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The non-negative Q_1-matrix completion problem

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Abstract
A matrix is a Q_1-matrix if it is a Q-matrix with positive diagonal entries. A matrix is a nonnegative matrix if it is a matrix with nonnegative entries. A digraph D is said to have nonnegative Q_1-completion if every partial nonnegative Q_1-matrix specifying D can be completed to a nonnegative Q_1-matrix. In this paper, some necessary and sufficient conditions for a digraph to have nonnegative Q_1-completion are provided. Later on the relationship among the completion problems of nonnegative Q_1-matrix and some other class of matrices are shown. Finally, the digraphs of order at most four that include all loops and have nonnegative Q_1-completion are singled out.

Keywords
Partial matrix, Nonnegative Q_1-matrix, Digraph, Matrix completion, Nonnegative Q_1-completion problem.

AMS Subject Classification
15A48.

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1. Introduction

A real $n \times n$ matrix $B = [b_{ij}]$ is a Q_1-matrix if all diagonal entries are positive and for every $k \in \{1,2,\dots,n\}$, $S_k(B) > 0$, where $S_k(B)$ is the sum of all $k \times k$ principal minors of B. The matrix B is a *Q-matrix* if for every $k \in \{1,2,\dots,n\}$, $S_k(B) > 0$. A nonnegative Q_1-matrix is a Q_1-matrix in which all off diagonal entries are nonnegative. A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A partial matrix M is *fully specified* if all entries of M are specified, i.e., if M is a matrix. A *partial nonnegative* (*positive*) *matrix* is a partial matrix whose specified entries are nonnegative (positive).

For a subset α of $\langle n \rangle = \{1,2,\dots,n\}$, the *principal partial submatrix* $M(\alpha)$ is the partial matrix obtained from M by deleting all rows and columns not indexed by α. A *principal minor* of M is the determinant of a fully specified principal submatrix of M. For a given class Γ of matrices (e.g., Q, Q_1-matrices) a *partial* Γ*-matrix* is a partial matrix for which the specified entries satisfy the properties of a Γ-matrix. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *matrix completion problem* asks which partial matrices have completions with a given property. A Γ*-completion* of a partial Γ-matrix M is a completion of M which is a Γ-matrix.

A number of researchers studied matrix completion problems for different classes of matrices ([5–13]). In 2009, DeAlba *et al.* [2] solved the Q-matrix completion problem. For literature survey and complete updated results, one can see [3].

1.1 Digraphs

Any standard reference, for example, [1] and [4] can be use for graph theoretic terminologies. A *directed graph* or *digraph* $D = (V_D, A_D)$ of order $n > 0$ is a finite nonempty set V_D, with $|V_D| = n$ of objects called *vertices* together with a (possibly empty) set A_D of ordered pairs of vertices, called *arcs*. We write $v \in D$ (resp. $(u,v) \in D$) to imply $v \in V_D$ (resp. $(u,v) \in A_D$). If $x = (u,u)$, then x is called a *loop* at the vertex u.

A (*directed*) *u-v path* P of length $k \geq 0$ in D is an alternating sequence $(u = v_0, x_1, v_1, \ldots, x_k, v_k = v)$ of vertices and arcs, where v_i, $1 \leq i \leq k$, are distinct vertices and $x_i = (v_{i-1}, v_i)$. Further, if $k \geq 2$ and $u = v$, then a u-v path is a *cycle* of length k. The vertices v_i and the arcs x_i are said to be on P. We then write $C_k = \langle v_1, v_2, \ldots, v_k \rangle$ and call C_k a k-cycle in D. A digraph without any cycle is said to be *acyclic*. A 1-cycle consists of a vertex v and a loop at v.

A cycle C is *odd* (resp. *even*) if its length is odd (resp. even). A digraph $H = (V_H, A_H)$ is a *subdigraph of order* k of the digraph D if $|V_H| = k$ and $V_H \subseteq V_D$, $A_H \subseteq A_D$. A subdigraph H of D is an *induced subdigraph* if $A_H = (V_H \times V_H) \cap A_D$ (*induced by* V_H) and is a *spanning subdigraph* if $V_H = V_D$. A digraph D is said to be connected (resp. strongly connected) if for every pair u, v of vertices, D contains a u-v path (resp. both a u-v path and a v-u path). The maximal connected (resp. strongly connected) subdigraphs of D are called *components* (resp. *strong components*) of D.

The *complement of a digraph* D is the digraph $\overline{D}$, where $V_{\overline{D}} = V_D$ and $(u,v) \in A_{\overline{D}}$ if and only if $(u,v) \notin A_D$. A digraph D is said to be *symmetric* if $(u,v) \in D$ implies $(v,u) \in D$. On the other hand, D is *asymmetric* if $(u,v) \in D$ implies $(v,u) \notin D$. A *complete symmetric digraph* on n vertices, denoted by K_n, is the digraph having all possible arcs (including all loops).

Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic*, if there is a bijection $\phi : V_1 \to V_2$ such that $A_2 = \{(\phi(u), \phi(v)) : (u,v) \in A_1\}$. An *unlabelled* digraph is an equivalent class of isomorphic digraphs. Choosing a particular member of an unlabelled digraph is referred as a *labelling* of the unlabelled digraph.

1.2 Digraphs with matrices

Let π be a permutation of a nonempty finite set V. The digraph $D_\pi = (V, A_\pi)$, where $A_\pi = \{(v, \pi(v)) : v \in V\}$ is called a *permutation digraph*. Clearly, each component of a permutation digraph is a loop or a cycle. The digraph D_π is said to be *positive* (resp. *negative*) if π is an even permutation (resp. an odd permutation). It is clear that D_π is negative if and only if it has odd number of even cycles.

A *permutation subdigraph* H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k). A digraph D is *stratified* if D has a permutation subdigraph of order k for every $k = 2, 3, \ldots, |D|$.

Let $B = [b_{ij}]$ be an $n \times n$ matrix. We have

$$\det(B) = \sum (\operatorname{sgn} \pi) b_{1\pi(1)} \cdots b_{n\pi(n)}$$

where the sum is taken over all permutations π of $\langle n \rangle$.

2. Partial nonnegative Q_1-matrix and the nonnegative Q_1-matrix completion problem

A partial nonnegative matrix is a partial matrix in which all specified entries are nonnegative. A *partial Q_1-matrix* is a partial Q-matrix with all specified diagonal entries are positive. Thus, a *partial nonnegative Q_1-matrix* is a partial nonnegative matrix M with all specified positive diagonal entries and $S_k(M) > 0$ for every $k \in \{1, 2, \ldots, n\}$, whenever all $k \times k$ principal submatrices are fully specified. Now, a partial nonnegative Q_1-matrix is characterized as follows.

Proposition 2.1. *Suppose $M = [a_{ij}]$ is a partial nonnegative matrix. Then M is a partial nonnegative Q_1-matrix if and only if exactly one of the following holds:*
(i) At least one diagonal entry of M is unspecified, all specified diagonal entries are positive.
(ii) All diagonal entries are specified and positive; at least one off-diagonal entry is unspecified.
(iii) All entries of M are specified and M is a nonnegative Q_1-matrix.

For any partial nonnegative Q_1-matrix M, a completion B of M is called a *nonnegative Q_1-completion* of M, if B is a nonnegative Q_1-matrix. Since permutation similarity of a matrix to a nonnegative Q_1-matrix is a nonnegative Q_1-matrix, it is quite clear that if a partial nonnegative Q_1-matrix M has a nonnegative Q_1-completion, so does any partial matrix which is permutation similar to M.

One can easily verify that any partial nonnegative matrix M with all unspecified diagonal entries has nonnegative Q_1-completion. By choosing sufficiently large values for the unspecified diagonal entries, a nonnegative Q_1-completion can be obtained. Suppose M be a partial nonnegative Q_1-matrix in which the diagonal entries at (i,i) positions ($i = k+1, \ldots, n$) are unspecified. If $M[1, \ldots, k]$ is fully specified, M may not have a nonnegative Q_1-completion. For example, the partial nonnegative matrix,

$$M = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & ? \end{bmatrix},$$

where ? denotes an unspecified entry, does not have nonnegative Q_1-completion. In fact for any completion B of M, $S_3(B) = 0$. On the other hand, if $M[1, \ldots, k]$ has an unspecified entry and has a nonnegative Q_1-completion, then M has a nonnegative Q_1-completion. A completion of M can be obtained by choosing sufficiently large values for the unspecified diagonal entries. These above observations are listed in the following results.

Theorem 2.2. *If a nonnegative matrix M omits all diagonal entries, then M has nonnegative Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial nonnegative Q_1-matrix. For any $s > 1$, consider a completion $B = [b_{ij}]$ of M by setting all diagonal entries equal to s and rest of the off diagonal entries to be equal to zero. Then, any $r \times r$ principal minor will be of the form $s^r + p(s)$, where $p(s)$ is a polynomial of degree $\leq r-1$. Now by choosing s large enough, we have $S_r(B) > 0$ for all $r \times r$ principal minors of B. Since only finitely many principal minors are to be considered, thus for sufficiently large s, M has nonnegative Q_1-completion. □

Theorem 2.3. *Suppose M be a partial nonnegative Q_1-matrix in which the diagonal entry at $(r+1, r+1)$ position is unspecified. If the principal submatrix $M[1,\ldots,r]$ of M is not fully specified and has nonnegative Q_1-completion, then M has nonnegative Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial nonnegative Q_1-matrix which omits the diagonal entry at $(r+1, r+1)$ position. Then, M is of the form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where, $M_{11} = M[1,\ldots,r]$ and $M_{22} = M[r+1, r+1]$.

Consider B_1 be the nonnegative Q_1-matrix completion of $M[1,\ldots,r]$. Then,

$$M' = \begin{bmatrix} B_1 & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

is a partial nonnegative Q_1-matrix, since M_{22} has an unspecified diagonal entry. Now for $s > 0$, consider a completion $B = [b_{ij}]$ of M' obtained by choosing $b_{ii} = s, \ \ i = r+1$ and $b_{ij} = 0$ against all other unspecified entries in M'. Then B is of the form,

$$B = \begin{bmatrix} B_1 & B_{12} \\ B_{21} & s \end{bmatrix}.$$

Since B_1 is a nonnegative Q_1-matrix, $S_i(B_1) > 0$ for $1 \leq i \leq r$. For $2 \leq j \leq r+1$,

$$S_j(B) \quad = \quad S_j(B_1) + sS_{j-1}(B_1) + s_j,$$

where s_j is a constant. Now $S_j(B) > 0$ for sufficiently large values of s and clearly B is nonnegative Q_1-matrix. □

Corollary 2.4. *Suppose M be a partial nonnegative Q_1-matrix in which the diagonal entries at (i,i) positions $(i = r+1,\ldots,n)$ are unspecified. If the principal submatrix $M[1,\ldots,r]$ of M is not fully specified and has nonnegative Q_1-completion, then M has nonnegative Q_1-completion.*

The following example shows that the converse of Corollary 2.4 is not true.

Example 2.5. *Consider the partial nonnegative matrix,*

$$M = \begin{bmatrix} d_1 & a_{12} & ? & a_{14} \\ a_{21} & d_2 & ? & ? \\ ? & a_{32} & d_3 & ? \\ a_{41} & ? & a_{43} & ? \end{bmatrix},$$

where ? denotes the unspecified entries. Here we have $d_i > 0$, $\forall i = 1,2,3$. We show that for any choice of values of the specified entries M has nonnegative Q_1-completions, but there are occasions when $M[1,2,3]$ does not have nonnegative Q_1-completion. For $x > 0$, consider the completion $B(x)$ of M defined as follows:

$$B(x) = \begin{bmatrix} d_1 & a_{12} & \frac{1}{x} & a_{14} \\ a_{21} & d_2 & 1 & 0 \\ 0 & a_{32} & d_3 & \frac{1}{x^2} \\ a_{41} & x^4 & a_{43} & x \end{bmatrix}.$$

Then,

$$\begin{aligned} S_1(B(t)) &= x + \sum d_i, \\ S_2(B(t)) &= x(d_1+d_2+d_3) - \frac{a_{43}}{x^2} + f_0(x), \\ S_3(B(t)) &= a_{14}a_{21}x^4 + x^2 + f_1(x), \\ S_4(B(t)) &= a_{14}a_{21}d_3x^4 + d_1x^2 + f_1(x), \end{aligned}$$

where $f_i(x)$ is a polynomial in x of degree at most i, $i = 0,1$. Consequently, $B(x)$ is a nonnegative Q_1-matrix for sufficiently large x, and therefore M has nonnegative Q_1-completion. On contrast, the partial nonnegative Q_1-matrix

$$M[1,2,3] = \begin{bmatrix} 1 & 10 & ? \\ 10 & 1 & ? \\ ? & 0 & 1 \end{bmatrix},$$

with unspecified entries ? is the principal submatrix of M induced by its diagonal $\{1,2,3\}$. Now one can verify that $M[1,2,3]$ does not have nonnegative Q_1-completion, because $S_2(M) < 0$ for any completion of $M[1,2,3]$.

3. Digraphs and the nonnegative Q_1-completion problem

An $n \times n$ partial matrix M *specifies* a digraph $D = (\langle n \rangle, A_D)$ if for $1 \leq i, j \leq n$, $(i,j) \in A_D$ if and only if the (i,j)-th entry of M is specified. For example, the partial nonnegative Q_1-matrix M in Example 2.5 specifies the digraph D in Figure 1. We say that a digraph D has nonnegative Q_1-*completion*, if every partial nonnegative Q_1-matrix specifying D can be completed to a nonnegative Q_1-matrix. The nonnegative Q_1-*matrix completion problem* aims at studying and classifying all digraphs D which have nonnegative Q_1-completion.

The property of being a nonnegative Q_1-matrix is preserved under similarity and transposition, but it is not inherited by principal submatrices, as it can easily be verified. Also it is clear that if a digraph D has nonnegative Q_1-completion, then any digraph which is isomorphic to D has nonnegative Q_1-completion.

Theorem 3.1. *Suppose M is a partial nonnegative Q_1-matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subgraph of D has nonnegative Q_1-completion, then M has nonnegative Q_1-completion.*

Proof. We prove the result for the case when D has two strong components D_1 and D_2. The general result will then follow by induction. By a relabeling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial nonnegative Q_1-matrix specifying D_i, $i = 1,2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a nonnegative Q_1-completion B_{ii}. Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in X as well as all unspecified entries in M_{12} as 0. Then, for $2 \le k \le |D|$ we have,

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11}) S_{k-r}(B_{22}) \ge 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. Thus M can be completed to a nonnegative Q_1-matrix. □

The proof of the following result is similar.

Theorem 3.2. *Suppose M is a partial nonnegative Q_1-matrix specifying the digraph D. If the partial submatrix of M induced by each component of D has a nonnegative Q_1-completion, then M has a nonnegative Q_1-completion.*

Consider the digraph D in the Figure 1. We show that D has nonnegative Q_1-completion, but the subgraph D_1 induced by vertices $\{1,2,3\}$ does not have nonnegative Q_1-completion (See Example 2.5). The property of having nonnegative Q_1-completion is not inherited by induced subdigraphs. This can be also seen from the Example 2.5.

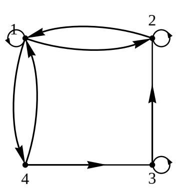

Figure 1. The Digraph D

3.1 Sufficient conditions for nonnegative Q_1-matrix completion

Theorem 3.3. *If a digraph $D \neq K_n$ of order n has nonnegative Q_1-completion, then any spanning subdigraph D_0 of D has nonnegative Q_1-completion.*

Proof. Suppose M_{D_0} be a partial nonnegative Q_1-matrix specifying the digraph D_0. Consider a partial matrix M_D obtained from M_{D_0} by specifying the entries corresponding to $(i,j) \in A_D \setminus A_{D_0}$ as 0 and $(i,i) \in A_D \setminus A_{D_0}$ as 1. Since $D \neq K_n$, M_D is a partial nonnegative Q_1-matrix specifying D (By Proposition 2.1). Suppose B be a nonnegative Q_1-completion of M_D which is also nonnegative Q_1-completion of M_{D_0}. Hence the result follows.

□

Theorem 3.4. *A digraph has nonnegative Q_1-completion if it does not contain an cycle of even length.*

Proof. Suppose M be a partial nonnegative Q_1-matrix specifying a digraph D which has no cycles of even length. For $t > 0$, consider a completion B of M by assigning all the unspecified diagonal entries as t and all unspecified off diagonal entries as 0. Then for each $1 \le k \le n$, $S_k(B)$ contains a positive constant. On the other hand, for each $k \in \{1,2,\ldots,n\}$, $S_k(B)$ contains no negative terms, because D does not contain an even cycle. Hence the result follows. □

Corollary 3.5. *An acyclic digraph has nonnegative Q_1-completion.*

However the converse of the Theorem 3.4 is not true which can be seen from the Example 3.6.

Example 3.6. *Consider the digraph D_1 in Figure 2. Now*

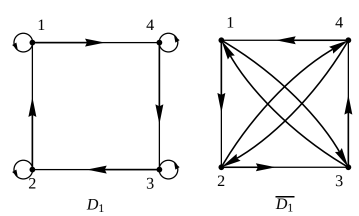

Figure 2. The digraph D_1 having nonnegative Q_1-completion

consider a partial nonnegative Q_1-matrix

$$M = \begin{bmatrix} d_1 & ? & ? & a_{14} \\ a_{21} & d_2 & ? & ? \\ ? & a_{32} & d_3 & ? \\ ? & ? & a_{43} & d_4 \end{bmatrix}$$

specifying the digraph D_1 with unspecified entries as ?. Now being a partial nonnegative Q_1 matrix M, all the specified off-diagonal entries are nonnegative and $d_i > 0$, $\forall i = 1,2,3,4$. If any one off-diagonal specified entries are zero, then by *putting all unspecified entries as zero, we get the desired result. Suppose $a_{21}a_{14}a_{43}a_{32} \neq 0$. For $t > 0$, consider a completion*

$$B == \begin{bmatrix} d_1 & 0 & t & a_{14} \\ a_{21} & d_2 & 0 & 0 \\ 0 & a_{32} & d_3 & 0 \\ 0 & 0 & a_{43} & d_4 \end{bmatrix},$$

of M. Then we have a positive term $ta_{32}a_{21}$ in $S_3(B)$ and $d_4ta_{32}a_{21}$ in $\det B$. By choosing t sufficiently large, we have $S_3(B) > 0$ and $S_4(B) > 0$. Again for positive diagonal entries $d_i, i = 1, \ldots, 4$, we have $S_1(B) > 0$ and $S_2(B) > 0$. Hence the result follows.

Now we have the following result:

Theorem 3.7. *Suppose $D \neq K_4$ be a digraph with all loops and without any 2-cycle. Suppose D has one even cycle C of length 4. If $\overline{D}$ contains a 2-cycle $\langle u, v\rangle$ such that either $C + (u, v)$ or $C + (v, u)$ has a 3-cycle, then D has nonnegative Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial nonnegative Q_1-matrix specifying the digraph D. Suppose (u, v) forms a 3-cycle in $C + (u, v)$. For $t > 0$, consider a completion $B = [b_{ij}]$ of M as follows:

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } (i,j) \in D \\ t, & \text{if } (i,j) = (u,v) \in \overline{D} \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily seen that $S_1(B)$ and $S_2(B)$ are positive. If any one of the specified off diagonal entries are zero, then we are done. If not, then $S_3(B)$ contains a positive term $ta_{ij}a_{jk}$ specifying the 3-cycle of $C + (u, v)$. Again $S_4(B)$ contains a positive term $d_l ta_{ij}a_{jk}$ as well as a negative term $\prod_{i \neq j} a_{ij}$. By choosing t sufficiently large, we have $S_k(B) > 0$ for $k = 3, 4$. Hence the result follows. □

The digraph D_1 in Figure 2 satisfies the Theorem 3.7. The digraph D_1 contains a 4 cycle $C = \langle 1, 4, 3, 2\rangle$. Also the digraph $\overline{D}$ contains a 2-cycle $\langle 1, 3\rangle$. Now $C + (3, 1)$ contains a 3-cycle $\langle 1, 4, 3\rangle$. Hence D_1 has nonnegative Q_1-completion by Theorem 3.7.

3.2 Necessary conditions for nonnegative Q_1-matrix completion

Theorem 3.8. *If a digraph $D \neq K_n$ of order $n \geq 2$ contains two vertices v_1 and v_2 with indegree or outdegree n, then D does not have nonnegative Q_1-completion.*

Proof. Suppose a digraph D of order $n \geq 2$ contains two vertices v_1 and v_2 with indegree or outdegree n. Consider a partial nonnegative Q_1-matrix M specifying D with all specified entries are exactly 1. Then two columns or rows of M are equal and for any completion B of M, we have $\det B = 0$. Hence the result follows. □

Theorem 3.9. *Suppose $D \neq K_n$ be a digraph which includes all loops and has nonnegative Q_1-completion, then D does not have a 2-cycle.*

Proof. Suppose that D has a 2-cycle $\langle v_1, v_2\rangle$. Consider a partial nonnegative Q_1-matrix $M = [a_{ij}]$ specifying D such that $a_{ii} = 1$ $(1 \leq i \leq n)$ and $a_{v_1v_2}a_{v_2v_1} > \binom{n}{2}$ and rest of all specified entries are zero. Let $B = [b_{ij}]$ be any completion of M. Then

$$S_2(B) = \sum_{i \neq j} b_{ii}b_{jj} - \sum_{i \neq j} b_{ij}b_{ji} \; < \; - \sum_{i,j \notin \{v_1,v_2\}} b_{ij}b_{ji} \; < \; 0,$$

and, therefore, B is not a nonnegative Q_1-matrix. □

Example 3.10. *Consider the digraph D_2 in Figure 3. Here D_2 has a 2-cycle $\langle 1, 3\rangle$. Thus by Theorem 3.9, D_2 does not have nonnegative Q_1-completion. To see this consider a partial nonnegative Q_1-matrix*

$$M = \begin{bmatrix} 1 & ? & 10 & 0 \\ 0 & 1 & ? & 0 \\ 10 & 0 & 1 & 0 \\ ? & ? & ? & 1 \end{bmatrix},$$

specifying the digraph D_2. Then for any completion B of M, we have $S_2(B) < 0$. Hence, M cannot be completed to a nonnegative Q_1-matrix.

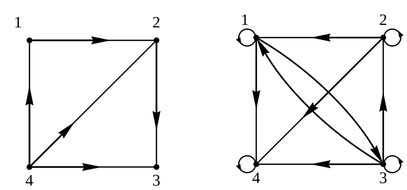

Figure 3. The Digraphs D_2 and $\overline{D_2}$

Remark 3.11. *If a digraph D of order n includes all loops has nonnegative Q_1-completion, then D has less than $\frac{1}{2}n(n+1)$ arcs. In case D has more than $\frac{1}{2}n(n+1)$ arcs, then D must have a 2-cycle.*

Theorem 3.12. *Let $D \neq K_n$ be a digraph of order n that includes all loops and contains an even cycle C of length 4. If D has nonnegative Q_1-completion, then $\overline{D}$ has a 2-cycle.*

Proof. Suppose $M = [a_{ij}]$ be a partial nonnegative Q_1-matrix specifying the digraph D. For $t > 1$, consider the partial nonnegative Q_1-matrix $M(t)$ with the specified entries as follows

$$a_{ij} = \begin{cases} t, & \text{if } (i,j) \in A_C \\ 1, & \text{if } (i,i) \in A_D \\ 0, & \text{otherwise.} \end{cases}$$

Let $B = [b_{ij}]$ be a completion of $M(t)$, where $b_{ij} = x_{ij} \geq 0$ for $(i, j) \notin A_D$. Now we have,

$$0 \; < \; S_2(B) = 6 - \sum b_{ij} b_{ji},$$

and this implies each of x_{ij} to be bounded above by 6. On the other hand we have,

$$S_4(B(t)) = -t^4 + p(t, x_{ij}),$$

where $p(t, x_{ij})$ is a polynomial and have degree at most 3 in t. Consequently, for a large value of t, $S_4(B(t)) < 0$ for any nonnegative choices of x_{ij} within their bounds. For such a value of t, $B(t)$ is not a Q_1-matrix. □

Example 3.13. *Consider the digraph D_3 and its complement $\overline{D_3}$ in Figure 4. The digraph D_3 satisfies the conditions of the Theorem 3.12. Hence it does not have nonnegative Q_1-completion.*

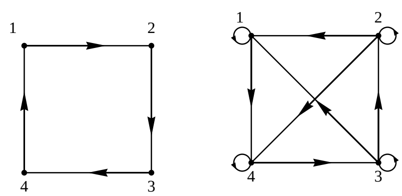

Figure 4. The Digraphs D_3 and $\overline{D_3}$

4. Relationship theorems

4.1 Q-completion and nonnegative Q_1-completion

It is easily seen that a nonnegative Q_1 matrix is a Q-matrix but not vice versa. However their completion problems are not related.

(i) Consider the digraph D_4 and its complement $\overline{D_4}$ in Figure 5. Here D_4 is acyclic and contains all loops. Hence by Corollary 3.5, D_4 has nonnegative Q_1-completion. On the other hand $\overline{D_4}$ is not stratified, thus it does not have Q-completion. (See Theorem 2.8, [2]).

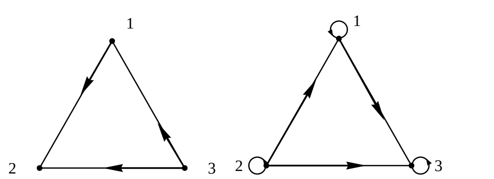

Figure 5. The Digraph D_4 and its complement $\overline{D_4}$

(ii) Consider the digraph D_5 and its complement $\overline{D_5}$ in Figure 6. Here the digraph D_5 does not have nonnegative Q_1-completion (by Theorem 3.9). But since $\overline{D_5}$ is weakly stratified, D_5 has Q-completion (See Theorem 2.12, [2]).

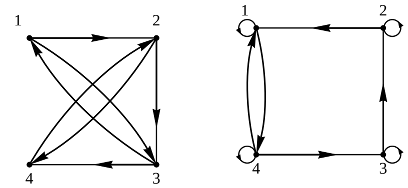

Figure 6. The Digraphs D_5 and $\overline{D_5}$

4.2 Nonnegative Q-completion and non-negative Q_1 -completion

Although a nonnegative Q_1-matrix is a nonnegative Q-matrix, but their completion problem are partially different. Now we have the following:

Proposition 4.1. *If a digraph D has nonnegative Q-completion then it has nonnegative Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial nonnegative Q_1-matrix specifying the digraph D. Then M is also a partial nonnegative Q-matrix specifying the digraph D. Since M has nonnegative Q-completion, thus M can be completed to a nonnegative Q-matrix B by assigning the unspecified diagonal entries (if any) as a positive real number t. Clearly B is a nonnegative Q_1-completion of M. □

However the converse of the Proposition 4.1 is not true. The digraph D_6 in Figure 7 has nonnegative Q_1-completion but does not have nonnegative Q-completion. Consider a partial nonnegative Q-matrix

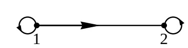

Figure 7. The Digraph D_6

$$M = \begin{bmatrix} 1 & 0 \\ ? & 0 \end{bmatrix},$$

specifying the digraph D_6. It is easily seen that M cannot be completed to a nonnegative Q-matrix since for any completion B of M we have $\det B = 0$. On the other hand, the digraph D_6 has nonnegative Q_1-completion by Theorem 3.4.

4.3 Positive Q-completion and nonnegative Q_1-completion

In this subsection, we will compare the nonnegative Q_1-completion problem with the positive Q-completion problem.

Proposition 4.2. *If a digraph D has nonnegative Q_1-completion, then D has positive Q-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial positive Q-matrix specifying the digraph D. Then M is a partial nonnegative Q_1-matrix specifying D. Let B be a nonnegative Q-completion of M. Then, perturbing the zero entries in B by small positive quantities, a positive Q-completion of M can be obtained. □

However, the converse is not true which can be seen from the following example.

Example 4.3. *Consider the digraph D_7 in Figure 8. The complement of the digraph D_7 i.e. $\overline{D_7}$ contains a 2-cycle $\langle 2,4\rangle$ such that the arc $(4,2)$ in $\overline{D_7}$ satisfies the Theorem 2.10,[12]. Hence the digraph D_7 has positive Q-completion. On the*

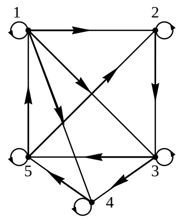

Figure 8. The Digraph D_7

other hand, consider a partial nonnegative Q_1 matrix

$$M(t)=\begin{bmatrix} 1 & t & 1 & 1 & x_1 \\ x_2 & 1 & t & x_3 & x_4 \\ x_5 & x_6 & 1 & 0 & t \\ x_7 & x_8 & x_9 & 1 & 1 \\ t & 1 & x_{10} & x_{11} & 1 \end{bmatrix},$$

specifying the digraph D_7, where $t>1$ and x_i are unspecified entries. Now for any completion $B(t)$, we have

$$S_2(B(t))=\begin{pmatrix} 5 \\ 2 \end{pmatrix}-\left(t\sum x_i+\sum x_i+x_3x_8\right)>0, \quad (4.1)$$

which implies that x_i and x_3x_8 are bounded by $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. However x_3 and x_8 can take any arbitrary values. Again we have,

$$S_4(B(t))=-t^4+c_1t^2+c_2t+c_3+x_3(-t^2+c_4)+x_4(-c_6t^2+c_7), \quad (4.2)$$

where c_r are polynomials in x_i. Consequently, for large values of t, $S_4(B(t))<0$ for any completion $B(t)$ of $M(t)$

5. Classification of digraphs of small order having nonnegative Q_1-completion

In this section we will classify all the digraphs of order at most four as to nonnegative Q_1-completion. For this purpose we will apply the previously obtained results on the digraphs. The nomenclature of the digraphs has been considered from the list in [4, Appendix, pp. 233]. Here, $D_p(q,n)$ is the one obtained by attaching a loop at each of the vertices to the n-th member in the list of digraphs with p vertices and q (non-loop) arcs in the list.

Now permutation similarity of nonnegative Q_1-matrix implies that if a digraph D has nonnegative Q_1-completion, then any digraph which is isomorphic to D has nonnegative Q_1-completion. Thus any digraph which is obtained by labelling the unlabelled digraph associated to D has nonnegative Q_1-completion.

Theorem 5.1. *For $1\le p\le 4$, the digraphs $D_p(q,n)$ which are listed below have nonnegative Q_1-completion.*

$p=2;$	$q=0,1,2;$	$n=1$
$p=3;$	$q=0,1;$	$n=1$
	$q=2;$	$n=2$–4
	$q=3;$	$n=2,3$
	$q=6;$	$n=1$
$p=4;$	$q=0,1;$	$n=1$
	$q=2;$	$n=2$–5
	$q=3;$	$n=4$–13
	$q=4;$	$n=16$–27
	$q=5;$	$n=29$–38
	$q=6;$	$n=46$–48
	$q=12;$	$n=1.$

Proof. It can be easily seen that $D_p(q,n)$ has nonnegative Q_1-completion if $q=0$ or it is a complete digraph.

The digraphs $D_2(q,n), q=1, n=1$; $D_3(q,n), q=1, n=1$; $q=2, n=2$–4; $q=3, n=2,3$, $D_4(q,n), q=1, n=1$; $q=2, n=2$–5; $q=3, n=3$–13; $q=4, n=17$–27; $q=5, n=29,30,31,33$–38; $q=6, n=46$–48 do not contain a cycle of even length and hence each of the digraph has nonnegative Q_1-completion by Theorem 3.4.

Each of the digraph $D_4(q,n), q=4, n=16$; $q=5, n=32$; $q=6, n=45$ satisfies the statement of the Theorem 3.7, and hence each digraph has nonnegative Q_1-completion.

The digraph $D_4(q,n), q=6, n=45$ satisfies the statement of the Theorem 3.12, hence it does not have nonnegative Q_1-completion.

The rest of digraphs $D_p(q,n); 3\le p\le 4$, contains a 2-cycle and they do not have nonnegative Q_1-completion. □

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