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Application of quasi-subordination for certain subclasses of bi-univalent functions of complex order

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Abstract
In this present paper, the author construct a new class $S_{\lambda,\delta}^{k,\alpha}(\gamma,t,\Psi)$ of bi-univalent functions of complex order defined in the open unit disc. The second and the third coefficients of the Taylor-Maclaurin series for functions in the new subclass are determined. Several special consequences of the results are also pointed out.

Keywords
Bi-univalent functions, coefficient bounds, subordination, quasi-subordination.

AMS Subject Classification
30C45.

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Contents

1. Introduction and Preliminaries

Let A indicate an analytic function family, which is normalized under the condition of $f(0)=f'(0)-1=0$ in $\Delta=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$ and given by the following Taylor-Maclaurin series:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0)=0, \quad |w(z)|<1 \quad (z\in\Delta),$$

such that

$$f(z)=g(w(z)) \quad (z\in\Delta).$$

We denote this subordination by

$$f\prec g \text{ or } f(z)\prec g(z) \quad (z\in\Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0)=g(0), \quad f(\Delta)\subset g(\Delta).$$

In the year 1970, Robertson [19] introduced the concept of quasi-subordination. For two analytic functions f and g, the function f is said to be quasi-subordinate to g in Δ and written as

$$f(z)\prec_q g(z) \quad (z\in\Delta),$$

if there exists an analytic function $|h(z)|\le 1$ such that $\frac{f(z)}{h(z)}$ analytic in Δ and

$$\frac{f(z)}{h(z)}\prec g(z) \quad (z\in\Delta),$$

that is, there exists a Schwarz function $w(z)$ such that $f(z) = h(z)g(w(z))$. Observe that if $h(z) = 1$, then $f(z) = g(w(z))$ so that $f(z) \prec g(z)$ in Δ. Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said that is majorized by g and written $f(z) \ll g(z)$ in Δ. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. (see, e.g. [19], [18], [14] for works related to quasi-subordination).

The Koebe-One Quarter Theorem [9] ensures that the image of Δ under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1.1). For a brief history and interesting examples in the class Σ, see [25] (see also [5], [6], [13], [16]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [11], [23] and [24]). Not much is known about the bounds on the general coefficient $|a_n|$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([4], [10], [21], [26]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\};\ \mathbb{N} = \{1,2,3,...\})$ is still an open problem.

The study of operators plays an important role in the Geometric Function Theory and its related fields. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better (see, for example [2], [3], [7], [12] and [15]). Recently, Darus and İbrahim [8] introduced a differential operator

$$D^{k,\alpha}_{\lambda,\delta} : A \to A$$

by

$$D^{k,\alpha}_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [n^\alpha + (n-1)n^\alpha \lambda]^k \begin{pmatrix} n+\delta-1 \\ \delta \end{pmatrix} a_n z^n$$

where $z \in \Delta$ and $k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda, \delta \geq 0$.

It should be remarked that the operator $D^{k,\alpha}_{\lambda,\delta}$ is a generalization of many other linear operators studied by earlier researchers. Namely:

- for $\alpha = 1,\ \lambda = 0;\ \delta = 0$ or $\alpha = \ \delta = 0;\ \lambda = 1$, the operator $D^{k,1}_{0,0} \equiv D^{k,0}_{1,0} \equiv D^k$ is the popular Salagean operator [22],
- for $\alpha = 0,\quad \delta = 0$, the operator $D^{k,0}_{\lambda,0} \equiv D^k_\lambda$ has been studied by Al-Oboudi (see [1]),
- for $\alpha = 0$, the operator $D^{k,0}_{\lambda,\delta} \equiv D^k_{\lambda,\delta}$ has been studied by Darus and İbrahim (see [8]),
- for $k = 0$, the operator $D^{k,\alpha}_{\lambda,\delta} \equiv D^\delta$ has been studied by Ruscheweyh (see [20]).

Making use of the differential operator $D^{k,\alpha}_{\lambda,\delta}$, we introduce a new class of analytic bi-univalent functions as follows:

Definition 1.1. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $S^{k,\alpha}_{\lambda,\delta}(\gamma, t, \Psi)$, if the following conditions are satisfied:*

$$\frac{1}{\gamma}\left[\frac{z\left(D^{k,\alpha}_{\lambda,\delta}f(z)\right)'}{(1-t)D^{k,\alpha}_{\lambda,\delta}f(z) + tz\left(D^{k,\alpha}_{\lambda,\delta}f(z)\right)'} - 1\right] \prec_q (\Psi(z) - 1)$$

and

$$\frac{1}{\gamma}\left[\frac{w\left(D^{k,\alpha}_{\lambda,\delta}g(w)\right)'}{(1-t)D^{k,\alpha}_{\lambda,\delta}g(w) + tw\left(D^{k,\alpha}_{\lambda,\delta}g(w)\right)'} - 1\right] \prec_q (\Psi(w) - 1),$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq t < 1;\ k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda, \delta \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

On specializing the parameters t, k, δ one can define the various new subclasses of Σ as illustrated in the following examples.

Example 1.2. *For $t = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $S^{k,\alpha}_{\lambda,\delta}(\gamma, \Psi)$, if the following conditions are satisfied:*

$$\frac{1}{\gamma}\left[\frac{z\left(D^{k,\alpha}_{\lambda,\delta}f(z)\right)'}{D^{k,\alpha}_{\lambda,\delta}f(z)} - 1\right] \prec_q (\Psi(z) - 1)$$

and

$$\frac{1}{\gamma}\left[\frac{w\left(D^{k,\alpha}_{\lambda,\delta}g(w)\right)'}{D^{k,\alpha}_{\lambda,\delta}g(w)} - 1\right] \prec_q (\Psi(w) - 1),$$

where $k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda, \delta \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

Example 1.3. *For $t = k = \delta = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $S_\Sigma(\gamma, \Psi)$, if the following conditions are satisfied:*

$$\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec_q (\Psi(z) - 1)$$

and

$$\frac{1}{\gamma}\left[\frac{wg'(w)}{g(w)} - 1\right] \prec_q (\Psi(w) - 1),$$

where $z, w \in \Delta$ and the function g is given by (1.2).

2. Main Result and its Consequences

Firstly, we will state the Lemma 2.1 to obtain our result.

Lemma 2.1. *(See [17]) If $p \in P$, then $|p_i| \leq 1$ for each i, where P is the family all functions p, analytic in Δ, for which*

$$\Re\{p(z)\} > 0,$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots.$$

Through out this paper it is assumed that Ψ is analytic in Δ with $\Psi(0) = 1$ and let

$$\Psi(z) = 1 + C_1 z + C_2 z^2 + \cdots \quad (C_1 > 0). \tag{2.1}$$

Also let

$$h(z) = D_0 + D_1 z + D_2 z^2 + \cdots \quad (|h(z)| \leq 1,\ z \in \Delta). \tag{2.2}$$

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $S_{\lambda,\delta}^{k,\alpha}(\gamma,t,\Psi)$ proposed by Definition 1.1.

Theorem 2.2. *Let f of the form (1.1) be in the class $S_{\lambda,\delta}^{k,\alpha}(\gamma,t,\Psi)$. Then*

$$\begin{aligned}|a_2| &\\ &\leq |\gamma|\,|D_0|\,C_1\sqrt{C_1}\,[(1-t)(1+\delta)]^{-\frac{1}{2}}\\ &\quad (\times)\Big|(1+\delta)\,[2^{\alpha}(1+\lambda)]^{2k}\,[(1-t)(C_1-C_2)-(1+t)\gamma C_1^2 D_0]\\ &\quad + [3^{\alpha}(1+2\lambda)]^{k}\,(2+\delta)\gamma C_1^2 D_0\Big|^{-\frac{1}{2}}\end{aligned}$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2} + \frac{|\gamma D_1| C_1}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)} + \frac{|\gamma D_0| C_1}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

Proof. If $f \in S_{\lambda,\delta}^{k,\alpha}(\gamma,t,\Psi)$ then, there are analytic functions $u, v : \Delta \to \Delta$ with $u(0) = v(0) = 0$, $|u(z)| < 1, |v(w)| < 1$ and a function h given by (2.2), such that

$$\frac{1}{\gamma}\left[\frac{z\left(D_{\lambda,\delta}^{k,\alpha}f(z)\right)'}{(1-t)D_{\lambda,\delta}^{k,\alpha}f(z)+tz\left(D_{\lambda,\delta}^{k,\alpha}f(z)\right)'} - 1\right] = h(z)\left(\Psi(w(z)) - 1\right) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left[\frac{w\left(D_{\lambda,\delta}^{k,\alpha}g(w)\right)'}{(1-t)D_{\lambda,\delta}^{k,\alpha}g(w)+tw\left(D_{\lambda,\delta}^{k,\alpha}g(w)\right)'} - 1\right] = h(z)\left(\Psi(w(z)) - 1\right). \tag{2.4}$$

Determine the functions p_1 and p_2 in P given by □

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + \cdots.$$

Thus,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right] \tag{2.5}$$

and

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right]. \tag{2.6}$$

The fact that p_1 and p_2 are analytic in Δ with $p_1(0) = p_2(0) = 1$. Since $u, v : \Delta \to \Delta$, the functions p_1, p_2 have a positive real part in Δ, and the relations $|p_i| \leq 2$ and $|q_i| \leq 2$ are true. Using (2.5) and (2.6) together with (2.1) and (2.2) in the right hands of the relations (2.3) and (2.4), we obtain

$$h(z)\left[\Psi(u(z)) - 1\right] = \tfrac{1}{2}D_0 C_1 p_1 z + \left\{\tfrac{1}{2}D_1 C_1 p_1 + \tfrac{1}{2}D_0 C_1\left(p_2 - \tfrac{p_1^2}{2}\right) + \tfrac{1}{4}D_0 C_2 p_1^2\right\}z^2 + \cdots$$

and

$$h(w)\left[\Psi(v(w)) - 1\right] = \tfrac{1}{2}D_0 C_1 q_1 w + \left\{\tfrac{1}{2}D_1 C_1 q_1 + \tfrac{1}{2}D_0 C_1\left(q_2 - \tfrac{q_1^2}{2}\right) + \tfrac{1}{4}D_0 C_2 q_1^2\right\}w^2 + \cdots.$$

In the light of (2.3) and (2.4), we get

$$\frac{(1-t)[2^{\alpha}(1+\lambda)]^{k}(1+\delta)}{\gamma}a_2 = \frac{D_0 C_1 p_1}{2}, \tag{2.7}$$

$$\begin{aligned}&\frac{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)a_3 - (1-t)(1+t)[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2 a_2^2}{\gamma}\\ &= \frac{D_1 C_1 p_1}{2} + \frac{D_0 C_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{D_0 C_2 p_1^2}{4}\end{aligned} \tag{2.8}$$

and

$$-\frac{(1-t)[2^{\alpha}(1+\lambda)]^{k}(1+\delta)}{\gamma}a_2 = \frac{D_0 C_1 q_1}{2}, \tag{2.9}$$

$$\begin{aligned}&\frac{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)(2a_2^2 - a_3) - (1-t)(1+t)[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2 a_2^2}{\gamma}\\ &= \frac{D_1 C_1 q_1}{2} + \frac{D_0 C_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{D_0 C_2 q_1^2}{4}.\end{aligned}$$

$$(2.10)$$

Now, (2.7) and (2.9) give

$$p_1 = -q_1 \tag{2.11}$$

and

$$8(1-t)^2\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2 a_2^2 = \gamma^2 D_0^2 C_1^2\left(p_1^2+q_1^2\right). \tag{2.12}$$

Adding (2.8) and (2.10), we get

$$\frac{2(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)-2(1-t)(1+t)[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}{\gamma}a_2^2$$

$$= \frac{D_0C_1(p_2+q_2)}{2}+\frac{D_0(C_2-C_1)\left(p_1^2+q_1^2\right)}{4}. \tag{2.13}$$

By using (2.11), (2.12) and Lemma 2.1 in (2.13), we obtain the desired result.

Next, to find the bound on $|a_3|$, by using subtracting (2.10) and (2.8), we have

$$\frac{2(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}{\gamma}\left(a_3-a_2^2\right) = \frac{D_0C_1(p_2-q_2)}{2}+\frac{D_1C_1(p_1-q_1)}{2}. \tag{2.14}$$

It follows from (2.11), (2.12) and (2.14) that

$$a_3 = \frac{\gamma^2D_0^2C_1^2\left(p_1^2+q_1^2\right)}{8(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{\gamma D_1C_1(p_1-q_1)}{4(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$

$$+\frac{\gamma D_0C_1(p_2-q_2)}{4(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \le \frac{|\gamma D_0|^2C_1^2}{(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{|\gamma D_1|C_1}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$

$$+\frac{|\gamma D_0|C_1}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

This completes the proof of Theorem 2.1.

Putting $t=0$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let f of the form (1.1) be in the class* $S_{\lambda,\delta}^{k,\alpha}(\gamma,\Psi)$.

$$|a_2| \le \frac{|\gamma||D_0|C_1\sqrt{C_1}}{\sqrt{(1+\delta)\left|(1+\delta)[2^{\alpha}(1+\lambda)]^{2k}\left[(C_1-C_2)-\gamma C_1^2D_0\right]+[3^{\alpha}(1+2\lambda)]^{k}(2+\delta)\gamma C_1^2D_0\right|}}$$

and

$$|a_3| \le \frac{|\gamma D_0|^2C_1^2}{[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{|\gamma D_1|C_1}{[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$

$$+\frac{|\gamma D_0|C_1}{[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

Corollary 2.4. *Let f of the form (1.1) be in the class* $S_{\Sigma}(\gamma,\Psi)$. *Then*

$$|a_2| \le \frac{|\gamma D_0|C_1\sqrt{C_1}}{\sqrt{\left|C_1-C_2+\gamma C_1^2D_0\right|}}$$

and

$$|a_3| \le |\gamma D_0|^2C_1^2+\frac{(|D_1|+|D_0|)|\gamma|C_1}{2}.$$

For the function Ψ is given by

$$\Psi(z)=\left(\frac{1+z}{1-z}\right)^{\xi}=1+2\xi z+2\xi^2z^2+\cdots\ (0<\xi\le 1)$$

which gives

$$C_1=2\xi,\ \ C_2=2\xi^2,$$

Theorem 2.1 reduces to:

Corollary 2.5. *Let* $f\in S_{\lambda,\delta}^{k,\alpha}\left[\gamma,t,\left(\frac{1+z}{1-z}\right)^{\xi}\right]$. *Then*

$$|a_2| \le 2|\gamma||D_0|\xi\left[(1-t)(1+\delta)\right]^{-\frac{1}{2}}\Big|(1+\delta)[2^{\alpha}(1+\lambda)]^{2k}[(1-t)(1-\xi)$$

$$-2(1+t)\gamma\xi D_0]+2[3^{\alpha}(1+2\lambda)]^{k}(2+\delta)\gamma\xi D_0\Big|^{-\frac{1}{2}}$$

and

$$|a_3| \le \frac{4|\gamma D_0|^2\xi^2}{(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{2|\gamma D_1|\xi}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$

$$+\frac{2|\gamma D_0|\xi}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

If we set

$$\Psi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+\cdots\ (-1\le B\le A<1)$$

which gives

$$C_1=(A-B),\ \ C_2=-B(A-B),$$

Theorem 2.1 reduces to:

Corollary 2.6. *Let* $f\in S_{\lambda,\delta}^{k,\alpha}\left[\gamma,t,\frac{1+Az}{1+Bz}\right]$. *Then*

$$|a_2| \le |\gamma||D_0|(A-B)\left[(1-t)(1+\delta)\right]^{-\frac{1}{2}}$$

$$(\times)\Big|(1+\delta)[2^{\alpha}(1+\lambda)]^{2k}[(1-t)(1+B)-(1+t)\gamma(A-B)D_0]$$

$$+[3^{\alpha}(1+2\lambda)]^{k}(2+\delta)\gamma(A-B)D_0\Big|^{-\frac{1}{2}}$$

and

$$|a_3| \le \frac{|\gamma D_0|^2(A-B)^2}{(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{|\gamma D_1|(A-B)}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$

$$+\frac{|\gamma D_0|(A-B)}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

Finally, if we set

$$\Psi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots \quad (0<\xi\le 1)$$

which gives

$$C_1=C_2=2(1-\beta),$$

Theorem 2.1 reduces to:

Corollary 2.7. *Let* $f\in S^{k,\alpha}_{\lambda,\delta}\left[\gamma,t,\frac{1+(1-2\beta)z}{1-z}\right]$. *Then*

$$|a_2|\le|\gamma|\,|D_0|\,(A-B)\left[(1-t)(1+\delta)\right]^{-\frac{1}{2}}$$
$$\times\left|\left[3^{\alpha}(1+2\lambda)\right]^{k}(2+\delta)\gamma D_0-(1+\delta)\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+t)\gamma D_0\right|^{-\frac{1}{2}}$$

and

$$|a_3| \le \frac{4|\gamma D_0|^2(1-\beta)^2}{(1-t)^2[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2}+\frac{2|\gamma D_1|(1-\beta)}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}$$
$$+\frac{2|\gamma D_0|(1-\beta)}{(1-t)[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}.$$

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