



Approximate controllability of multi-term time-fractional stochastic differential inclusions with nonlocal conditions

Ashish Kumar^{1*} and Dwijendra N Pandey²

Abstract

In this paper, approximate controllability results are discussed for a class of multi-term time-fractional inclusion differential systems with state-dependent delay. A set of sufficient conditions for the set of non-local non-linear multi-term differential inclusion system has been discussed. An example is also given to verify the derived result.

Keywords

Fractional calculus, approximate controllability, multi-term time-fractional delay differential system, (β, γ_j) -resolvent family, fixed point theorems, differential inclusions, state-dependent delay.

AMS Subject Classification

34A08, 34G20, 34K30, 47H10, 93B05, 93E03.

^{1,2}Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India.

*Corresponding author: ¹ akumar2@ma.iitr.ac.in; ² dwij.iitk@gmail.com

Article History: Received 24 June 2019; Accepted 18 September 2019

©2019 MJM.

Contents

1	Introduction	687
2	Preliminaries	688
3	Main Result	690
4	Example	698
5	Conclusion	698
	References	698

1. Introduction

Fractional calculus is the study of integrals and derivatives of arbitrary order (both real and complex). Although it was introduced to the end of the seventeenth century, the main contributions have been made during the last few decades due to its applications in fluid mechanics, electromagnetics, biological population models, optics and signals processing. There are several definitions of fractional derivatives such as Hadamard derivative, Grunwald Letnikov derivative, Riemann Liouville fractional derivative, Caputo fractional derivative etc. For the basics of fractional calculus, one can refer to the monographs [10, 15, 21, 22] and for recent developments in this field, we can make references to the papers [2, 6, 11, 20, 32] and references cited therein.

The concept of controllability plays a vital role in the control theory and engineering. There are two basic concepts of controllability namely, exact and approximate controllability which are equivalent in finite-dimensional systems and different in the case of infinite-dimensional systems. Klamka [16] discussed the controllability of linear systems in finite-dimensional spaces and controllability of fractional evolution dynamical systems in a finite-dimensional space are well established in [4, 31]. The concept of approximate controllability of the several types of nonlinear systems under different conditions is discussed in [5, 19] and reference therein.

Stochastic differential equations are widely used in physics, biology, chemistry, probability theory, mathematical finance, ecology, neuroscience, image processing, signal processing, information theory, computer science, cryptography and telecommunications etc. and these equations can be solved analytically as well as numerically. Mathematically, it can be viewed as a generalization of the dynamical systems theory to models with noise. For fundamentals and recent development of stochastic differential equations, follow the articles [3, 7, 13, 19, 23, 25, 29] and the references therein.

In recent years multi-term time-fractional differential equations have been a fruitful field of research in mathematics and engineering. For instance, the time-fractional telegraph equation has been studied in the article [14]. Pardo et al. [2] studied

the existence of mild solutions of multi-term differential systems via the method of measure of noncompactness. Singh et al. [27, 28] investigated the existence and controllability results for multi-term time-fractional differential equations.

Mahmudov et al. [18] investigated the controllability of linear stochastic systems in finite-dimensional space. Balasubramanian et al. [5] established sufficient conditions for the approximate controllability of neutral stochastic functional differential systems with infinite delay in Hilbert spaces. Kumar et al. [17] investigated the approximate controllability of fractional order semilinear systems with bounded delay. Vijayakumar [30] explored the approximate controllability results for the inclusion differential systems with infinite delay in Hilbert spaces. However, to the best of our knowledge, there are no results on the approximate controllability of multi-term time-fractional differential stochastic differential inclusions as treated in the current paper.

So, in this paper, we study the controllability results for the following class of multi-term time-fractional differential inclusions

$${}^c D^{1+\beta} y(t) + \sum_{j=1}^n \alpha_j {}^c D^{\gamma_j} y(t) \in Ay(t) + Bu(t) + F(t, y(t)) + G(t, y(t)) \frac{dw(t)}{dt}, t \in \mathcal{I}, \tag{1.1}$$

$$y(0) + h(y) = \sigma, \quad y'(0) = \chi, \tag{1.2}$$

where $\gamma_j \in \mathbb{R}^+, \forall j = 1, 2, \dots, n$ and $0 < \beta \leq \gamma_n \leq \dots \leq \gamma_1 \leq 1$ and $\frac{1}{2} + \beta > \gamma_j$, ${}^c D^\eta$ stands for the Caputo fractional derivative of order η for $\eta > 0$, and A is a closed linear operator on a separable Hilbert space \mathbb{H} with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Assume that \mathbb{K} is another separable Hilbert space with norm $\|\cdot\|_{\mathbb{K}}$ and inner product $\langle \cdot, \cdot \rangle_{\mathbb{K}}$. Let $w(t)$ be a given Wiener process or Brownian motion with finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The control function $u(\cdot)$ belongs to the space $L^2_{\mathcal{F}}(\mathcal{I}, \mathbb{U})$, \mathbb{U} is a Hilbert space and $B : \mathbb{U} \rightarrow \mathbb{H}$ is a bounded linear operator. We denote the space of bounded and linear operators from \mathbb{K} into \mathbb{H} , by $\mathcal{L}(\mathbb{K}, \mathbb{H})$. The random variable $\chi \in \mathbb{H}$ with $E\|\chi\|^2 < \infty$. The functions F and G are multivalued and h is a nonlinear function.

In section 2, we have presented some basic notations and preliminaries. Section 3 is devoted to the approximate controllability for multi-term time-fractional stochastic inclusion systems. In the final section, an example is given to verify the theoretical result.

2. Preliminaries

Let \mathbb{N} and \mathbb{R} be the usual notations of natural and real numbers, respectively. For a linear operator A on \mathbb{H} , let $\mathcal{D}(A)$, $\mathcal{R}(A)$ and $\rho(A)$ be the notations for the domain, range and resolvent of A , respectively. Let $w = \{w(t)\}_{t \geq 0}$ represent a Q -Wiener process on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the

usual conditions (i.e right continuous and $\{\mathcal{F}_0\}$ containing all P -null sets) and the linear bounded covariance operator Q such that $tr(Q) < \infty$, where $tr(Q)$ denotes the trace of Q . Further, we assume that there exists an orthonormal system $\{e_n\}_{n \geq 1}$ which is complete in \mathbb{K} , a sequence $\{\zeta_n\}_{n \geq 1}$ of independent Brownian motions such that $\langle w(t), e \rangle_{\mathbb{K}} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e \rangle \zeta_n(t)$, $e \in \mathbb{K}, t \geq 0$ and a sequence of non-negative real numbers $\{\lambda_n\}_{n \geq 1}$ such that $Qe_n = \lambda_n e_n, n = 1, 2, 3, \dots$. We assume that $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}} \mathbb{K}, \mathbb{H})$ represents the space of all Hilbert Schmidt operators from $Q^{\frac{1}{2}} \mathbb{K}$ to \mathbb{H} with inner product $\langle \phi, \varphi \rangle = tr[\phi Q \varphi^*] < \infty$. Let $\mathcal{L}_2(\Omega, \mathcal{F}_T, \mathbb{H})$ be the Banach space of all \mathcal{F}_T measurable \mathbb{H} valued square integrable random variables. Moreover, let $\mathcal{L}_2^{\mathcal{F}}(\mathcal{I}, \mathbb{H})$ be the Hilbert space of all square integrable and \mathcal{F}_t adapted processes with value in \mathbb{H} . We denote the Banach space of all continuous functions $y : [0, T] \rightarrow \mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{H})$, by \mathcal{C} which satisfy $\sup_{t \in [0, T]} E\|y(t)\|^2 < \infty$.

Now, let

$\mathcal{P}(\mathbb{H})$ denote the power set of \mathbb{H} ,

$\mathcal{P}_{cl}(\mathbb{H})$ denote all the closed subsets of \mathbb{H} ,

$\mathcal{P}_{bd}(\mathbb{H})$ denote all the bounded subsets of \mathbb{H} ,

$\mathcal{P}_{cv}(\mathbb{H})$ denote all the convex subsets of \mathbb{H} ,

$\mathcal{P}_{cp}(\mathbb{H})$ denote all the compact subsets of \mathbb{H} .

A multivalued map $f : \mathbb{H} \rightarrow \mathcal{P}(\mathbb{H})$ is said to be convex valued if $f(y)$ is convex for all $y \in \mathbb{H}$ and is said to be closed if $f(y)$ is closed for all $y \in \mathbb{H}$. Multivalued function f is said to be bounded on bounded sets if $f(C) = \cup_{y \in C} f(y)$ is bounded in \mathbb{H} i.e. $\sup_{y \in C} \{\sup\{\|x\| : x \in f(y)\}\} < \infty$.

Definition 2.1. [31] For all bounded subsets C of \mathbb{H} , if $f(C)$ is relatively compact, then the map f is called completely continuous.

Definition 2.2. [31] A map f is called upper semicontinuous (u.s.c.) on \mathbb{H} if for each $y_0 \in \mathbb{H}$, the set $f(y_0)$ is a nonempty closed subset of \mathbb{H} , and if for each open subset C of \mathbb{H} containing $f(y_0)$, there exists an open neighborhood U of y_0 such that $f(U) \subseteq C$.

If the multivalued map f is completely continuous with non empty compact values, then f is u.s.c. if and only if f has a closed graph i.e. $x^n \rightarrow x_0, y^n \rightarrow y_0, y^n \in f(x_0)$, implies $y_0 \in f(x_0)$. We call that f admits a fixed point if there is a $y \in \mathbb{H}$ such that $y \in f(y)$.

Definition 2.3. [24] A multivalued map $f : \mathcal{I} \times \mathbb{H} \rightarrow \mathcal{P}_{bd,cl,cv}(\mathbb{H})$ is called L^2 -Caratheodory if

- (i) $t \mapsto f(t, y)$ is measurable for each $y \in \mathbb{H}$.
- (ii) $y \mapsto f(t, y)$ is u.s.c. for almost all $t \in \mathcal{I}$.



(iii) for each $r > 0$, there exists $j_r \in L^1(\mathcal{I}, \mathbb{R}^+)$ such that

$$\|f(t, y)\|^2 = \sup_{h \in f(t, y)} E\|h\|^2 \leq j_r(t),$$

for all $\|y\|_{\mathcal{C}}^2 \leq r$ and for a.e. $t \in \mathcal{I}$.

Lemma 2.4. [24] Let \mathbb{H} be a Hilbert space and \mathcal{I} be a compact real interval. Assume that F be a L^2 -Caratheodory multivalued map and for each $y \in \mathcal{C}$ the set $S_{F, y} = \{f \in L^2(L(\mathbb{K}, \mathbb{H})) : f(t) \in F(t, y(t)), \text{ for all } t \in \mathcal{I}\}$ is nonempty. Let Φ be a continuous mapping from $L^2(\mathcal{I}, \mathbb{H})$ to $C(\mathcal{I}, \mathbb{H})$, then the operator

$$\begin{aligned} \Phi \circ S_F : C(\mathcal{I}, \mathbb{H}) &\rightarrow \mathcal{P}_{cv, cp}(C(\mathcal{I}, \mathbb{H})), \\ y &\mapsto (\Phi \circ S_F)(y) = \Phi(S_{F, y}), \end{aligned}$$

is a closed graph operator in $C(\mathcal{I}, \mathbb{H}) \times C(\mathcal{I}, \mathbb{H})$.

For detailed study of multivalued maps and its properties, see [12].

Definition 2.5. [2] Let A be a closed linear operator on a Hilbert space \mathbb{H} with the domain $\mathcal{D}(A)$ and let $\beta > 0, \gamma_j, \alpha_j \in \mathcal{N}$ for $j = 1, 2, \dots, n$. If there exists $\omega \geq 0$ and a strongly continuous function $\mathcal{S}_{\beta, \gamma_j} : \mathbb{R}^+ \rightarrow \mathcal{L}(\mathbb{H})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} : \text{Re } \lambda > \omega\} \subset \rho(A)$ and

$$\begin{aligned} \lambda^\beta \left(\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} - A \right)^{-1} y = \\ \int_0^\infty e^{-\lambda t} \mathcal{S}_{\beta, \gamma_j}(t) y dt, \quad \text{Re } \lambda > \omega, y \in \mathbb{H}. \end{aligned} \quad (2.1)$$

Then A is called the generator of (β, γ_j) -resolvent family.

Theorem 2.6. [2] Let $0 < \beta \leq \gamma_n \leq \dots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0, j = 1, 2, \dots, n$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t \in \mathbb{R}}$. Then, A generates a bounded (β, γ_j) -resolvent family $\{\mathcal{S}_{\beta, \gamma_j}(t)\}_{t \geq 0}$.

Now, we recall few basic definitions of fractional calculus (for detailed study, see [2, 22]). Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0; \\ 0, & t \leq 0, \end{cases}$$

where Γ stands for gamma function. The function g_η satisfies $(g_a * g_b)(t) = g_{a+b}(t)$, for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and $\text{Re } \lambda > 0$, where $\widehat{(\cdot)}$ denotes the Laplace transformation and $(\cdot * \cdot)(\cdot)$ denotes the convolution.

Definition 2.7. For a function $f \in L^1_{loc}([0, \infty), \mathbb{R})$, the Riemann-Liouville fractional integral of order $\eta > 0$ is

$$\begin{aligned} I^\eta f(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s) ds \\ &= \int_0^t g_\eta(t-s) f(s) ds, \quad t > 0, \end{aligned}$$

and $I^0 f(t) := f(t)$.

The integral defined above satisfies $I^\eta \circ I^b = I^{\eta+b}$ for $b > 0, I^\eta f(t) = (g_\eta * f)(t)$ and $\widehat{I^\eta f}(\lambda) = \frac{1}{\lambda^\eta} \widehat{f}(\lambda)$ for $\text{Re } \lambda > 0$.

Definition 2.8. Caputo fractional derivative of order $\eta > 0$ for a function $f \in \mathcal{C}^m([0, \infty), \mathbb{R})$, where $m = \lceil \eta \rceil$ is given by

$${}^c D^\eta f(t) = I^{m-\eta} D^m f(t) = \int_0^t g_{m-\eta}(t-s) D^m f(s) ds,$$

and ${}^c D^0 f(t) := f(t)$, where $D^m = \frac{d^m}{dt^m}$. In addition, we have ${}^c D^\eta f(t) = (g_{m-\eta} * D^m f)(t)$.

Remark 2.9. If $f^{(i)}(0) = 0$, for $i = 1, 2, 3, \dots, n-1$, then $(I^\eta \circ {}^c D^\eta) f(t) = f(t)$ and ${}^c D^\eta f(t) = \lambda^\eta \widehat{f}(\lambda)$.

Note. In definitions of Riemann-Liouville fractional integral and Caputo fractional derivative, we have considered zero as a lower limit of the integral.

Definition 2.10. [28] Let $0 < \beta \leq \gamma_n \leq \dots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be the generator of a bounded (β, γ_j) -resolvent family $\{\mathcal{S}_{\beta, \gamma_j}(t)\}_{t \geq 0}$. Then a stochastic process $y(t) \in \mathcal{C}$ is called the mild solution of the system (1.1) – (1.2) if $h, \sigma, \chi \in \mathcal{L}^2_0(\Omega, \mathcal{C})$ and there exists $f(t) \in F(t, y(t))$ and $g(t) \in G(t, y(t))$ such that

$$y(t) = \begin{cases} \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y)) + (g_1 * \mathcal{S}_{\beta, \gamma_j})(t) \chi \\ + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y)) ds \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) B u(s) ds \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) f(s) ds \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) g(s) dw(s). \end{cases}$$

where $\mathcal{T}_{\beta, \gamma_j}(t) = \int_0^t \frac{1}{\Gamma(\beta)} (t-s)^{\beta-1} \mathcal{S}_{\beta, \gamma_j}(s) ds$.

Remark 2.11. $\mathcal{T}_{1, \gamma_j} = g_1 * \mathcal{S}_{\beta, \gamma_j}$.

We denote by $y_T(\sigma, \chi, u)$ the state value of system (1.1) – (1.2) corresponding to the control u and initial values σ and χ at the terminal time T . The set $\mathfrak{R}(T, \sigma, \chi) := \{y_T(\sigma, \chi, u) : u(\cdot) \in L^2_{\mathcal{F}}(\mathcal{I}, \mathbb{U})\}$ is called the reachable set of the system (1.1) – (1.2).

Definition 2.12. The system (1.1) – (1.2) is said to be approximately controllable on the time interval \mathcal{I} if $\mathfrak{R}(T, \sigma, \chi) = \mathbb{H}$ where $\mathfrak{R}(T, \sigma, \chi)$ is the closure of $\mathfrak{R}(T, \sigma, \chi)$ in \mathbb{H} .

In order to establish the approximately controllability results for the system (1.1) – (1.2), first we consider approximate controllability of its linear part

$${}^c D^{1+\beta} y(t) + \sum_{j=1}^n \alpha_j {}^c D^{\gamma_j} y(t) \in A y(t) + B u(t) \quad (2.2)$$

$$y(0) + h(y) = \sigma, \quad y'(0) = \chi. \quad (2.3)$$



For this event, we are required to introduce operator associated with (2.2) – (2.3) as

$$\Gamma_\tau^T = \int_\tau^T \mathcal{T}_{\beta,\gamma_j}(T-s)BB^* \mathcal{T}_{\beta,\gamma_j}^*(T-s)ds, \quad (2.4)$$

where B^* and $\mathcal{T}_{\beta,\gamma_j}^*(t)$ represent the adjoint of B and $\mathcal{T}_{\beta,\gamma_j}(t)$, respectively. It is straight forward that the operator Γ_τ^T is a linear operator. Let $\mathcal{R}(\rho, \Gamma_\tau^T) = (\rho I + \Gamma_\tau^T)^{-1}$.

Lemma 2.13. [19] *The linear system (2.2) – (2.3) is approximately controllable in $[\tau, T]$, $0 \leq \tau < T$ if and only if $\rho(\rho I + \Gamma_\tau^T)^{-1} \rightarrow 0$ as $\rho \rightarrow 0^+$ in strongly operator topology.*

Now, we introduce the fixed point theorems, which are to be used in obtaining the main results.

Theorem 2.14. [9] *Let \mathbb{H} be a Hilbert space containing open ball $\mathbb{B}(0, r)$ and closed ball $\mathbb{B}[0, r]$ centered at origin and of radius r . Assume that $\Phi : \mathbb{B}[0, r] \rightarrow \mathcal{P}_{cp,cv}(\mathbb{H})$ is an u.s.c. and completely continuous. Then, either*

- (i) Φ has a fixed point, or
- (ii) there exists a $y \in \mathbb{H}$ with $\|y\| = r$ such that $\theta y \in \Phi(y)$ for some $\theta > 1$.

Theorem 2.15. [1] *For a nonempty, open and convex subset \mathbb{V} of Hilbert space \mathbb{H} with $0 \in \mathbb{V}$. If $\Psi : \mathbb{V} \rightarrow \mathcal{P}_{cp,cv}(\mathbb{H})$ is an u.s.c. and completely continuous. Then, either*

- (i) Ψ has a fixed point, or
- (ii) there exists a $y \in \partial\mathbb{V}$ and $\vartheta \in (0, 1)$ with $y = \vartheta\Phi(y)$.

3. Main Result

In this section, under the following assumptions (A₁) – (A₅), we first prove the existence of mild solution for the system (1.1) – (1.2). Secondly, by assuming the approximate controllability of linear system (2.2) – (2.3) and boundedness of nonlinear functions F and G in their respective domains, we show the approximate controllability of the system (1.1) – (1.2). So, In order to obtain our results, we consider the following assumptions:

- (A₁) The operators $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t \geq 0}$ and $\{\mathcal{T}_{\beta,\gamma_j}(t)\}_{t \geq 0}$ are compact, and $\sup_{t \in \mathcal{I}} \|\mathcal{S}_{\beta,\gamma_j}(t)\| = S_0, S_0 > 0$.
- (A₂) The nonlinear function $h : \mathcal{C} \rightarrow \mathbb{H}$ is completely continuous and there exist constants $\eta_1, \eta_2 > 0$ such that

$$\|h(y)\| \leq \eta_1 \|y\|_{\mathcal{C}}^2 + \eta_2, \quad \text{for } y \in \mathcal{C}.$$

- (A₃) The multivalued map $F : \mathcal{I} \times \mathbb{H} \rightarrow \mathcal{P}_{bd,cl,cv}(\mathbb{H})$ is a L^2 –Caratheodory function satisfies the following conditions

- (i) For $y \in \mathbb{H}$ the function $F(\cdot, y)$ is measurable, and for each $t \in \mathcal{I}$ the function $F(t, \cdot)$ is u.s.c.. Moreover, the set $S_{F,y} =$

$$\{f \in L^2(\Omega, \mathbb{H}) : f(t) \in F(t, y), \text{ for a.e. } t \in \mathcal{I}\}$$

is nonempty for each fixed $y \in \mathcal{C}$.

- (ii) For each $r_0 > 0$, there exists a function $N_f(r_0) > 0$ depending on r_0 such that

$$\sup_{E\|y\|^2 \leq r_0} \|F(t, y)\|^2 \leq N_f(r_0), \quad \text{for a.e. } t \in \mathcal{I},$$

$$\text{where } \|F(t, y)\|^2 = \sup_{f \in F(t, y)} E\|f\|^2.$$

- (A₄) The multivalued map $G : \mathcal{I} \times \mathbb{H} \rightarrow \mathcal{P}_{bd,cl,cv}(\mathcal{L}(\mathbb{K}, \mathbb{H}))$ is a L^2 –Caratheodory function satisfies the following conditions

- (i) For $y \in \mathbb{H}$ the function $G(\cdot, y)$ is measurable, and for each $t \in \mathcal{I}$ the function $f(t, \cdot)$ is u.s.c.. Moreover, the set $S_{G,y} =$

$$\{g \in L^2(\mathcal{L}(\mathbb{K}, \mathbb{H})) : g(t) \in G(t, y), \text{ for a.e., } t \in \mathcal{I}\}$$

is nonempty for each fixed $y \in \mathcal{C}$.

- (ii) For each $r_0 > 0$, there exists a function $N_g(r_0) > 0$ depending on r_0 such that

$$\sup_{E\|y\|^2 \leq r_0} \|G(t, y)\|^2 \leq N_g(r_0), \quad \text{for a.e. } t \in \mathcal{I},$$

$$\text{where } \|G(t, y)\|^2 = \sup_{g \in G(t, y)} E\|g\|^2.$$

- (A₅) There exists a real number $k_0 > 0$ such that $k_0 >$

$$k_0 > \frac{1}{1-L_2} \left\{ \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \left(12E\|\tilde{y}_T\|^2 + 12 \int_0^T E\|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \right) + \left(1 + \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \right) L_1 \right\} \quad (3.1)$$



where

$$\begin{aligned}
 L_1 &= 12S_0^2(E\|\sigma\|^2 + \eta_2) + 12n \\
 &\times \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\
 &\times (E\|\sigma\|^2 + \eta_2) \\
 &+ 6TS_0^2 E\|\chi\|^2 + \frac{6S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_f(k_0) \\
 &+ \frac{6Tr(Q)S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_g(k_0), \\
 L_2 &= \left(1 + \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \right) \\
 &\times \left(12S_0^2 \eta_1 + 12n \right. \\
 &\left. \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j} \eta_1}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \right).
 \end{aligned}$$

$$\Upsilon(y) = \begin{cases} w \in \mathcal{C} : w(t) = \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y)) \\ + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s) \\ \times (\sigma - h(y)) ds \\ + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) B u_y^p(s) ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) f(s) ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) g(s) dw(s), \\ f \in S_{F,y}, g \in S_{G,y}. \end{cases}$$

Further, in several steps, we will show that Υ has a fixed point.

Step 1 : For each $y \in \mathcal{C}$ and $\rho > 0$, the operator $\Upsilon(y)$ is convex.

In fact, for $t \in \mathcal{I}$ and $w_1, w_2 \in \Upsilon(y)$, then there exist $f_1, f_2 \in S_{F,y}$ and $g_1, g_2 \in S_{G,y}$ such that

Lemma 3.1. [19] For any $\tilde{y}_T \in L^2(\mathcal{F}_T, \mathbb{H})$ there exists $\tilde{\phi} \in L^2_{\mathcal{F}}(L^2([0, T], \mathcal{L}^0))$ such that $\tilde{y}_T = E\tilde{y}_T + \int_0^T \tilde{\phi}(s) dw(s)$.

Now for any $\rho > 0$ and $\tilde{y}_T \in L^2(\mathcal{F}_T, \mathbb{H})$, we define the control function

$$u_y^{\rho}(\xi) = \begin{cases} B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \left[(\rho I + \Gamma_0^T)^{-1} \left(E\tilde{y}_T \right. \right. \\ \left. \left. - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y)) - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \right. \\ \left. \left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s) \right. \right. \\ \left. \left. \times (\sigma - h(y)) ds \right) \right. \\ \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) dw(s) \right] \\ - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \\ \times \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{T}_{\beta,\gamma_j}(T-s) f(s) ds \\ - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \\ \times \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{T}_{\beta,\gamma_j}(T-s) g(s) dw(s). \end{cases}$$

$$w_i(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y)) + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \\ \times \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y)) ds + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-\xi) B \left[B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \right. \\ \times \left\{ (\rho I + \Gamma_0^T)^{-1} \right. \\ \times \left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y)) \right. \\ \left. \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \right. \\ \left. \left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s) \right. \right. \\ \left. \left. \times (\sigma - h(y)) ds \right) \right. \\ \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) dw(s) \right\} \\ - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \\ \times \mathcal{T}_{\beta,\gamma_j}(T-s) f_i(s) ds \\ - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \\ \times \mathcal{T}_{\beta,\gamma_j}(T-s) g_i(s) dw(s) \Big] d\xi \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) f_i(s) ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) g_i(s) dw(s), \quad i = 1, 2. \end{cases}$$

Theorem 3.2. Assume that the assumptions $(A_1) - (A_5)$ are satisfied. Then, the multi-term time-fractional stochastic system (1.1) – (1.2) has a mild solution on \mathcal{I} .

Proof. For $\rho > 0$, we define a multivalued operator $\Upsilon : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ by

Let $0 \leq \lambda \leq 1$, then for each $t \in \mathcal{I}$, we have



For $t \in \mathcal{I}$, we have

$$\begin{aligned} \lambda w_1(t) + (1 - \lambda)w_2(t) &= \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y)) \\ &+ \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y))ds \\ &+ (g_1 * \mathcal{S}_{\beta, \gamma_j})(t)\chi + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-\xi)B \\ &\times \left[B^* \mathcal{T}_{\beta, \gamma_j}^*(T-\xi) \right. \\ &\left. \left\{ (\rho I + \Gamma_0^T)^{-1} \left(E\bar{y}_T - \mathcal{S}_{\beta, \gamma_j}(T)(\sigma - h(y)) \right. \right. \right. \\ &- (g_1 * \mathcal{S}_{\beta, \gamma_j})(T)\chi \\ &- \left. \left. \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y))ds \right) \right. \right. \\ &\left. \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s)dw(s) \right\} - B^* \mathcal{T}_{\beta, \gamma_j}^*(T-\xi) \right. \\ &\times \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{T}_{\beta, \gamma_j}(T-s) \\ &\times [\lambda f_1(s) + (1-\lambda)f_2(s)]ds \\ &- B^* \mathcal{T}_{\beta, \gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{T}_{\beta, \gamma_j}(T-s) \\ &\times [\lambda g_1(s) + (1-\lambda)g_2(s)]dw(s) \Big] d\xi \\ &+ \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]ds \\ &+ \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s)[\lambda g_1(s) + (1-\lambda)g_2(s)]dw(s). \end{aligned}$$

By convexity of $S_{F,y}$ and $S_{G,y}$ we obtain that $\lambda f_1 + (1-\lambda)f_2 \in S_{F,y}$ and $\lambda g_1 + (1-\lambda)g_2 \in S_{G,y}$. Hence $\lambda w_1 + (1-\lambda)w_2 \in \Upsilon(y)$.

Step 2 : Υ maps bounded sets into bounded sets in \mathcal{C} .

For each $r_0 > 0$, we define $\mathbb{B}_{r_0} = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}}^2 \leq r_0, t \in \mathcal{I}\}$. Obviously, \mathbb{B}_{r_0} is bounded, closed and convex subset of \mathcal{C} . Indeed, it is enough to show that for each $w \in \Upsilon(y)$, $y \in \mathbb{B}_{r_0}$, there exists a constant $l_0 > 0$ such that $E\|y\|^2 \leq l_0$. Let $w \in \Upsilon(y)$ and $y \in \mathbb{B}_{r_0}$. Then for each $t \in \mathcal{I}$, there exists $f \in S_{F,y}$ and $g \in S_{G,y}$ such that

$$w(t) = \begin{pmatrix} \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y)) + \sum_{j=1}^n \alpha_j \\ \times \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y))ds \\ + (g_1 * \mathcal{S}_{\beta, \gamma_j})(t)\chi \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s)Bu_y^p(s)ds \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s)f(s)ds \\ + \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s)g(s)dw(s). \end{pmatrix}$$

$$\begin{aligned} E\|u_y^p(\xi)\|^2 &\leq \frac{S_B^2 S_0^2}{\rho^2 [\Gamma(1+\beta)]^2} (T-\xi)^{2\beta} \\ &\times \left\{ 6 \left\| E\bar{y}_T + \int_0^\xi \tilde{\phi}(s)dw(s) \right\|^2 \right. \\ &+ 6E \left\| \mathcal{S}_{\beta, \gamma_j}(T)(\sigma - h(y)) \right\|^2 \\ &+ 6E \left\| (g_1 * \mathcal{S}_{\beta, \gamma_j})(T)\chi \right\|^2 \\ &+ 6E \left\| \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right. \\ &\times \left. \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y))ds \right\|^2 \\ &+ 6E \left\| \int_0^\xi \mathcal{T}_{\beta, \gamma_j}(T-s)f(s)ds \right\|^2 \\ &\left. + 6E \left\| \int_0^\xi \mathcal{T}_{\beta, \gamma_j}(T-s)g(s)dw(s) \right\|^2 \right\} \end{aligned}$$

$$\leq \left\{ \begin{aligned} &\frac{6S_B^2 S_0^2}{\rho^2 [\Gamma(1+\beta)]^2} (T-\xi)^{2\beta} \left\{ 2E\|\bar{y}_T\|^2 \right. \\ &+ 2 \int_0^T E\|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \\ &+ 2S_0^2(E\|\sigma\|^2 + E\|h(y)\|^2) + TS_0^2 E\|\chi\|^2 \\ &+ n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ &\times (2E\|\sigma\|^2 + 2E\|h(y)\|^2) \\ &+ \frac{S_0^2}{[\Gamma(1+\beta)]^2} \int_0^T (T-s)^{2\beta} E\|f(s)\|^2 ds \\ &\left. + Tr(Q) \int_0^T E\|(g_\beta * \mathcal{S}_{\beta, \gamma_j})(T-s)g(s)\|^2 ds \right\} \end{aligned} \right.$$

$$\leq (T-\xi)^{2\beta} N_u,$$

where

$$N_u = \left\{ \begin{aligned} &\frac{6S_B^2 S_0^2}{\rho^2 [\Gamma(1+\beta)]^2} \left(2E\|\bar{y}_T\|^2 \right. \\ &+ 2 \int_0^T E\|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \\ &+ 2S_0^2(E\|\sigma\|^2 + E\|h(y)\|^2) + TS_0^2 E\|\chi\|^2 \\ &+ n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ &\times (2E\|\sigma\|^2 + 2E\|h(y)\|^2) \\ &+ \frac{S_0^2}{[\Gamma(1+\beta)]^2} \int_0^T (T-s)^{2\beta} E\|f(s)\|^2 ds \\ &\left. + Tr(Q) \int_0^T E\|(g_\beta * \mathcal{S}_{\beta, \gamma_j})(T-s)g(s)\|^2 ds \right). \end{aligned} \right.$$



Now, we have

$$E\|w(t)\|^2 \leq \begin{cases} 6E\|\mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y))\|^2 \\ +6E\left\|\sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right. \\ \times \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y))ds \Big\|^2 \\ +6E\|(g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi\|^2 \\ +6E\left\|\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)Bu_y^\rho(s)ds\right\|^2 \\ +6E\left\|\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)f(s)ds\right\|^2 \\ +6E\left\|\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)g(s)dw(s)\right\|^2 \end{cases}$$

$$\leq 12S_0^2(E\|\sigma\|^2 + \eta_1 r_0 + \eta_2) + 12n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \times (E\|\sigma\|^2 + \eta_1 r_0 + \eta_2) + 6TS_0^2 E\|\chi\|^2$$

$$+ 6 \frac{S_B^2 S_0^2 T^{4\beta+1}}{[\Gamma(1+\beta)]^2 4\beta+1} N_u + 6 \frac{S_0^2 T^{2\beta+1}}{[\Gamma(1+\beta)]^2 (2\beta+1)} N_f(r_0) + 6Tr(Q) \frac{S_0^2 T^{2\beta+1}}{[\Gamma(1+\beta)]^2 (2\beta+1)} N_g(r_0)$$

$\leq l_0$.

Thus, Υ maps bounded sets into bounded sets in \mathcal{C} .

Step 3 : Υ maps bounded sets into equicontinuous sets of \mathcal{C} .

For each $y \in \mathbb{B}_{r_0}$, $w \in \Upsilon(y)$ and for $\varepsilon > 0$, $0 < l_1 < l_2 \leq T$, we have

$$E\|w(l_2) - w(l_1)\|^2 \leq$$

$$\begin{aligned} & 14E\|\mathcal{S}_{\beta,\gamma_j}(l_2) - \mathcal{S}_{\beta,\gamma_j}(l_1)(\sigma - h(y))\|^2 \\ & + 14E\left\|\sum_{j=1}^n \alpha_j \int_0^{l_1-\varepsilon} \left[\frac{(l_2-s)^{\beta-\gamma_j} - (l_1-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\right] \right. \\ & \times \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y))ds \Big\|^2 \\ & + 14E\left\|\sum_{j=1}^n \alpha_j \int_{l_1-\varepsilon}^{l_1} \left[\frac{(l_2-s)^{\beta-\gamma_j} - (l_1-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\right] \right. \\ & \times \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y))ds \Big\|^2 \\ & + 14E\left\|\sum_{j=1}^n \alpha_j \int_{l_1}^{l_2} \frac{(l_2-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y))ds \right\|^2 \\ & + 14E\left\|\int_{l_1}^{l_2} \mathcal{S}_{\beta,\gamma_j}(s)\chi ds\right\|^2 \\ & + 14E\left\|\int_0^{l_1-\varepsilon} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]Bu_y^\rho(s)ds\right\|^2 \\ & + 14E\left\|\int_{l_1-\varepsilon}^{l_1} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]Bu_y^\rho(s)ds\right\|^2 \\ & + 14E\left\|\int_{l_1}^{l_2} \mathcal{T}_{\beta,\gamma_j}(l_2-s)Bu_y^\rho(s)ds\right\|^2 \\ & + 14E\left\|\int_0^{l_1-\varepsilon} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]f(s)ds\right\|^2 \\ & + 14E\left\|\int_{l_1-\varepsilon}^{l_1} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]f(s)ds\right\|^2 \\ & + 14E\left\|\int_{l_1}^{l_2} \mathcal{T}_{\beta,\gamma_j}(l_2-s)f(s)ds\right\|^2 \\ & + 14E\left\|\int_0^{l_1-\varepsilon} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]g(s)dw(s)\right\|^2 \end{aligned}$$

$$\begin{aligned} & + 14E\left\|\int_{l_1-\varepsilon}^{l_1} [\mathcal{T}_{\beta,\gamma_j}(l_2-s) - \mathcal{T}_{\beta,\gamma_j}(l_1-s)]g(s)dw(s)\right\|^2 \\ & + 14E\left\|\int_{l_1}^{l_2} \mathcal{T}_{\beta,\gamma_j}(l_2-s)g(s)dw(s)\right\|^2 \end{aligned}$$



$$\leq \left\{ \begin{aligned} &14E\|\mathcal{S}_{\beta,\gamma_j}(l_2) - \mathcal{S}_{\beta,\gamma_j}(l_1)\|(\sigma - h(y))\|^2 + 14nS_0^2 \\ &\times \sum_{j=1}^n \alpha_j^2 \int_0^{l_1-\varepsilon} \left[\frac{(l_2-s)^{\beta-\gamma_j} - (l_1-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right]^2 \\ &\times E\|(\sigma - h(y))\|^2 ds + 14nS_0^2 \sum_{j=1}^n \alpha_j^2 \\ &\times \int_{l_1-\varepsilon}^{l_1} \left[\frac{(l_2-s)^{\beta-\gamma_j} - (l_1-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right]^2 \\ &\times E\|(\sigma - h(y))\|^2 ds + 14nS_0^2 \sum_{j=1}^n \alpha_j^2 \\ &\times \int_{l_1}^{l_2} \left[\frac{(l_2-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right]^2 E\|(\sigma - h(y))\|^2 ds \\ &+ 14S_0^2(l_2 - l_1)E\|\chi\|^2 \\ &+ 14S_B^2 N_u T \int_0^{l_1-\varepsilon} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14S_B^2 N_u \varepsilon \int_{l_1-\varepsilon}^{l_1} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14S_B^2 N_u (l_2 - l_1) \int_{l_1}^{l_2} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s)\|^2 ds \\ &+ 14N_f(r_0)T \int_0^{l_1-\varepsilon} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14N_f(r_0)\varepsilon \int_{l_1-\varepsilon}^{l_1} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14N_f(r_0)(l_2 - l_1) \int_{l_1}^{l_2} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s)\|^2 ds \\ &+ 14Tr(Q)N_g(r_0) \int_0^{l_1-\varepsilon} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14Tr(Q)N_g(r_0) \int_{l_1-\varepsilon}^{l_1} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s) \\ &- \mathcal{T}_{\beta,\gamma_j}(l_1 - s)\|^2 ds \\ &+ 14Tr(Q)N_g(r_0) \int_{l_1}^{l_2} \|\mathcal{T}_{\beta,\gamma_j}(l_2 - s)\|^2 ds. \end{aligned} \right.$$

Moreover, compactness of $\mathcal{S}_{\beta,\gamma_j}(\cdot)$ and $\mathcal{T}_{\beta,\gamma_j}(\cdot)$ gives the continuity in uniform operator topology, by which we conclude that the right side of above inequality tends to zero as $l_2 \rightarrow l_1$. Hence Υ is equicontinuous.

Step 4 : The set $\mathbb{U}(t) = \{w(t) : w \in \Upsilon(\mathbb{B}_{r_0}), t \in \mathcal{I}\}$ is relatively compact.

Obviously, the set $\mathbb{U}(0)$ is relatively compact. For each $y \in \mathbb{B}_{r_0}$, $w \in \Upsilon(y)$ and for $t \in (0, T]$, $\varepsilon \in (0, t)$, we define

$$w_\varepsilon(t) = \left\{ \begin{aligned} &\mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y)) \\ &+ \sum_{j=1}^n \alpha_j \int_0^{t-\varepsilon} \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s) \\ &\times (\sigma - h(y)) ds \\ &+ (g_1 * \mathcal{S}_{\beta,\gamma_j})(t - \varepsilon)\chi \\ &+ \int_0^{t-\varepsilon} \mathcal{T}_{\beta,\gamma_j}(t-s)Bu_y^\rho(s) ds \\ &+ \int_0^{t-\varepsilon} \mathcal{T}_{\beta,\gamma_j}(t-s)f(s) ds \\ &+ \int_0^{t-\varepsilon} \mathcal{T}_{\beta,\gamma_j}(t-s)g(s)dw(s). \end{aligned} \right.$$

We observe by the compactness of $\mathcal{S}_{\beta,\gamma_j}(t)$ and $\mathcal{T}_{\beta,\gamma_j}(t-s)$ that the set $\mathbb{U}_\varepsilon(t) = \{w_\varepsilon(t) : w_\varepsilon \in \Upsilon(\mathbb{B}_{r_0})\}$ is relative compact in \mathbb{H} . On the other hand, we have

$$\begin{aligned} &E\|w(t) - w_\varepsilon(t)\|^2 \leq \\ &5E \left\| \sum_{j=1}^n \alpha_j \int_{t-\varepsilon}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s) \right. \\ &\times (\sigma - h(y)) ds \left. \right\|^2 + 5E \left\| \int_{t-\varepsilon}^t \mathcal{S}_{\beta,\gamma_j}(s)\chi ds \right\|^2 \\ &+ 5E \left\| \int_{t-\varepsilon}^t \mathcal{T}_{\beta,\gamma_j}(t-s)Bu_y^\rho(s) ds \right\|^2 \\ &+ 5E \left\| \int_{t-\varepsilon}^t \mathcal{T}_{\beta,\gamma_j}(t-s)f(s) ds \right\|^2 \\ &+ 5E \left\| \int_{t-\varepsilon}^t \mathcal{T}_{\beta,\gamma_j}(t-s)g(s)dw(s) \right\|^2 \\ &\leq 5n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 \varepsilon^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ &\times 2(E\|\sigma\|^2 + \eta_1 r_0 + \eta_2) + 5\varepsilon S_0^2 E\|\chi\|^2 \\ &+ \frac{5S_0^2 S_B^2 N_u \varepsilon^{1+4\beta}}{(1+4\beta)[\Gamma(1+\beta)]^2} \\ &+ \frac{5S_0^2 N_f(r_0) \varepsilon^{3+2\beta}}{(1+2\beta)[\Gamma(1+\beta)]^2} \\ &+ \frac{5Tr(Q)S_0^2 N_g(r_0) \varepsilon^{1+2\beta}}{(1+2\beta)[\Gamma(1+\beta)]^2} \end{aligned} \tag{3.2}$$

Now, letting $\varepsilon \rightarrow 0$ we observe that there exist relatively compact set arbitrary close to $\mathbb{U}(t)$ for each $t \in (0, T]$. Thus, for all $t \in (0, T]$, $\mathbb{U}(t)$ is relatively compact in \mathbb{H} . Hence, the set $\mathbb{U}(t)$ is relatively compact for all $t \in \mathcal{I}$.

Step 5 : Υ has a closed graph.

Let $y^n \rightarrow y^0$ and $w^n \rightarrow w^0$ as $n \rightarrow \infty$. Now we will show that $w^0 \in \Upsilon(y^0)$. In fact, for $w^n \in \Upsilon(y^n)$ and $t \in \mathcal{I}$, there exists



$f^n \in S_{F,y^n}$ and $g^n \in S_{G,y^n}$ such that

$$w^n(t) = \left\{ \begin{array}{l} \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y^n)) + \sum_{j=1}^n \alpha_j \\ \times \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^n))ds \\ + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-\xi)B \\ \times \left[B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \left\{ (\rho I + \Gamma_0^T)^{-1} \right. \right. \\ \times \left. \left. \left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y^n)) \right) \right. \right. \\ \left. \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi - \sum_{j=1}^n \alpha_j \right. \right. \\ \left. \left. \times \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^n))ds \right. \right. \\ \left. \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s)dw(s) \right\} \right. \\ \left. - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \right. \\ \left. \times \mathcal{T}_{\beta,\gamma_j}(T-s)f^n(s)ds \right. \\ \left. - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \right. \\ \left. \times \mathcal{T}_{\beta,\gamma_j}(T-s)g^n(s)dw(s) \right] d\xi \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)f^n(s)ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)g^n(s)dw(s). \end{array} \right.$$

Next, we must show that there exists $f^0 \in S_{F,y^0}$ and $g^0 \in S_{G,y^0}$ such that

$$w^0(t) = \left\{ \begin{array}{l} \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y^0)) + \sum_{j=1}^n \alpha_j \\ \times \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^0))ds \\ + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-\xi) \\ \times B \left[B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \left\{ (\rho I + \Gamma_0^T)^{-1} \right. \right. \\ \times \left. \left. \left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y^0)) \right) \right. \right. \\ \left. \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi - \sum_{j=1}^n \alpha_j \right. \right. \\ \left. \left. \times \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^0))ds \right. \right. \\ \left. \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s)dw(s) \right\} \right. \\ \left. - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \right. \\ \left. \times \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{T}_{\beta,\gamma_j}(T-s)f^0(s)ds \right. \\ \left. - B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \int_0^T (\rho I + \Gamma_s^T)^{-1} \right. \\ \left. \times \mathcal{T}_{\beta,\gamma_j}(T-s)g^0(s)dw(s) \right] d\xi \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)f^0(s)ds \\ + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)g^0(s)dw(s). \end{array} \right.$$

By $w^0 \in Y(y^0)$ and continuity of h , we obtain

$$E \left\| \left(\begin{array}{l} w^n(t) - \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y^n)) \\ - \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^n))ds \\ - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi \\ - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-\xi)B \left[B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \left\{ (\rho I + \Gamma_0^T)^{-1} \right. \right. \right. \\ \times \left. \left. \left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y^n)) \right) \right. \right. \\ \left. \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \right. \\ \left. \left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^n))ds \right. \right. \\ \left. \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s)dw(s) \right\} \right] d\xi \right. \\ \left. - \left(w^0(t) - \mathcal{S}_{\beta,\gamma_j}(t)(\sigma - h(y^0)) \right) \right. \\ \left. - \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^0))ds \right. \\ \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi \right. \\ \left. - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-\xi)B \left[B^* \mathcal{T}_{\beta,\gamma_j}^*(T-\xi) \right. \right. \\ \left. \left. \times \left\{ (\rho I + \Gamma_0^T)^{-1} \left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(y^0)) \right) \right. \right. \right. \\ \left. \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \right. \\ \left. \left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^0))ds \right. \right. \\ \left. \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s)dw(s) \right\} \right] d\xi \right\|^2 \rightarrow 0. \end{array} \right.$$

Consider the linear operator

$$\Phi : L^2(\Omega, \mathbb{H}) \times L^2(\mathcal{L}(\mathbb{K}, \mathbb{H})) \rightarrow C(\mathcal{I}, \mathbb{H}),$$

$$(f, g) \mapsto \Phi(f, g) = \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s) \times \left[f^0(s) - BB^* \mathcal{T}_{\beta,\gamma_j}^*(T-s) \times \int_0^T (\rho I + \Gamma_0^T)^{-1} \mathcal{T}_{\beta,\gamma_j}(T-\xi)f^0(\xi)d\xi \right] ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s) \left[g^0(s) - BB^* \mathcal{T}_{\beta,\gamma_j}^*(T-s) \int_0^T (\rho I + \Gamma_0^T)^{-1} \mathcal{T}_{\beta,\gamma_j}(T-\xi)g^0(\xi)dw(\xi) \right] dw(s).$$



It follows from Lemma 2.4, that $\Phi \circ S_{(F,G)}$ is a closed graph operator, where

$$S_{(F,G)} = \{f \in F(t, y(t))\} \times \{g \in G(t, y(t))\}.$$

Moreover, by the definition of Φ , we have

$$\begin{aligned} & \left(w^n(t) - \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y^n)) \right. \\ & - \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y^n)) ds \\ & - (g_1 * \mathcal{S}_{\beta, \gamma_j})(t)\chi \\ & - \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-\xi) B \left[B^* \mathcal{T}_{\beta, \gamma_j}^*(T-\xi) \right. \\ & \times \left. \left. \left((\rho I + \Gamma_0^T)^{-1} \left(E \tilde{y}_T - \mathcal{S}_{\beta, \gamma_j}(T)(\sigma - h(y^n)) \right) \right. \right. \right. \\ & - (g_1 * \mathcal{S}_{\beta, \gamma_j})(T)\chi \\ & - \left. \left. \left. \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y^n)) ds \right) \right] \right. \\ & \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) dw(s) \right\} d\xi \in S_{(F,G,y^n)}. \end{aligned}$$

Since $y^n \rightarrow y^0$, It conclude by Lemma 2.4, that

$$\begin{aligned} & \left(w^0(t) - \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y^0)) \right. \\ & - \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y^0)) ds \\ & - (g_1 * \mathcal{S}_{\beta, \gamma_j})(t)\chi \\ & - \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-\xi) B \\ & \times \left[B^* \mathcal{T}_{\beta, \gamma_j}^*(T-\xi) \left\{ (\rho I + \Gamma_0^T)^{-1} \left(E \tilde{y}_T \right. \right. \right. \\ & - \mathcal{S}_{\beta, \gamma_j}(T)(\sigma - h(y^0)) \\ & - (g_1 * \mathcal{S}_{\beta, \gamma_j})(T)\chi \\ & - \left. \left. \left. \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y^0)) ds \right) \right\} \right. \\ & \left. + \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) dw(s) \right\} d\xi \in S_{(F,G,y^0)}. \end{aligned}$$

This implies that $w^0 \in Y(y^0)$. Hence Y has closed graph. Now, by Arzela-Ascoli theorem and by steps 1 – 5, the multivalued map Y is compact, u.s.c. with convex closed values.

Step 6 : The operator Y has a mild solution.

For $k_0 > 0$ given by (3.1), define a open ball $\mathbb{B}_{k_0} \subset \mathcal{C}$. We know by steps 1 – 5 that Y satisfies all the conditions of

Lemma 2.14. Let y be a possible solution for $\theta y \in Y(y)$ for some for some $\theta > 1$ satisfying $E\|y\|^2 = k_0$. Then, we obtain

$$y(t) = \begin{cases} \theta^{-1} \mathcal{S}_{\beta, \gamma_j}(t)(\sigma - h(y)) \\ + \theta^{-1} \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \\ \times \mathcal{S}_{\beta, \gamma_j}(s)(\sigma - h(y)) ds \\ + \theta^{-1} (g_1 * \mathcal{S}_{\beta, \gamma_j})(t)\chi \\ + \theta^{-1} \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) B u_y^0(s) ds \\ + \theta^{-1} \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) f(s) ds \\ + \theta^{-1} \int_0^t \mathcal{T}_{\beta, \gamma_j}(t-s) g(s) dw(s). \end{cases}$$

Then, parallel to the step 2, we have

$$\begin{aligned} E\|y(t)\|^2 & \leq 12S_0^2(E\|\sigma\|^2 + \eta_1 E\|y\|^2 + \eta_2) \\ & + 12n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ & \times (E\|\sigma\|^2 + \eta_1 E\|y\|^2 + \eta_2) + 6TS_0^2 E\|\chi\|^2 \\ & + \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \int_0^t (T-\xi)^{4\beta} \left(12E\|\tilde{y}_T\|^2 \right. \\ & + 12 \int_0^T E\|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \\ & + 12S_0^2(E\|\sigma\|^2 + \eta_1 E\|y\|^2 + \eta_2) \\ & + 12n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ & \times (E\|\sigma\|^2 + \eta_1 E\|y\|^2 + \eta_2) + 6TS_0^2 E\|\chi\|^2 \\ & + \frac{6S_0^2}{[\Gamma(1+\beta)]^2} \int_0^T (T-s)^{2\beta} N_f(E\|y\|^2) ds \\ & + \frac{6Tr(Q)S_0^2}{[\Gamma(1+\beta)]^2} \int_0^T (T-s)^{2\beta} N_g(E\|y\|^2) ds \Big) d\xi \\ & + \frac{6S_0^2}{[\Gamma(1+\beta)]^2} \int_0^t (t-s)^{2\beta} N_f(E\|y\|^2) ds \\ & + \frac{6Tr(Q)S_0^2}{[\Gamma(1+\beta)]^2} \int_0^t (t-s)^{2\beta} N_g(E\|y\|^2) ds. \end{aligned}$$

Taking supremum over t , we get $E\|y\|^2 = k_0$, then

$$\begin{aligned} k_0 & \leq 12S_0^2(E\|\sigma\|^2 + \eta_1 k_0 + \eta_2) \\ & + 12n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\ & \times (E\|\sigma\|^2 + \eta_1 k_0 + \eta_2) + 6TS_0^2 E\|\chi\|^2 \\ & + \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \left(12E\|\tilde{y}_T\|^2 \right. \end{aligned}$$



$$\begin{aligned}
 &+ 12 \int_0^T E \|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \\
 &+ 12S_0^2(E\|\sigma\|^2 + \eta_1 k_0 + \eta_2) \\
 &+ 12n \sum_{j=1}^n \frac{\alpha_j^2 S_0^2 T^{1+2\beta-2\gamma_j}}{(1+2\beta-2\gamma_j)[\Gamma(1+\beta-\gamma_j)]^2} \\
 &\times (E\|\sigma\|^2 + \eta_1 k_0 + \eta_2) \\
 &+ 6TS_0^2 E\|\chi\|^2 + \frac{6S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_f(k_0) \\
 &+ \frac{6Tr(Q)S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_g(k_0) \\
 &+ \frac{6S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_f(k_0) \\
 &+ \frac{6Tr(Q)S_0^2}{[\Gamma(1+\beta)]^2} \frac{T^{1+2\beta}}{(1+2\beta)} N_g(k_0).
 \end{aligned}$$

Calculating the value of k_0 , we get

$$\begin{aligned}
 k_0 \leq & \frac{1}{1-L_2} \left[\frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \left(12E\|\tilde{y}_T\|^2 \right. \right. \\
 & \left. \left. + 12 \int_0^T E \|\tilde{\phi}(s)\|_{\mathcal{L}_0^2}^2 ds \right) \right. \\
 & \left. + \left(1 + \frac{6S_B^4 S_0^4}{\rho^2 [\Gamma(1+\beta)]^4} \frac{T^{1+4\beta}}{(1+4\beta)} \right) L_1 \right]
 \end{aligned}$$

which is a contradiction to (3.1). Thus, in view of Lemma 2.14, the inclusion operator Υ admits a fixed point that turns out as a mild solution of (1.1) – (1.2) on \mathcal{I} . \square

Theorem 3.3. *Let the multivalued functions F and G be uniformly bounded and the assumptions $(A_1) - (A_5)$ are fulfilled. Moreover, if the linear fractional stochastic system (2.2) – (2.3) is approximately controllable, then the fractional stochastic system (1.1) – (1.2) is approximately controllable on \mathcal{I} .*

Proof. Let y^ρ be a mild solution of the system (1.1) – (1.2) i.e. fixed point of Υ . Using the stochastic Fubini theorem, we conclude that y^ρ satisfies

$$y^\rho(T) = \begin{cases} \tilde{y}_T - \rho(\rho I + \Gamma_0^T)^{-1} \left(E\tilde{y}_T \right. \\ \left. - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(\rho y)) \right. \\ \left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \\ \left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \right. \\ \left. \times \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^\rho)) ds \right) \\ \left. - \rho \int_0^T (\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) dw(s) \right. \\ \left. + \rho \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) f^\rho(s) ds \right. \\ \left. + \rho \int_0^T (\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) g^\rho(s) dw(s), \right. \end{cases}$$

where

$$f^\rho \in S_{F,y^\rho} = \{f^\rho \in L^2(\Omega, \mathbb{H}) : f^\rho(t) \in F(t, y^\rho), \text{ for a.e. } t \in \mathcal{I}\}$$

and $g^\rho \in S_{G,y^\rho}$, where

$$S_{G,y^\rho} = \{g^\rho \in L^2(\mathcal{L}(\mathbb{K}, \mathbb{H})) : g^\rho(t) \in F(t, y^\rho), \text{ for a.e. } t \in \mathcal{I}\}.$$

By the uniformly boundedness of F and G , there exists a constant $\Delta > 0$ such that $\|f^\rho(s)\|^2 + \|g^\rho(s)\|^2 \leq \Delta$. Then, there is a subsequence still denoted by $\{f^\rho(s), g^\rho(s)\}$ weakly converging to, say, $\{f(s), g(s)\}$. Now, it follows by the compactness of $\mathcal{S}_{\beta,\gamma_j}(t)$ that

$$\begin{aligned}
 \mathcal{S}_{\beta,\gamma_j}(T-s)f^\rho(s) &\rightarrow \mathcal{S}_{\beta,\gamma_j}(T-s)f(s), \\
 \mathcal{S}_{\beta,\gamma_j}(T-s)g^\rho(s) &\rightarrow \mathcal{S}_{\beta,\gamma_j}(T-s)g(s).
 \end{aligned}$$

Now,

$$\begin{aligned}
 E\|y^\rho(T) - \tilde{y}_T\|^2 &= 6E \left\| \rho(\rho I + \Gamma_0^T)^{-1} \right. \\
 &\left(E\tilde{y}_T - \mathcal{S}_{\beta,\gamma_j}(T)(\sigma - h(\rho y)) \right. \\
 &\left. - (g_1 * \mathcal{S}_{\beta,\gamma_j})(T)\chi \right. \\
 &\left. - \sum_{j=1}^n \alpha_j \int_0^T \frac{(T-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)(\sigma - h(y^\rho)) ds \right) \Big\|^2 \\
 &+ 6E \left(\int_0^T \left\| \rho(\rho I + \Gamma_s^T)^{-1} \tilde{\phi}(s) \right\|_{\mathcal{L}_0^2}^2 ds \right) \\
 &+ 6E \left(\int_0^T \left\| \rho(\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) \right. \right. \\
 &\left. \left. \times [f^\rho(s) - f(s)] \right\| ds \right)^2 \\
 &+ 6E \left(\int_0^T \left\| \rho(\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) f(s) \right\| ds \right)^2 \\
 &+ 6E \left(\int_0^T \left\| \rho(\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) \right. \right. \\
 &\left. \left. \times [g^\rho(s) - g(s)] \right\|_{\mathcal{L}_2^0}^2 ds \right) \\
 &+ 6E \left(\int_0^T \left\| \rho(\rho I + \Gamma_s^T)^{-1} \mathcal{S}_{\beta,\gamma_j}(T-s) g^\rho(s) \right\|_{\mathcal{L}_2^0}^2 ds \right).
 \end{aligned}$$

Further, by Lemma 2.13, the operator $\rho(\rho I + \Gamma_s^T)^{-1} \rightarrow 0$ strongly as $\rho \rightarrow 0^+$, and moreover $\|\rho(\rho I + \Gamma_s^T)^{-1}\| \leq 1$. Now, using Lebesgue dominated convergence theorem, we deduce that $E\|y^\rho(T) - \tilde{y}_T\|^2 \rightarrow 0$ as $\rho \rightarrow 0^+$. This proves the approximate controllability of the system (1.1) – (1.2). \square



4. Example

Let $\beta, \gamma_j > 0, j = 1, 2, 3, \dots, n$ be given such that $0 < \beta \leq \gamma_n \leq \dots \leq \gamma_1 \leq 1$. We consider the following system

$$\begin{aligned}
 {}^c D^{1+\beta} z(t, x) + \sum_{j=1}^n \alpha_j {}^c D^{\gamma_j} z(t, x) &= \frac{\partial^2}{\partial x^2} z(t, x) + \mu(t, x) \\
 + Q_1(t, z(s, x)) + Q_2(t, z(s, x)) \frac{dw(t)}{dt}, \quad t \in \mathcal{I} = [0, 1], \\
 z(t, 0) = z(t, \pi) &= 0, \\
 z(0, x) + \sum_{i=1}^m c_i z(t_i, x) &= \xi_0(x), \quad 0 \leq x \leq \pi, \\
 \frac{\partial z(t, x)}{\partial t} \Big|_{t=0} &= z_0,
 \end{aligned} \tag{4.1}$$

where $w(t)$ represents a standard cylindrical Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ and $\xi_0, z_0 \in L^2(0, \pi)$. $\mu : [0, 1] \times (0, \pi) \rightarrow (0, \pi)$ is continuous in t and $c_i > 0$ for all $i = 1, 2, \dots, m$. The functions $Q_1, Q_2 : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ are continuous. We consider the space $\mathbb{H} = \mathbb{U} = L^2([0, \pi])$ equipped with norm $\|\cdot\|_{L^2}$ and define A by $\mathcal{D}(A) :=$

$$\begin{aligned}
 \{w \in \mathbb{X} : w, w' \text{ are absolutely continuous,} \\
 w'' \in \mathbb{X}, w(0) = w(\pi) = 0\}, \\
 \text{and } Aw = w''
 \end{aligned}$$

Then the operator A is given by

$$Aw = \sum_{n=1}^{\infty} -n^2 \langle w, w_n \rangle w_n,$$

where $w_n(t) = \sqrt{\frac{2}{\pi}} \sin nt, n = 1, 2, \dots$, is orthonormal set of eigenfunctions corresponding to the eigenvalues $\lambda_n = -n^2$ of A . Then A will be a generator of cosine family such that

$$C(t)w = \sum_{n=1}^{\infty} \cos nt \langle w, w_n \rangle w_n,$$

Now by Theorem 2.6, A generates a bounded (β, γ_j) -resolvent family $\{\mathcal{S}_{\beta, \gamma_j}(t)\}_{t \geq 0}$. Let $z(t)x = z(t, x)$ and define the bounded linear operator $B : \mathbb{U} \rightarrow \mathbb{H}$ by $Bu(t)x = \mu(t, x)$. Further, we assume $F(t, z(t))(x) = Q_1(t, z(t, x)) = \frac{e^{-t}}{1+e^{-t}} \sin(z(t, x))$, $G(t, z(t))(x) = Q_2(t, z(t, x)) = \frac{e^{-t}}{1+e^{-t}} \sin(z(t, x))$ and $h(z)(x) = \sum_{i=1}^m c_i z(t_i, x)$. Then the assumptions $(A_2) - (A_4)$ are satisfied. Moreover, the system can be written in the abstract form (1.1)-(1.2) and the corresponding linear fractional stochastic system (4.1) is approximately controllable, so by theorem 3.3, the fractional stochastic system (4.1) is approximately controllable on $[0, 1]$.

5. Conclusion

This paper has investigated the approximate controllability results for the multi-term time-fractional stochastic differential systems with non-local conditions. The results obtained are quite useful for studying physical problems which are characterized by fractional systems. The authors are interested to study the approximate controllability of multi-term time-fractional differential systems fractional order Sobolev-type stochastic integrodifferential systems via measure of non-compactness in order to drop the compactness condition on the operator $\{\mathcal{S}_{\beta, \gamma_j}(t)\}_{t \geq 0}$.

Acknowledgment

The work of the first author is supported by the “University Grant Commission, India”.

References

- [1] R. P. Agarwal, M. Benchohra and S. Hamani. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Applicandae Mathematicae*, 109(3):973-1033, 2010.
- [2] E. Alvarez-Pardo and C. Lizama. Mild solutions for multi-term time-fractional differential equations with nonlocal initial conditions. *Electronic Journal of Differential Equations*, 2014(39):1-10, 2014.
- [3] L. Arnold, Stochastic differential equations. *New York*, 1974.
- [4] K. Balachandran, V. Govindaraj, G. L. Rodriguez, and J.J. Trujillo. Controllability of nonlinear higher order fractional dynamical systems. *Nonlinear Dynamics*, 71(4):605-612, 2013.
- [5] P. Balasubramaniam, J.Y. Park and P. Muthukumar. Approximate controllability of neutral stochastic functional differential systems with infinite delay. *Stochastic Analysis and Applications*, 28(2):389-400, 2010.
- [6] J. Dabas and A. Chauhan. Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay. *Mathematical and Computer Modelling*, 57(3-4):754-763, 2013.
- [7] J.P. Dauer and N. Mahmudov. Controllability of stochastic semilinear functional differential equations in hilbert spaces. *Journal of Mathematical Analysis and Applications*, 290(2):373-394, 2004.
- [8] K. Deimling. Multivalued differential equations, *vol. 1 of de gruyter series in nonlinear analysis and applications*, 1992.
- [9] B.C. Dhage. Multi-valued mappings and fixed points ii. *Tamkang Journal of Mathematics*, 37(1):27-46, 2006.
- [10] K. Diethelm and N.J. Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265(2):229-248, 2002.
- [11] M. Feckan, Y. Zhou and J. R. Wang. Response to comments on the concept of existence of solution for im-



- pulsive fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 19(12):4213-4215, 2014.
- [12] S. Hu and N. S. Papageorgiou. Handbook of multivalued analysis: Volume II: Applications, volume 500. *Springer Science & Business Media*, 2013.
- [13] N. Ikeda and S. Watanabe. Stochastic differential equations and diffusion processes, volume 24. *Elsevier*, 2014.
- [14] H. Jiang, F. Liu, I. Turner and K. Burrage. Analytical solutions for the multi-term time-space caputo-riesz fractional advection-diffusion equations on a finite domain. *Journal of Mathematical Analysis and Applications*, 389(2):1117-1127, 2012.
- [15] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, volume 204. *Elsevier Science Limited*, 2006.
- [16] J. Klamka. Controllability of linear dynamical systems. 1963.
- [17] S. Kumar and N. Sukavanam. Approximate controllability of fractional order semilinear systems with bounded delay. *Journal of Differential Equations*, 252(11):6163-6174, 2012.
- [18] N. Mahmudov and A. Denker. On controllability of linear stochastic systems. *International Journal of Control*, 73(2):144-151, 2000.
- [19] N. I. Mahmudov. Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. *SIAM journal on control and optimization*, 42(5):1604-1622, 2003.
- [20] A. Meraj and D. N. Pandey. Existence of mild solutions for fractional non-instantaneous impulsive integro-differential equations with nonlocal conditions. *Arab Journal of Mathematical Sciences*, 2018.
- [21] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.
- [22] I. Podlubny. *Fractional differential equations*, Academic Press, New York 1999.
- [23] P. E. Protter. Stochastic differential equations. In *Stochastic integration and differential equations*, pages 249-361. Springer, 2005.
- [24] Y. Ren, L. Hu, and R. Sakthivel. Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay. *Journal of Computational and Applied Mathematics*, 235(8):2603-2614, 2011.
- [25] R. Sakthivel, J. J. Nieto, N. Mahmudov, et al. Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay. *Taiwanese Journal of Mathematics*, 14(5):1777-1797, 2010.
- [26] R. Sakthivel, S. Suganya, and S. M. Anthoni. Approximate controllability of fractional stochastic evolution equations. *Computers & Mathematics with Applications*, 63(3):660-668, 2012.
- [27] V. Singh and D. N. Pandey. Existence results for multi-term time-fractional impulsive differential equations with fractional order boundary conditions. *Malaya Journal of Matematik (MJM)*, 5(4, 2017):625-635, 2017.
- [28] V. Singh and D. N. Pandey. Controllability of multi-term time-fractional differential systems. *Journal of Control and Decision*, pages 1-17, 2018.
- [29] D. Talay. Numerical solution of stochastic differential equations. 1994.
- [30] V. Vijayakumar. Approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in hilbert spaces. *IMA Journal of Mathematical Control and Information*, 35(1):297-314, 2016.
- [31] J. Wang and Y. Zhou. Existence and controllability results for fractional semilinear differential inclusions. *Nonlinear Analysis: Real World Applications*, 12(6):3642-3653, 2011.
- [32] J. Wang, Y. Zhou, M. Fec, et al. On recent developments in the theory of boundary value problems for impulsive fractional differential equations. *Computers & Mathematics with Applications*, 64(10):3008-3020, 2012.

 ISSN(P):2319 – 3786
 Malaya Journal of Matematik
 ISSN(O):2321 – 5666

