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A modified viscosity implicit rule for a variational inequality problem and a uniformly L-Lipschitzian asymptotically pseudocontractive mapping in a Banach space

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Abstract

In this paper, we propose a modified implicit rule for the uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in a Banach space.Related strong convergence theorems are established under the assumptions on certain parameters.Furthermore, it also provides solution to an appropriate variational inequality problem. Our main result improves and extends many known results of the recent literature.

Keywords

Variational inequality, asymptotically pseudocontractive mapping, viscosity implicit rule, Banach space.

AMS Subject Classification

47H05; 47H09, 47H10

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1. Introduction

The class of pseudocontractive operator is introduced by Browder and Petryshyn[13] in 1967 in Hilbert space who said that U is pseudocontractive operator if and only if T=I-U is monotone operator.They also proved the existence results and convergence results for this class of mappings in Hilbert space using Krasnoselskij[14] iteration .In the same year, Browder[15] independently gave the existence of fixed points of pseudocontractive mapping in real uniformly convex Banach space and real Banach space with uniform structure.He assreted that the class of pseudocontractive operators includes the important class of nonexpansive operators and shown that T is pseudocontractive if A=I-T is accretive.

It is well known if T is a nonexpansive mapping then U:I-T is monotone in Hilbert space for any subset D of H into H and accretive operator in Banach space into itself.However, converse is not true i.e.if U is monotone or accretive operator then T=I-U is not nonexpansive (see Browder[16]. In fact, this was the reason why pseudocontractive operator was introduced.

The class of pseudocontractive mappings plays an important role in the theory of nonlinear mappings because of its firm connection with the accretive mappings.(see Kirk and Shoneberg[17].Browder[15] and Kato[18]. Independent of each other ,these authors have tried to characterize pseudocontractive mappings as the mapping T for which the mapping A=I-T is accretive.Consequently, several methods of approximating the equilibrium points of the initial value problems

$$x'(t)+Ax(t)=0, x(0)=x_0$$

have been evolved so for proving the existence and approximation using pseudocontractive operators.One of them is based on Viscosity Implicit Rule.

The viscosity iterative algorithms has been investigated extensively by many authors to find the common element of the set of fixed point of pseudocontractive mappings and the set of solution of variational inequality problem (see [11],[21],[22]and the references therein).On this line of investigation, in 2000 , Moudafi [1] introduced the viscosity iterative algorithm for proving the strong convergence of non-expansive mappings in real Hilbert space. Later in 2004, Xu [2] extended the result of Moudafi [1] to a Banach space and introduced the following viscosity technique for non expansive mapping in a uniformly smooth Banach space :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0$$

where f is a contraction and $\|\alpha_n\|$ is a sequence in $[0,1]$.

The implicit midpoint rule is a powerful method for solving ordinary differential equations ; (see [[8],[9]] and the references therein).Recently, in 2015 , Xu et. al [4] applied the viscosity technique fo rthe non expansive mapping and introduced the following viscosity implicit midpoint rule

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \quad (1.1)$$

They proved that the sequence generated by equation (1.1) converges strongly to a fixed point of T ,which also solves the following variational inequality in Hilbert space ,

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in F(T) \quad (1.2)$$

In 2017, Luo et. al. [7] proved strong convergence for strict pseudocontractive mapping with some appropriate conditions on parameters by using the above [4] implicit midpoint rule of non expansive mappings in uniformly smooth Banach space which also solves some variational inequality problem. Recently,Yan et. al. [10] extended the result of Luo et.al. [7] from non expansive mapping to asymptotically non expansive mapping and gave the generalized viscosity implicit rule for asymptotically non expansive mapping in Hilbert space as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0$$

Further, Yao et. al. [12] introduced another semi-implicit midpoint rule as follows :

$$x_{n+1} = \alpha_n f(x_n) + \beta_n f(x_n) + \gamma_n T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0$$

In 2016, Yu et. al. [19] extended the work of Yao et. al.[12] and gave following generalized viscosity implicit rule for non expansive mapping in Hilbert space :

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\delta_n x_n + (1-\delta_n)x_{n+1}), \quad n \geq 0$$

. Its sequence converges strongly to fixed point T. In 2017, Wang et. al.[20] extended the work of [19]to a uniformly L-Lipschitzian asymptotically pesudocontractive mapping in Banach space and introduced the following modified viscosity implicit rule

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T^n(\zeta_n x_n + (1-\zeta_n)x_{n+1}), \quad n \geq 0$$

Now in this paper, motivated by above results of [[19],[20]] we introduce the following modified iterative algorithm based on viscosity implicit rule for a uniformly L-Lipschitzian asymptotically pseudocontractive mapping in Banach space, which is more general than Theorem 2.1 of [20],

$$x_{n+1} = \alpha_n(1-\delta_n)x_n + \beta_n f(x_n) + \gamma_n T^n(s_n x_n + (1-s_n)x_{n+1}), \quad n \geq 0$$

with some suitable assumptions imposed on parameters and prove strong convergence theorem for asymptotically pseudocontractive mapping in Banach space. It extends the main result of Wang et. al.[20] and improve many such other results.

2. Preliminaries

Through out this paper, we assume that E is a real Banach space and E^* is the dual space of E. Let C be a subset of E and let J denote the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2$$

for all $x \in E$, where $\langle .,.\rangle$ denotes the generalized duality pairing.We denote the single valued duality pairing by j.In a Hilbert space H, j is the identity mapping.We recall here some useful definitions.

Definition 2.1. *A mapping $f : C \to C$ is said to be a strict contraction if there exists a constant $\lambda \in (0,1)$ satisfying*

$$\|f(x) - f(y)\| \leq \lambda \|x-y\|, \quad \forall x,y \in C$$

Definition 2.2. *A mapping $T : C \to C$ is said to be an asymptotically non-expansive if there exists a sequence $\{k_n\}$ with $k_n \to 1$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x-y\|, \quad \forall x,y \in C$$

Definition 2.3. *A mapping $T : C \to C$ is said to be an asymptotically pseudocontractive in Banach space if there exists a sequence $\{k_n\}$ with $k_n \to 1$ and $j(x-y) \in J(x-y)$ for which the following inequality holds*

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2, \quad \forall x,y \in C, n \geq 1$$

Definition 2.4. *A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists some $L > 0$ such that*

$$\|T^n x - T^n y\| \leq L\|x-y\|, \quad \forall x,y \in C, \quad n \geq 1$$

We can easily see that if T is an asymptotically non-expansive mapping then it is both asymptotically pseudocontractive and uniformly L-Lipschitzian but the converse need not to be true in general.
We shall use here the following lemmas :

Lemma 2.5 ([2]). *Assume $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* R *such that*

1. $\sum_{n=0}^{\infty} \alpha_n = \infty$.
2. $\limsup_{n\to\infty} \dfrac{\delta_n}{\alpha_n} \leq 0 \quad$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 ([11])**.** *Let* E *be a reflexive smooth Banach space with a weakly sequential continuous duality mapping* J*. Let* C *be a nonempty bounded and closed convex subset of* E *and* $T : C \to C$ *be a uniformly L-Lipschitzian and asymptotically pseudocontraction. Then* $(I-T)$ *is demiclosed at zero, where* I *is the identity mapping, i.e., if* $x_n \rightharpoonup x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$*.*

Lemma 2.7 ([12])**.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two bounded sequence in a Banach space* E *and* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n$*. Suppose that*

$$x_{n+1} = (1-\beta_n) + \beta_n y_n \forall n \geq 0$$

and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$*.*
Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$*.*

3. Main result

Theorem 3.1. *Let* E *be a reflexive smooth Banach space with a weakly sequentially continuous duality mapping. Let* J, C *be a nonempty bounded and closed convex subset of* E*, and let* $T : C \to C$ *be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence* k_n *such that* $F(T) \neq \phi$ *and* $f : C \to C$ *be a contraction with coefficient* $\lambda \in (0,1)$*. Pick any* $x_0 \in C$*, let* x_n *be a sequence generated by*

$$x_{n+1} = \alpha_n(1-\delta_n)x_n + \beta_n f(x_n) + \gamma_n T^n(s_n x_n + (1-s_n)x_{n+1}) \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *and* $\{s_n\} \subset (0,1)$ *satisfying the conditions*

1. $\alpha_n + \beta_n + \gamma_n = 1, \quad \lim_{n\to\infty} \beta_n = 0,$
   $\gamma_n = \eta\beta_n, \lim_{n\to\infty} \delta_n = 0;$
   $0 < \eta < \dfrac{(s_{n+1}) - \lambda}{L - (s_{n+1})}$
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1,$
   $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \mid \alpha_{n+1} - \alpha_n \mid = 0$
   , $\lim_{n\to\infty} \mid \beta_{n+1} - \beta_n \mid = 0$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.
3. $\sum_{n=0}^{\infty} \beta_n = \infty$, $0 < s_n < s_{n+1} < 1$, $\sum_{n=0}^{\infty} \delta_n = \infty$,
   $\gamma_n(1-s_n) < \frac{1}{L}$;
4. $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0, x \in C'$

where C' *is any bounded subset of* C *for all* $n \geq 0$*.Then* $\{x_n\}$ *defined by* 3.1 *converges strongly to a fixed point* p *of the asymptotically pseudocontractive mapping* T*, which solves the variational inequality :*

$$\langle (I-f)p, j(p-y) \rangle \leq 0, \quad \forall y \in F(T)$$

Proof. We divide the proof into five steps
Step 1: First we show that $\{x_n\}$ is bounded. Take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned}
||x_{n+1} - p|| &= ||\alpha_n(1-\delta_n)x_n + \beta_n f(x_n) \\
&\quad + \gamma_n T^n(s_n x_n + (1-s_n)x_{n+1}) - p|| \\
&= ||\alpha_n x_n - \alpha_n\delta_n x_n + \beta_n f(x_n) \\
&\quad + \gamma_n T^n(s_n x_n + (1-s_n)x_{n+1}) - p|| \\
&= ||\alpha_n[(1-\delta_n)(x_n - p) + \delta_n(-p)] \\
&\quad + \gamma_n[T^n(s_n x_n + (1-s_{n+1})x_{n+1}) - p] \\
&\quad + \beta_n(f(x_n) - f(p)) + \beta_n(f(p) - p)|| \\
&\leq \alpha_n(1-\delta_n)||x_n - p|| + \alpha_n\delta_n||p|| \\
&\quad + \beta_n||(f(x_n) - f(p)|| + \beta_n||(f(p) - p)|| \\
&\quad + \gamma_n||T^n(s_n x_n + (1-s_n)x_{n+1}) - p|| \\
&\leq \alpha_n(1-\delta_n)||(x_n - p)|| + \alpha_n\delta_n||p|| \\
&\quad + \beta_n\lambda||x_n - p|| + \beta_n||(f(p) - p)|| \\
&\quad + \gamma_n L||(s_n x_n + (1-s_n)x_{n+1}) - p|| \\
&\leq \alpha_n(1-\delta_n)||(x_n - p)|| + \alpha_n\delta_n||p|| \\
&\quad + \beta_n\lambda||x_n - p|| + \beta_n||(f(p) - p)|| \\
&\quad + \gamma_n L s_n||x_n - p|| \\
&\quad + \gamma_n L(1-\delta_n)||x_{n+1} - p|| \\
&= (\alpha_n(1-\delta_n) + \beta_n\lambda + \gamma_n L s_n)||(x_n - p)|| \\
&\quad + \beta_n||(f(p) - p)|| + \alpha_n\delta_n||p|| \\
&\quad + \gamma_n L(1-s_n)||x_{n+1} - p||.
\end{aligned}$$

which implies that

$$\begin{aligned}
[1 - \gamma_n L(1-s_n)]||x_{n+1} - p|| &\leq (\alpha_n(1-\delta_n) + \beta_n\lambda \\
&\quad + \gamma_n L s_n)||x_n - p|| + \beta_n||f(p) - p|| \\
&\quad + \alpha_n\delta_n||p||
\end{aligned} \tag{3.2}$$

Since $\gamma_n, 1 - s_n \in (0,1), \gamma_n(1-s_n) < \dfrac{1}{L}$
we get $1 - \gamma_n L(1-s_n) > 0$.
By (3.2) and condition (1).
$[\alpha_n + \beta_n + \gamma_n = 1, \lim_{n\to\infty} \beta_n = 0, \gamma_n = \eta\beta_n,$
$0 < \eta < \dfrac{(s_n+1) - \lambda}{L - ((s_n+1))}]$

It follows that

$$
\begin{aligned}
&||x_{n+1}-p|| \\
&\leq \frac{\alpha_n(1-s_n)+\lambda\beta_n+\gamma_n L s_n}{1-\gamma_n L(1-s_n)}||x_n-p|| \\
&+\frac{\beta_n}{1-\gamma_n L(1-s_n)}||f(p)-p|| \\
&+\frac{\alpha_n s_n}{1-\gamma_n L(1-s_n)}||p|| \\
&=[1-\frac{1-\alpha_n(1-s_n)-\lambda\beta_n-\gamma_n L}{1-\gamma_n L(1-s_n)}]||x_n-p|| \\
&+\frac{\beta_n}{1-\gamma_n L(1-s_n)}||f(p)-p|| \\
&+\frac{\alpha_n s_n}{1-\gamma_n L(1-s_n)}||p||. \\
&=[1-\frac{\beta_n[1+s_n-\lambda]-\gamma_n(L-(s_n+1))}{1-\gamma_n L(1-s_n)}]||x_n-p|| \\
&+\frac{\beta_n}{1-\gamma_n L(1-s_n)}||f(p)-p|| \\
&+\frac{\alpha_n s_n}{1-\gamma_n L(1-s_n)}||p|| \\
&=[1-\frac{\beta_n[(1+s_n)-\lambda]-\eta[L-(s_n+1)}{1-\gamma_n L(1-s_n)}]||x_n-p|| \\
&+\frac{\beta_n[(s_n+1)-\lambda-\eta[L-(s_n+1)]]}{1-\gamma_n L(1-s_n)} \\
&\frac{||f(p)-p||}{(s_n+1)-\lambda-\eta[L-(s_n+1)]} \\
&+\frac{\alpha_n s_n[(s_n+1)-\lambda-\eta[L-(s_n+1)]]}{1-\gamma_n L(1-s_n)} \\
&\frac{||p||}{\alpha_n s_n[(s_n+1)-\lambda-\eta[L-(s_n+1)]]}
\end{aligned}
$$

Consequently , we get

$$
\begin{aligned}
&||x_{n+1}-p|| \\
&\leq max\Big\{||x_n-p||,\frac{1}{(s_n+1)-\lambda-\eta[L-(s_n+1)]} \\
&||f(p)-p||,\frac{1}{(s_n+1)-\lambda-\eta[L-(s_n+1)]}||p||\Big\}, \\
&\forall n\geq 0.
\end{aligned}
$$

By induction we readily obtained

$$
\begin{aligned}
&||x_n-p|| \\
&\leq max\Big\{||x_0-p||,\frac{1}{(s_n+1)-\lambda-\eta[L-(s_n+1)]} \\
&||f(p)-p||,\frac{1}{(s_n+1)-\lambda-\eta[L-(s_n+1)]}||p||\Big\}, \\
&\forall n\geq 0.
\end{aligned}
$$

Hence we can observe that x_n is bounded . Consequently, $f(x_n)$ and $T^n(s_nx_n+(1-s_n)x_{n+1})$ are also bounded.

Step 2: $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
To see this , we set

$$y_n=\frac{x_{n+1}-\alpha_n(1-\delta_n)x_n}{1-\alpha_n(1-\delta_n)}$$

Thus we have

$$
\begin{aligned}
y_{n+1}-y_n=&\frac{x_{n+2}-\alpha_{n+1}(1-\delta_{n+1})x_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})} \\
&-\frac{x_{n+1}-\alpha_n(1-\delta_n)x_n}{1-\alpha_n(1-\delta_n)}
\end{aligned}
$$

from equation(3.1), we get

$$
\begin{aligned}
&y_{n+1}-y_n \\
&=\frac{1}{1-\alpha_{n+1}(1-\delta_{n+1})}\big\{\beta_{n+1}f(x_{n+1}) \\
&+\gamma_{n+1}T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+1})\big\} \\
&-\frac{\beta_n f(x_n)+\gamma_n T^n(s_nx_n+(1-s_n)x_{n+1})}{1-\alpha_n(1-\delta_n)} \\
&=\frac{1}{1-\alpha_{n+1}(1-\delta_{n+1})}\big\{\beta_{n+1}f(x_{n+1}) \\
&+\gamma_{n+1}T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2})\big\} \\
&-\frac{\beta_n f(x_n)+(1-\alpha_n-\beta_n)T^n(s_nx_n+(1-s_n)x_{n+1})}{1-\alpha_n(1-\delta_n)} \\
&=\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}[f(x_{n+1})-f(x_n)] \\
&+(\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)})f(x_n) \\
&-(\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}). \\
&[T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}[T^{n+1}(s_{n+1}x_{n+1} \\
&+(1-s_{n+1})x_{n+2})-T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&+[T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}) \\
&-T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&=\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}[f(x_{n+1})-f(x_n)] \\
&+(\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)})[f(x_n) \\
&-T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}[T^{n+1}(s_{n+1}x_{n+1} \\
&+(1-s_{n+1})x_{n+2})-T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&+[T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}) \\
&-T^n(s_nx_n+(1-s_n)x_{n+1})]
\end{aligned}
$$

Thus,

$$\begin{aligned}
& y_{n+1}-y_n \\
&= \frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}[f(x_{n+1})-f(x_n)] \\
&+ (\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)})[f(x_n) \\
&- T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&+ (1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})})[T^{n+1}(s_{n+1}x_{n+1} \\
&+ (1-s_{n+1})x_{n+2})-T^n(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2})] \\
&+ (1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})})[T^n(s_{n+1}x_{n+1} \\
&+ (1-s_{n+1})x_{n+2})-T^n(s_nx_n+(1-s_n)x_{n+1})]
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|y_{n+1}-y_n\| \\
&\le \frac{\lambda\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}||(x_{n+1}-x_n|| \\
&+ |\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}|M \\
&+ \sup_{x\in C'}||T^{n+1}x-T^nx|| \\
&+ (1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})})L||(s_{n+1}x_{n+1} \\
&+ (1-s_{n+1})x_{n+2}-(s_nx_n+(1-s_n)x_{n+1})|| \\
&\le \frac{\lambda\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}||(x_{n+1}-x_n|| \\
&+ |\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}|M \\
&+ \sup_{x\in C'}||T^{n+1}x-T^nx|| \\
&+ (1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}) \\
& L||s_n(x_{n+1}-s_n)+(1-s_{n+1})(x_{n+2}-x_{n+1})|| \qquad (3.3)\\
&\le \frac{\lambda\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}||(x_{n+1}-x_n|| \\
&+ |\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}|M \\
&+ \sup_{x\in C'}||T^{n+1}x-T^nx||+(1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}) \\
& L[s_n||x_{n+1}-x_n||+(1-s_{n+1})||x_{n+2}-x_{n+1}||]
\end{aligned}$$

where C' contains sequence $\{s_nx_n+(1-s_n)x_{n+1}\}$ and $M>0$ is a constant such that

$$\begin{aligned}
M \ge \{ & \sup_{n\ge 0}||x_n-T^{n+1}(s_nx_n+(1-s_n)x_{n+1})||, \\
& ||f(x_n)-T^{n+1}(s_nx_n+(1-s_n)x_{n+1})||, \\
& ||f(x_n)-T^n(s_nx_n+(1-s_n)x_{n+1})||\}
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
& ||x_{n+2}-x_{n+1}|| \\
&= \|\alpha_{n+1}(1-\delta_{n+1})x_{n+1}+\beta_{n+1}f(x_{n+1}) \\
&+ \gamma_{n+1}T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}) \\
&- \alpha_n(1-\delta_n)x_n-\beta_nf(x_n) \\
&- \gamma_nT_n(s_nx_n+(1+s_n)x_{n+1})\| \\
&= \|\alpha_{n+1}(x_{n+1}-x_n)-\alpha_{n+1}\delta_{n+1}(x_{n+1}-x_n) \\
&+ (\alpha_{n+1}-\alpha_n)x_n-\alpha_nx_n(\delta_{n+1}-\delta_n) \\
&- (\alpha_{n+1}-\alpha_n)\delta_{n+1}x_n+\beta_{n+1}(f(x_{n+1}-f(x_n)) \\
&+ (\beta_{n+1}-\beta_n)f(x_n)+\gamma_{n+1}[T^{n+1}(s_{n+1}x_{n+1} \\
&+ (1-s_{n+1})x_{n+2})-T^{n+1}(s_nx_n+(1-s_n)x_{n+1})] \\
&+ (\gamma_{n+1}-\gamma_n)T^{n+1}(s_nx_n+(1-s_n)x_{n+1})+\gamma_n[T^{n+1} \\
& (s_nx_n+(1-s_n)x_{n+1})-T^n(s_nx_n+(1-s_n)x_{n+1})]\| \\
&= \|\alpha_{n+1}(x_{n+1}-x_n)-\alpha_{n+1}\delta_{n+1}(x_{n+1}-x_n) \\
&+ (\alpha_{n+1}-\alpha_n)x_n-\alpha_nx_n(\delta_{n+1}-\delta_n)-(\alpha_{n+1}-\alpha_n) \\
& .\delta_{n+1}x_n+\beta_{n+1}(f(x_{n+1}-f(x_n))+(\beta_{n+1}-\beta_n) \\
& f(x_n)z+\gamma_{n+1}[T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}) \\
&- T^{n+1}(s_nx_n+(1-s_n)x_{n+1})]-[(\alpha_{n+1}-\alpha_n) \\
&+ (\beta_{n+1}-\beta_n)]T^{n+1}(s_nx_n+(1-s_n)x_{n+1}) \\
&+ \gamma_n[T^{n+1}(s_nx_n+(1-s_n)x_{n+1}) \\
&- T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&= \|(\alpha_{n+1}(1+\delta_{n+1}))(x_{n+1}-x_n)+(\alpha_{n+1}-\alpha_n) \\
& [x_n-\delta_{n+1}x_n-T^{n+1}(s_nx_n+(1-\delta_n)x_{n+1})] \\
&- \alpha_nx_n(\delta_{n+1}-\delta_n)+(\beta_{n+1}-\beta_n)[f(x_n)-T^{n+1} \\
& (s_nx_n+(1-s_n)x_{n+1})]+\beta_{n+1}(f(x_{n+1}-f(x_n))) \\
&+ \gamma_{n+1}[T^{n+1}(s_{n+1}x_{n+1}+(1-s_{n+1})x_{n+2}) \\
&- T^{n+1}(s_nx_n+(1-s_n)x_{n+1})]+\gamma_n[T^{n+1}(s_nx_n \\
&+ (1-s_n)x_{n+1})-T^n(s_nx_n+(1-s_n)x_{n+1})] \\
&\le (\alpha_{n+1}-\alpha_{n+1}\delta_{n+1})||(x_{n+1}-x_n)||+|(\alpha_{n+1} \\
&- \alpha_n)|||x_n(1-\delta_{n+1})-T^{n+1}(s_nx_n+(1-s_n) \\
& x_{n+1})||-\alpha_n||x_n|||\delta_{n+1}-\delta_n|+|\beta_{n+1}-\beta_n| \\
& ||f(x_n)-T^{n+1}(s_nx_n+(1-s_n)x_{n+1}||+\lambda\beta_{n+1} \\
& ||(x_{n+1}-x_n)||+\gamma_{n+1}L||(s_{n+1}x_{n+1}+(1-s_{n+1}) \\
& x_{n+2}-s_nx_n-(1-s_n)x_{n+1}||+\gamma_n||T^{n+1}(s_nx_n \\
&+ (1-s_n)x_{n+1})-T^n(s_nx_n+(1-s_n)x_{n+1}||
\end{aligned}$$

$$
\begin{aligned}
&||x_{n+2}-x_{n+1}|| \\
&\le (\alpha_{n+1}-\alpha_{n+1}\delta_{n+1})+||(x_{n+1}-x_n)|| \\
&+|(\alpha_{n+1}-\alpha_n)|M+|\beta_{n+1}-\beta_n|M \\
&-\alpha_n|\delta_{n+1}-\delta_n|||x_n||+\lambda\beta_{n+1}||(x_{n+1}-x_n)|| \\
&+\gamma_{n+1}L||(1-s_{n+1})(x_{n+2}-x_{n+1})+s_n(x_{n+1}-x_n)|| \\
&+\gamma_n||T^{n+1}(s_nx_n+(1-s_n)x_{n+1}) \\
&-T^n(s_nx_n+(1-s_n)x_{n+1})|| \\
&\le(\alpha_{n+1}-\alpha_{n+1}\delta_{n+1}+\lambda\beta_{n+1}+\gamma_{n+1}Ls_n)||(x_{n+1}-x_n)|| \\
&+\gamma_{n+1}L(1-s_{n+1})||(x_{n+2}-x_{n+1})|| \\
&-\alpha_n|\delta_{n+1}-\delta_n|||x_n|| \\
&+(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)M+\sup_{x\varepsilon c'}||T^{n+1}x-T^nx||
\end{aligned}
$$

It turns out that

$$
\begin{aligned}
&||x_{n+2}-x_{n+1}|| \\
&\le\frac{(\alpha_{n+1}-\alpha_{n+1}\delta_{n+1}+\lambda\beta_{n+1}+\gamma_{n+1}Ls_n)}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&||x_{n+1}-x_n||+\frac{M}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)+\frac{1}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&\sup_{x\varepsilon c|}||T^{n+1}x-T^nx||-\frac{\alpha_n|\delta_{n+1}-\delta_n|}{1-\gamma_{n+1}L(1-s_{n+1})}||x|| \\
&=[1-\frac{\beta_{n+1}(1-\lambda)+\gamma_{n+1}L(s_{n+1}-s_n)-\gamma_{n+1}(L-1)}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&-\frac{\alpha_{n+1}\delta_{n+1}}{1-\gamma_{n+1}L(1-s_{n+1})}]||x_{n+1}-x_n|| \qquad (3.4)\\
&+\frac{M}{1-s_{n+1}L(1-s_{n+1})}(|\alpha_{n+1}-\alpha_n| \\
&+|\beta_{n+1}-\beta_n|)+\frac{1}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&\sup_{x\varepsilon c'}||T^{n+1}x-T^nx||-\frac{\alpha_n|\delta_{n+1}-\delta_n|}{1-\gamma_{n+1}L(1-s_{n+1})}||x_n|| \\
&\le[1-\frac{\beta_{n+1}[1-\lambda-\eta(L-1)]+\gamma_{n+1}L(s_{n+1}-s_n)}{1-\gamma_nL(1-s_{n+1})} \\
&-\frac{\alpha_{n+1}\delta_{n+1}}{1-\gamma_nL(1-s_{n+1})}]||x_{n+1}-x_n|| \\
&+\frac{M}{1-\gamma_{n+1}L(1-s_{n+1})}(|\alpha_{n+1}-\alpha_n| \\
&+|\beta_{n+1}-\beta_n|)+\frac{1}{1-\gamma_{n+1}L(1-s_{n+1})} \\
&\sup_{x\varepsilon c'}||T^{n+1}x-T^nx||-\frac{\alpha_n|\delta_{n+1}-\delta_n|}{1-\gamma_{n+1}L(1-s_{n+1})}||x_n||
\end{aligned}
$$

put (3.4) into (3.3), we get

$$
\begin{aligned}
&||y_{n+1}-y_n|| \\
&\le[\frac{\lambda\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}+(1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}) \\
&Ls_n+(1-\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})})L(1-s_{n+1})] \\
&||x_{n+1}-x_n|| \\
&+\frac{(\gamma_{n+1}+\alpha_{n+1}\delta_{n+1})L(1-s_{n+1})}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)+ \\
&\frac{(\gamma_{n+1}+\alpha_{n+1}\delta_{n+1})L(1-s_{n+1})}{[1-\alpha_{n+1}][1-\gamma_{n+1}(1-s_{n+1})]}\sup_{x\varepsilon c|}||T^{n+1}x-T^nx|| \\
&+|\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}| \qquad (3.5)\\
&=[\frac{\lambda\beta_{n+1}+(\gamma_{n+1}+\alpha_{n+1}\delta_{n+1})Ls_n}{1-\alpha_{n+1}(1-\delta_{n+1})} \\
&+\frac{(\gamma_{n+1}+\alpha_{n+1}\delta_{n+1})L(1-s_{n+1})}{1-\alpha_{n+1}(1-\delta_{n+1})}]||x_{n+1}-x_n|| \\
&+|\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}|M \\
&+\frac{1}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&\sup_{x\varepsilon c'}||T^{n+1}x-T^nx|| \\
&+\frac{\gamma_{n+1}L(1-s_{n+1})M}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|) \\
&\le[\frac{\lambda\beta_{n+1}+\gamma_{n+1}L+\alpha_{n+1}\delta_{n+1}L}{1-\alpha_{n+1}(1-\delta_{n+1})}]||x_{n+1}-x_n|| \\
&+|\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}|M \\
&+\frac{1}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&\sup_{x\varepsilon c'}||T^{n+1}x-T^nx|| \\
&+\frac{\gamma_{n+1}L(1-s_{n+1})M}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|) \\
&=[1-\frac{(1-\lambda)\beta_{n+1}+(\gamma_{n+1}+\alpha_{n+1}\delta_{n+1})(L-1)}{1-\alpha_{n+1}(1-\delta_{n+1})}] \\
&||x_{n+1}-x_n||+|\frac{\beta_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{\beta_n}{1-\alpha_n(1-\delta_n)}| \\
&.M+\frac{1}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&\sup_{x\in c'}||T^{n+1}x-T^nx|| \\
&+\frac{\gamma_{n+1}L(1-s_{n+1})M}{[1-\alpha_{n+1}(1-\delta_{n+1})][1-\gamma_{n+1}L(1-s_{n+1})]} \\
&(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)
\end{aligned}
$$

By conditions (i) , (ii) and (iii) we can get

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n-||x_{n+1}-x_n||)\leq 0.$$

Using lemma (2.7) , we get

$$\lim_{n\to\infty}||y_n-x_n||=0.$$

Note that

$$y_n-x_n=\frac{x_{n+1}-\alpha_n(1-\delta_n)x_n}{1-\alpha_n(1-\delta_n)}-x_n$$

therefore

$$y_n-x_n=\frac{x_{n+1}-x_n}{1-\alpha_n(1-\delta_n)}$$

So, we obtain

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0$$

Step 3: Next we prove that

$$\lim_{n\to\infty}||x_n-Tx_n||=0$$

In fact we observe

$$\begin{aligned}
&||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\\
&=\|\alpha_n(1-s_n)x_n+\beta_nf(x_n)+\gamma_nT^n(s_nx_n\\
&+(1-s_n)x_{n+1})-T^n(s_nx_n+(1-s_n)x_{n+1})\|\\
&=\|\alpha_n(1-s_n)x_n+\beta_nf(x_n)-\alpha_nT^n(s_nx_n\\
&+(1-s_n)x_{n+1})-\beta_nT^n(s_nx_n+(1-s_n)x_{n+1})\|\\
&=\|\alpha_n[x_n-\delta_nx_n-T^n(s_nx_n+(1-s_n)x_{n+1})]\\
&+\beta_n[f(x_n)-T^n(s_nx_n+(1-s_n)x_{n+1})]\|\\
&=\|\alpha_nx_n-\alpha_nx_{n+1}+\alpha_nx_{n+1}-\alpha_n\delta_nx_n+\alpha_n\delta_nx_{n+1}\\
&-\alpha_n\delta_nx_{n+1}-\alpha_nT^n(s_nx_n+(1-s_n)x_{n+1})+\beta_n(f(x_n)\\
&-T^n(s_nx_n+(1-s_n)x_{n+1}))\|\\
&\leq\alpha_n||x_n-x_{n+1}||+\alpha_n||x_{n+1}-T^n(s_nx_n\\
&+(1-s_n)x_{n+1})||+\alpha_n\delta_n||x_n-x_{n+1}||-\alpha_n\delta_nx_{n+1}\\
&+\beta_n||f(x_n)-T^n(s_nx_n+(1-s_n)x_{n+1})||.
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\alpha_n)||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\\
&\leq\alpha_n||x_n-x_{n+1}||+\alpha_n\delta_n||x_{n+1}-x_n||-\alpha_n\delta_nx_{n+1}+\\
&\beta_n||f(x_n)-T^n(s_nx_n+(1-s_n)x_{n+1})||.
\end{aligned}$$

That is

$$\begin{aligned}
&||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1}||\\
&\leq\frac{\alpha_n+\alpha_n\delta_n}{(1-\alpha_n)}||x_n-x_{n+1}||+\frac{\beta_n}{(1-\alpha_n)}\\
&||f(x_n)-T^n(s_nx_n+(1-s_n)x_{n+1})||-\frac{\alpha_n\delta_n}{(1-\alpha_n)}||x_{n+1}||.
\end{aligned}$$

By condition (1) and (2) and using step (2) we get

$$||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\to 0,\quad as\quad n\to\infty \tag{3.7}$$

And , moreover we have

$$\begin{aligned}
&\|x_n-T^nx_n\|\\
&=\|x_n-x_{n+1}+x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})\\
&+T^n(s_nx_n+(1-s_n)x_{n+1})-T^nx_n\|\\
&\leq||x_n-x_{n+1}||+||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\\
&+||T^n(s_nx_n+(1-s_n)x_{n+1})-T^nx_n||\\
&\leq||x_n-x_{n+1}||+||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\\
&+L||(s_nx_n+(1-s_n)x_{n+1})-x_n||\\
&\leq||x_n-x_{n+1}||+||x_{n+1}-T^n(s_nx_n+(1-s_n)x_{n+1})||\\
&+L(1-s_n)||x_n-x_{n+1}||\\
&\leq(1+L(1-s_n))||x_n-x_{n+1}||+||x_{n+1}\\
&-T^n(s_nx_n+(1-s_n)x_{n+1})||
\end{aligned}$$

In view of step (2) and (3.7), we have

$$||x_n-T^nx_n||\to 0,\quad as\quad n\to\infty \tag{3.8}$$

Since T is uniformly L-Lipschitzian, we derive

$$\begin{aligned}
&||x_n-Tx_n||\\
&=||x_n-T^nx_n+T^nx_n-Tx_n||\\
&\leq||x_n-T^nx_n||+||T^nx_n-Tx_n||\\
&\leq||x_n-T^nx_n||+L||T^{n-1}x_n-x_n||\\
&\leq||x_n-T^nx_n||+L||T^{n-1}x_n-T^{n-1}x_{n-1}\\
&+T^{n-1}x_{n-1}-x_{n-1}+x_{n-1}-x_n||\\
&\leq||x_n-T^nx_n||+L||T^{n-1}x_n-T^{n-1}x_{n-1}||\\
&+L||T^{n-1}x_{n-1}-x_{n-1}||+L||x_{n-1}-x_n||\\
&\leq||x_n-T^nx_n||+L^2||x_n-x_{n-1}||+L||T^{n-1}x_{n-1}\\
&-x_{n-1}||+L||x_{n-1}-x_n||\\
&\leq||x_n-T^nx_n||+L||T^nx_{n-1}-x_{n-1}||\\
&+(L^2+L)||x_{n-1}-x_n||
\end{aligned}$$

By step (2) and (3.8) , we have

$$||x_n-Tx_n||\to 0\quad as\quad n\to\infty$$

Step 4: we claim that

$$\limsup_{n\to\infty}\langle(I-f)p,j(p-x_n)\rangle\leq 0$$

Since x_n is bounded and C is a reflexive Banach space , there exists a subsequence of x_n which converges weakly to u, we assume that $x_{n_k}\to u$ and

$$\lim_{u\to\infty}\langle(I-f)p,j(p-x_{n_u})\rangle=\limsup_{n\to\infty}\langle(I-f)p,j(p-x_n)\rangle$$

Since C is a smooth Banach space, it follows from step (3) that

$$\lim_{k\to\infty}||x_{n_k}-Tx_{n_k}||=0$$

from lemma (2.6), we have $u \in F(T)$. On the other hand , since $p \in F(T)$ satisfies

$$\langle (I-f)p, j(p-u)\rangle \le 0, \forall u \in F(T)$$

by weakly sequential continuous duality mapping , we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle (I-f)p, j(p-x_n)\rangle \\
&= \lim_{k\to\infty}\langle (I-f)p, j(p-x_n)\rangle \\
&= \langle (I-f)p, j(p-x_n)\rangle \le 0
\end{aligned}$$

Step 5: Finally, we show that x_n converges strongly to $p \in F(T)$.

we set

$$\begin{aligned}
&||x_{n+1}-p||^2 \\
&= \langle \alpha_n x_n - \alpha_n\delta_n x_n + \beta_n f(x_n) + \gamma_n T^n(s_n x_n \\
&+ (1-s_n)x_{n+1}) - p, j(x_{n+1}-p)\rangle \\
&= \alpha_n\langle x_n - p, j(x_{n+1}-p)\rangle + \beta_n\langle f(x_n)-p, \\
&j(x_{n+1}-p)\rangle - \alpha_n\delta_n\langle x_n-p, j(x_{n+1}-p)\rangle \\
&+ \gamma_n\langle T^n(s_n x_n + (1-s_n)x_{n+1}) - p, j(x_{n+1}-p)\rangle \\
&\le \alpha_n\langle x_n-p, j(x_{n+1}-p)\rangle - \alpha_n\delta_n\langle x_n-p, j(x_{n+1}-p)\rangle \\
&+ \beta_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle \\
&+ \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle + \gamma_n\langle T^n(s_n x_n \\
&+ (1-s_n)x_{n+1}) - p, j(x_{n+1}-p)\rangle \\
&\le \alpha_n||x_n-p||||x_{n+1}-p|| - \alpha_n\delta_n||x_n-p||||x_{n+1}-p|| \\
&+ \beta_n\lambda||x_n-p||||x_{n+1}-p|| + \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&+ \gamma_n L||s_n x_n + (1-s_n)x_{n+1} - p||||x_{n+1}-p|| \\
&\le \alpha_n||x_n-p||||x_{n+1}-p|| - \alpha_n\delta_n||x_n-p|| \\
&||x_{n+1}-p|| + \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle + \\
&\gamma_n L s_n||x_n-p||||x_{n+1}-p|| + \gamma_n L(1-s_n)||x_{n+1}-p||^2 \\
&= [\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n]||x_n-p||||x_{n+1}-p|| \\
&+ \gamma_n L(1-s_n)||x_{n+1}-p||^2 + \\
&\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&\le \frac{\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n}{2}||x_n-p||^2 \\
&+ \frac{\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n}{2}||x_{n+1}-p||^2 \\
&+ \gamma_n L(1-s_n)||x_{n+1}-p||^2 + \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&= \frac{\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n}{2}||x_n-p||^2 \\
&+ \frac{\alpha_n + \beta_n\lambda + \gamma_n L(2-s_n)}{2}||x_{n+1}-p||^2 \\
&+ \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle
\end{aligned}$$

which implies

$$\begin{aligned}
&[1 - \frac{\alpha_n(1-\delta_n) + \beta_n\lambda + \gamma_n L(2-s_n)}{2}]||x_n-p||^2 \\
&\le \frac{\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n}{2}||x_n-p||^2 \\
&+ \beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle
\end{aligned}$$

That is

$$\begin{aligned}
&||x_{n+1}-p||^2 \\
&\le \frac{\alpha_n + \beta_n\lambda - \alpha_n\delta_n + \gamma_n L s_n}{2 - \alpha_n - \beta_n\lambda + \alpha_n\delta_n - \gamma_n L(2-s_n)}||x_{n+1}-p||^2 \\
&+ \frac{2\beta_n}{2 - \alpha_n - \beta_n\lambda + \alpha_n\delta_n - \gamma_n L(2-s_n)} \\
&\langle f(p)-p, j(x_{n+1}-p)\rangle \\
&= [1 - \frac{2(1 - \alpha_n - \beta_n\lambda + \alpha_n\delta_n - \gamma_n L)}{2 - \alpha_n - \beta_n\lambda + \alpha_n\delta_n - \gamma_n L(2-s_n)}] \\
&.||x_{n+1}-p||^2 \qquad (3.9) \\
&+ \frac{2\beta_n}{2 - \alpha_n - \beta_n\lambda + \alpha_n\delta_n - \gamma_n L(2-s_n)} \\
&\langle f(p)-p, j(x_{n+1}-p)\rangle
\end{aligned}$$

Let

$$\alpha_n = \frac{2(1 - \alpha_n - \beta_n\lambda - \alpha_n\delta_n - \gamma_n L)}{2 - \alpha_n - \beta_n\lambda - \alpha_n\delta_n - \gamma_n L(2-s_n)}$$

We have

$$\begin{aligned}
a_n &\ge 1 - \alpha_n - \beta_n\lambda - \alpha_n\delta_n - \gamma_n L \\
&= \beta_n(1-\lambda) - \gamma_n(L-1) \\
&\ge \beta_n(1-\lambda)
\end{aligned}$$

We claim that

$$\Sigma_{n=0}^{\infty}\alpha_n = \infty$$

By step (4), we get

$$\langle (I-f)p, j(p-y)\rangle \le 0 \qquad \forall\ y\ \in\ F(T)$$

Apply lemma (2.5) to (3.9),
we conclude that

$$\lim_{n\to\infty}||x_n-p|| = 0.$$

This finish the proof. □

Theorem 3.2. *Let E be a reflexive smooth Banach space with a weakly sequentially continuous duality mapping. Let J,C be a nonempty bounded and closed convex subset of E, and let $T: C \to C$ be an asymptotically nonexpansive mapping with a sequence k_n such that $F(T) \neq \phi$ and $f: C \to C$ be a contraction with coefficient $\lambda \in (0,1)$. Pick any $x_0 \in C$, let x_n be a sequence generated by*

$$x_{n+1} = \alpha_n(1-\delta_n)x_n + \beta_n f(x_n) + \gamma_n T^n(s_n x_n + (1-s_n)x_{n+1}) \qquad (3.10)$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ and $\{s_n\} \subset (0,1)$ satisfying the conditions

1. $\alpha_n + \beta_n + \gamma_n = 1, \quad \lim_{n\to\infty}\beta_n = 0,\ \gamma_n = \eta\beta_n,$
   $\lim_{n\to\infty}\delta_n = 0; \quad 0 < \eta < \dfrac{(s_{n+1}) - \lambda}{L - (s_{n+1})}$

2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \mid \alpha_{n+1} - \alpha_n \mid = 0$, $\lim_{n\to\infty} \mid \beta_{n+1} - \beta_n \mid = 0$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.
3. $\sum_{n=0}^{\infty} \beta_n = \infty$, $0 < s_n < s_{n+1} < 1$, $\sum_{n=0}^{\infty} \delta_n = \infty$, $\gamma_n(1 - s_n) < \frac{1}{L}$;
4. $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$, $x \in C'$

where C' is any bounded subset of C for all $n \geq 0$.Then $\{x_n\}$ defined by (3.10) converges strongly to a fixed point p of the asymptotically pseudocontractive mapping T, which solves the variational inequality :

$$\langle (I - f)p, j(p - y) \rangle \leq 0, \quad \forall y \in F(T)$$

Theorem 3.3. *Let E be a reflexive smooth Banach space with a weakly sequentially continuous duality mapping. Let J, C be a nonempty bounded and closed convex subset of E, and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence k_n such that $F(T) \neq \phi$ and $f : C \to C$ be a contraction with coefficient $\lambda \in (0,1)$. Pick any $x_0 \in C$, let x_n be a sequence generated by*

$$x_{n+1} = \alpha_n(1 - \delta_n)x_n + \beta_n f(x_n) + \gamma_n T^n(s_n x_n + (1 - s_n)x_{n+1}) \tag{3.11}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{s_n\} \subset (0,1)$ satisfying the conditions

1. $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \delta_n = 0$, $k_n - 1 = o(\beta_n)$;
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \mid \alpha_{n+1} - \alpha_n \mid = 0$, $\lim_{n\to\infty} \mid \beta_{n+1} - \beta_n \mid = 0$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.
3. $\sum_{n=0}^{\infty} \beta_n = \infty$, $0 < s_n < s_{n+1} < 1$, $\sum_{n=0}^{\infty} \delta_n = \infty$, $\gamma_n(1 - s_n) < \frac{1}{L}$;
4. $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0, x \in C'$

where C' is any bounded subset of C for all $n \geq 0$.Then $\{x_n\}$ defined by (3.11) converges strongly to a fixed point p of the asymptotically pseudocontractive mapping T, which solves the variational inequality :

$$\langle (I - f)p, j(p - y) \rangle \leq 0, \quad \forall y \in F(T)$$

Theorem 3.4. *Let E be a reflexive smooth Banach space with a weakly sequentially continuous duality mapping. Let J, C be a nonempty bounded and closed convex subset of E, and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence k_n such that $F(T) \neq \phi$. Pick any $x_0 \in C$, let x_n be a sequence generated by*

$$x_{n+1} = \alpha_n(1 - \delta_n)x_n + \beta_n \nu + \gamma_n T^n(s_n x_n + (1 - s_n)x_{n+1}) \tag{3.12}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{s_n\} \subset (0,1)$ satisfying the conditions

1. $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \beta_n = 0, \gamma_n = n\beta_n$; $\lim_{n\to\infty} \delta_n = 0$; $0 < \eta < \frac{(s_{n+1}) - \lambda}{L - (s_{n+1})}$
2. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$; $\lim_{n\to\infty} \beta_n = 0$; $\lim_{n\to\infty} \mid \alpha_{n+1} - \alpha_n \mid = 0$; $\lim_{n\to\infty} \mid \beta_{n+1} - \beta_n \mid = 0$, $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$.
3. $\sum_{n=0}^{\infty} \beta_n = \infty$, $0 < s_n < s_{n+1} < 1$, $\sum_{n=0}^{\infty} \delta_n = \infty$; $\gamma_n(1 - s_n) < \frac{1}{L}$;
4. $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0, x \in C'$

where C' is any bounded subset of C for all $n \geq 0$. Then $\{x_n\}$ defined by (3.12) converges strongly to a fixed point p of the asymptotically pseudocontractive mapping T. Proof: In this case, the mapping $f : C \to C$ defined by $f(x) = \nu, \forall x \in C$ is a strict contraction with constant $\lambda = 0$. The proof follows from Theorem 2.1 above.

4. Conclusion

We therefore conclude to say that $H(\cdot,\cdot) - \phi - \eta -$accretive operator are more general to establish the convergence of explicit iterative algorithm using the resolvent operator technique in uniformly convex Banach space. Also those could be the solution of certain variation inequality problem.

Remark 4.1.

1. Since every nonexpansive mapping is asymptotically nonexpansive and an asymptotically nonexpansive mapping is both asymptotically pseudocontractive and uniformly L-Lipschitzian, in Theorem 2.1 and 2.2, if T is a nonexpansive mapping in Hilbert spaces, then it is the main results of Yu et. al[19]. Thus, Theorem 2.1 improves and extends the Yu et. al's theorem in several aspects and improves some other results (see[1,5,7,8,11,12,14])

2. We note that in Theorems 2.2 and 2.3, we can choose condition $k_n - 1 = o(\beta_n)$; replacing the requirement $\gamma_n = \eta\beta_n$; $0 < \eta < \frac{(s_{n+1}) - \lambda}{L - (s_{n+1})}$. However, the proof is similar to Theorem 2.1 above.

3. In Theorem 2.4, if $s_n = 0$, then $\{x_n\}$ converges strongly to a fixed point of T. It is the main results of Yao et.al.[21].

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