\alpha_1+1}\right)\dfrac{1}{\tilde{p}} - 2 - \dfrac{\nu}{p} \\ \gamma_4 = \left(N+\dfrac{2}{\alpha_1+1}\right)\dfrac{1}{\tilde{p}} - \left(\beta_1+\dfrac{2}{\alpha_1+1}\right) - \dfrac{\nu}{p} \end{cases}$$

and

$$\begin{cases} \lambda_1 = \left(N+\dfrac{2}{\alpha_2+1}\right)\dfrac{1}{\tilde{q}} - \dfrac{4}{\alpha_2+1} - \dfrac{\mu}{q} \\ \lambda_2 = \left(N+\dfrac{2}{\alpha_2+1}\right)\dfrac{1}{\tilde{q}} - 2 - \dfrac{\mu}{q} \\ \lambda_3 = \left(N+\dfrac{2}{\alpha_2+1}\right)\dfrac{1}{\tilde{q}} - 2 - \dfrac{\mu}{q} \\ \lambda_4 = \left(N+\dfrac{2}{\alpha_2+1}\right)\dfrac{1}{\tilde{q}} - \left(\beta_2+\dfrac{2}{\alpha_2+1}\right) - \dfrac{\mu}{q} \end{cases}$$

we arrive at

$$\left(\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt\right)^{\frac{pq-1}{pq}} \le C\left[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}+R^{\gamma_4}\right] \times \left[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}+R^{\lambda_4}\right]^{\frac{1}{p}}, \tag{3.9}$$

similarly, we have

$$\left(\int_{Q_T} k\,|u|^p\,\zeta_2\,dx\,dt\right)^{\frac{pq-1}{pq}} \le C\left[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}+R^{\lambda_4}\right] \times \left[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}+R^{\gamma_4}\right]^{\frac{1}{q}}, \tag{3.10}$$

we observe that $\gamma_1 < \gamma_2 = \gamma_3$ and $\lambda_1 < \lambda_2 = \lambda_3$.
Set $\gamma = \left(N+\dfrac{2}{\alpha_1+1}\right)\dfrac{1}{\tilde{p}} - \rho - \dfrac{\nu}{p}$ and $\lambda = \left(N+\dfrac{2}{\alpha_2+1}\right)\dfrac{1}{\tilde{q}} - \sigma - \dfrac{\mu}{q}$,
where $\rho = \min\left\{2, \beta_1 + \dfrac{2}{\alpha_1+1}\right\}$ and $\sigma = \min\left\{2, \beta_2 + \dfrac{2}{\alpha_2+1}\right\}$.
Hence

$$\left(\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt\right)^{\frac{pq-1}{pq}} \le CR^{\gamma+\frac{\lambda}{p}} \tag{3.11}$$

and

$$\left(\int_{Q_T} k\,|u|^p\,\zeta_2\,dx\,dt\right)^{\frac{pq-1}{pq}} \le CR^{\lambda+\frac{\gamma}{q}}. \tag{3.12}$$

with the fact that

$$\frac{1}{p}+\frac{1}{\tilde{p}}=1 \ \text{ and } \ \frac{1}{q}+\frac{1}{\tilde{q}}=1 \tag{3.13}$$

by a simple computation,

$$\gamma+\frac{\lambda}{p} = N\Big(\frac{pq-1}{pq}\Big)+\frac{2}{\alpha_1+1}\Big(1-\frac{1}{p}\Big)+\frac{2}{\alpha_2+1}\Big(\frac{1}{p}-\frac{1}{pq}\Big) -\rho-\frac{\sigma}{p}-\frac{\nu}{p}-\frac{\mu}{pq}$$

and

$$\lambda+\frac{\gamma}{q} = N\Big(\frac{pq-1}{pq}\Big)+\frac{2}{\alpha_2+1}\Big(1-\frac{1}{q}\Big)+\frac{2}{\alpha_1+1}\Big(\frac{1}{q}-\frac{1}{pq}\Big) -\sigma-\frac{\rho}{q}-\frac{\mu}{q}-\frac{\nu}{pq}.$$

We conclude that

- If $\gamma+\dfrac{\lambda}{p}<0$, it yield

$$N<\frac{-\dfrac{2}{\alpha_1+1}(pq-q)-\dfrac{2}{\alpha_2+1}(q-1)+pq\rho+q\sigma+q\nu+\mu}{pq-1}.$$

Then the right hand side of (3.11) goes to 0, when R tends to infinity, we pass to the limit in the left hand side, as R goes to $+\infty$; we get

$$\lim_{R\to+\infty}\left(\int_{Q_T} h\,|v|^q\,\zeta_1\,dx\,dt\right)^{\frac{pq-1}{pq}}=0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of v and the fact that $\zeta_1(t,x)\to 1$ as $R\to+\infty$, we infer that

$$\left(\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q\,dx\,dt\right)^{\frac{pq-1}{pq}}=0.$$

This implies that $v\equiv 0$ *a. e.* on $\mathbb{R}^+\times\mathbb{R}^N$
Similarly, if $\lambda+\dfrac{\gamma}{q}<0$, it yield

$$N<\frac{-\dfrac{2}{\alpha_2+1}(pq-p)-\dfrac{2}{\alpha_1+1}(p-1)+pq\sigma+p\rho+p\mu+\nu}{pq-1},$$

by using also (3.12) to proceeding as above, we obtain $u\equiv 0$ *a. e.* on $\mathbb{R}^+\times\mathbb{R}^N$, which is a contradiction with (3.1).

- If $\gamma+\frac{\lambda}{p}=0$, we have

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q\,dx\,dt<\infty$$

Define

$$\Sigma=\left\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^N; t^{2(\alpha_1+1)}\le 2R^4, |x|^2\le 2R^2\right\}$$

From (3.11) we can get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q\,dx\,dt\le C\left(\int_\Sigma h\,|v|^q\,\zeta_1\,dx\,dt\right)^{\frac{1}{pq}}$$

we have

$$\lim_{R\to+\infty}\int_\Sigma h\,|v|^q\,\zeta_1\,dx\,dt=0,$$

hence, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h\,|v|^q\,dx\,dt=0,$$

this implies that $v\equiv 0$.
Similarly, if $\lambda+\frac{\gamma}{q}=0$, proceeding as above, we infer that $u\equiv 0$, we arrive again at a contradiction with (3.1). We deduce that no global weak solution is possible , which ends the proof.

□

Remark 3.2. *When* $\alpha_i\to 0$, $\beta_i\to 0$ *and* $\nu=\mu=0$ (i.e. $h=k=1$), *we recover the case who studied by A. Hakem (see [11]), when* $\alpha=\beta=0$. *Also we retrieve the same result obtained by F. Sun & M. Wang (see [19]).*

4. Conclusion

By using fractional calculus properties, applying the test function technique and under suitable conditions, we proved the nonexistence of global solution to the system above.

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ISSN(P):2319 − 3786
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