Existence of solutions for conformable fractional problems with nonlinear functional boundary conditions

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Abstract
In this article, we study the existence of solutions for nonlinear conformable fractional differential equations with nonlinear functional boundary conditions. We obtain the exact expression of the fractional Green's function related to the linear problem. Moreover, the method of upper and lower solutions together with Schauder's fixed point theorem is developed for the nonlinear conformable fractional problems with nonlinear functional boundary conditions.

Keywords
Conformable fractional derivative, nonlinear boundary conditions, Green's function, upper and lower solutions method, existence theorems, maximum principles.

AMS Subject Classification
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Contents

1. Introduction

It is very well known that fractional differential equations plays an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, aerodynamics or electrodynamics. For examples and details the reader can see the references [7, 8, 11–14, 17, 24–28, 33, 34]. Recently, a new fractional derivative, called the conformable fractional derivative, was introduced by Khalil et al. in [23]. For recent results on conformable fractional derivatives we refer the reader to [1–3, 5, 6, 15, 16, 19, 21–23]. Furthermore, in [3, 5, 6], the

authors proved the existence and uniqueness of solutions of initial value problems and boundary value problems for conformable fractional differential equations. In [16], the authors proved existence and uniqueness theorems for sequential linear conformable fractional differential equations. In [22], the authors proved the existence of solutions of boundary value problem involving conformable derivative by the method of upper and lower solutions.

This paper is concerned with the study of the existence of solutions for the nonlinear conformable fractional differential equations with nonlinear functional boundary conditions:

$$x^{(\alpha)}(t) = f(t, x(t)), \quad \text{for a.e. } t \in I = [0, b], \ b > 0, \ (1.1)$$

where $0 < \alpha \le 1$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a L^1_α-Carathéodory function, and $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α. We consider, depending on the circumstances, nonlinear functional boundary conditions of the type

$$L_1(x, x(b)) = 0 \ \text{ or } \ L_2(x(0), x) = 0,$$

with L_i $(i = 1, 2)$ a continuous function that satisfies suitable monotonicity assumptions. For this purpose, we use the

method of upper and lower solutions together with Schauder's fixed point theorem.

We point out that the method of lower and upper solutions has been applied by several authors to obtain the existence of solutions of initial value problems and boundary value problems for fractional differential equations, see [20, 29, 30, 32, 35].

Motivated by the previously mentioned papers, this is the first paper concerns the existence of solutions for the conformable fractional differential equations with linear and nonlinear functional boundary conditions. Existence results for these problems are obtained with new comparison results and new definitions of upper and lower solutions. For first order ordinary differential equations with nonlinear boundary conditions, we refer the reader to the papers [9, 18].

This paper is organized as follows. In Section 2, we introduce the definition of conformable fractional calculus and their important properties. In Section 3, we solve the general linear problem with conformable fractional derivative. We obtain the related Green's function and prove a comparison result. In Section 4, we prove the existence of solutions to nonlinear conformable fractional differential equation (1.1) coupled to nonlinear functional boundary conditions, some examples are shown to illustrate the obtained results.

2. Preliminaries

In this section, we introduce the definition of conformable fractional calculus and their important properties. The results can be seen in [23] and references therein.

Definition 2.1. *[23] Given a function $f : [0, \infty) \to \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of f of order α is defined by,*

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \qquad (2.1)$$

for all $t > 0$.

If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Theorem 2.2. *[23] Let $\alpha \in (0, 1]$ and $f : [0, \infty) \to \mathbb{R}$ a α-differentiable function at $t_0 > 0$, then f is continuous at t_0.*

Theorem 2.3. *[23] Let $\alpha \in (0, 1]$ and assume f, g to be α-differentiable at a point $t > 0$. Then,*

(i) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, for all $a, b \in \mathbb{R}$;

(ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$;

(iii) $(f/g)^{(\alpha)} = \dfrac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.

(iv) If, in addition, f is differentiable at a point $t > 0$, then

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

Additionaly, conformable fractional derivatives of certain functions as follow:

1. $(t^p)^{(\alpha)} = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.

2. $(\lambda)^{(\alpha)} = 0$, for all $\lambda \in \mathbb{R}$.

3. $(e^{ct})^{(\alpha)} = ct^{1-\alpha}e^{ct}$, for all $c \in \mathbb{R}$.

4. $(e^{\frac{p}{\alpha}t^\alpha})^{(\alpha)} = pe^{\frac{p}{\alpha}t^\alpha}$, for all $p \in \mathbb{R}$.

Remark 2.4. *It is not difficult to verify the following assertions:*

(i) The function $x : t \mapsto e^{\frac{p}{\alpha}t^\alpha}$, $p \in \mathbb{R}$, is the unique solution to the conformable fractional differential equation

$$x^{(\alpha)}(t) = px(t), \; t \in [0, \infty), \; x(0) = 1.$$

(ii) If f is differentiable at t, then f is α-differentiable at t.

We introduce the following spaces:

$$
\begin{aligned}
C_0(I) &= \{f : I \to \mathbb{R}, \text{ is continuous on } I \text{ with compact support in } I \}. \\
C^\alpha(I) &= \{f : I \to \mathbb{R}, \text{ is } \alpha\text{-differentiable on } I \text{ and } f^{(\alpha)} \in C(I)\}. \\
C_0^\alpha(I) &= \{f \in C^\alpha(I) : f(0) = f(b) = 0\}. \\
C_{0,b}^\alpha(I) &= \{f \in C^\alpha(I) : f(0) = f(b)\}.
\end{aligned}
$$

Definition 2.5. *[23] Let $\alpha \in (0, 1]$ and $f : [0, \infty) \to \mathbb{R}$. The conformable fractional integral of f of order α from 0 to t, denoted by $I_\alpha(f)(t)$, is defined by*

$$I_\alpha(f)(t) := I_1(t^{\alpha-1}f)(t) = \int_0^t f(s)d_\alpha s := \int_0^t f(s)s^{\alpha-1}ds.$$

The considered integral is the usual improper Riemann one.

Theorem 2.6. *[23] If f is a continuous function in the domain of I_α then, for all $t \geq 0$ we have*

$$(I_\alpha(f))^{(\alpha)}(t) = f(t).$$

Lemma 2.7. *[1, 23] Let $f : (0, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_\alpha(f^{(\alpha)})(t) = f(t) - f(0). \qquad (2.2)$$

Next, we develop the fractional Sobolev's spaces via conformable fractional calculus and their important properties. The basic definitions and relations based on [31](If $\mathbb{T}$ is a real interval $[0, \infty)$) are given:

Definition 2.8. *Let $B \subset I$. B is called null set if the measure of B is zero. Say that a property P holds almost everywhere (a.e.) on B if there is a null set $E_0 \subset B$ such that P holds for all $t \in B \setminus E_0$.*

Definition 2.9. *Let A a Lebesgue measurable subset of I. We say that function $f : I \to \mathbb{R}$, is a function α-integrable on A if and only if $t^{\alpha-1} f(t)$ is Lebesgue integrable on A. In such a case, we denote*

$$\int_A f(t) \, d_\alpha t = \int_A t^{\alpha-1} f(t) \, dt.$$

Definition 2.10. *Let $E \subset I$ be a Lebesgue measurable set and let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$ and let $\varphi : E \to \overline{\mathbb{R}}$ be a measurable function. We say that φ belongs to $L_\alpha^p(E)$ provided that either*

$$\int_E |\varphi(s)|^p \, d_\alpha s = \int_E |\varphi(s)|^p s^{\alpha-1} ds < +\infty \quad if \ p \in \mathbb{R},$$

or there exists a constant $C \in \mathbb{R}$ such that

$$|\varphi(s)| < C \text{ a.e. on } E \quad if \ p = +\infty.$$

Theorem 2.11. *[31] Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$. Then the set $L_\alpha^p(I)$ is a Banach space together with the norm defined for $\varphi \in L_\alpha^p(I)$ as*

$$\|\varphi\|_{L_\alpha^p(I)} := \begin{cases} (\int_I |\varphi(t)|^p \, d_\alpha t)^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf\{C \in \mathbb{R} : |\varphi(s)| < C \text{ a.e. on } I\}, & p = +\infty. \end{cases}$$

Remark 2.12. *It is not difficult to verify the following assertions for all $\alpha \in (0,1]$:*

(i) $L_\alpha^1(I) \subset L^1(I)$.

(ii) For $t \in I$, $t > 0$ and $\varphi : I \to \mathbb{R}$, it is satisfied that $\varphi^{(\alpha)} \in L_\alpha^1(I)$ if and only if $\varphi' \in L^1(I)$.

Theorem 2.13. *[31] Let $\alpha \in (0,1]$ and $f : I \to \mathbb{R}$ an absolutely continuous function on I, then f is conformable fractional differentiable of order α a.e. on I, and the following equality holds:*

$$f(t) = f(0) + \int_{[a,t]} f^{(\alpha)}(s) d_\alpha s \quad for \ all \ t \in I.$$

Definition 2.14. *Let $\alpha \in (0,1]$, $p \in \mathbb{R}$ be such that $p \geq 1$ and $u : I \to \mathbb{R}$. One says that $u \in W_{0,b}^{\alpha,p}(I)$ if and only if $u \in L_\alpha^p(I)$, there exists $g : I \to \mathbb{R}$ such that $g \in L_\alpha^p(I)$ and*

$$\int_I u(t)\phi^{(\alpha)}(t) d_\alpha t = -\int_I g(t)\phi(t) d_\alpha t, \quad for \ all \ \phi \in C_{0,b}^\alpha(I). \tag{2.3}$$

Theorem 2.15. *[31] Assume that $u \in W_{0,b}^{\alpha,p}(I)$ for some $p \in \overline{\mathbb{R}}$, with $p \geq 1$, and that equality (2.3) holds for some $g \in L_\alpha^p(I)$. Then, there exists a unique function $x \in V_{0,b}^{\alpha,p}(I)$ such that*

$$x = u, \ x^{(\alpha)} = g \ a.e. \ on \ [0,b].$$

For $p \in \mathbb{R}$, $p \geq 1$, we denote

$$V_{0,b}^{\alpha,p}(I) = \{u \in AC(I) : u^{(\alpha)} \in L_\alpha^p(I), u(0) = u(b)\}.$$

It is clear that for all $\alpha \in (0,1]$ and $p \in \mathbb{R}$, $p \geq 1$, we have that $V_{0,b}^{\alpha,p}(I) \subset W_{0,b}^{\alpha,p}(I)$.

Theorem 2.16. *[31] Let $p \in \mathbb{R}$ be such that $p \geq 1$. Then the set $W_{0,b}^{\alpha,p}(I)$ is a Banach space together with the norm defined as*

$$\|\varphi\|_{W_{0,b}^{\alpha,p}(I)} := \left(\int_I |\varphi(t)|^p d_\alpha t + \int_I |\varphi^{(\alpha)}(t)|^p d_\alpha t \right)^{\frac{1}{p}},$$

for every $\varphi \in W_{0,b}^{\alpha,p}(I)$.

We now define a notion of L_α^1-Carathéodory function.

Definition 2.17. *A function $f : I \times \mathbb{R} \to \mathbb{R}$ is called a L_α^1-Carathéodory function if the three following conditions hold:*

(i) for every $x \in \mathbb{R}$, the function $t \mapsto f(t,x)$ is Lebesgue measurable;

(ii) the function $x \mapsto f(t,x)$ is continuous almost every $t \in I$;

(iii) for every $r > 0$, there exists a function $h_r \in L_\alpha^1(I)$ such that $|f(t,x)| \leq h_r(t)$ for almost every $t \in I$ and all $x \in \mathbb{R}$ such that $|x| \leq r$.

3. Green's Functions and Comparison Results

In this section, we study the expression of the solutions of a linear conformable fractional differential equation of order $\alpha \in (0,1]$ coupled to two-point linear conditions. This study is mainly devoted to obtain the expression of the fractional Green's function related to the considered problem. Once we have such expression, we derive comparison results for the considered problems.

To be concise, we look for $x \in W_{0,b}^{\alpha,1}(I)$, the solution of the following linear problem:

$$x^{(\alpha)}(t) + p(t)x(t) = g(t), \quad a.e. \ t \in I, \quad a_0 x(0) - b_0 x(b) = \lambda_0, \tag{3.1}$$

with $p, g \in L_\alpha^1(I)$, and $a_0, b_0, \lambda_0 \in \mathbb{R}$.

Theorem 3.1. *If $a_0 \neq b_0 e^{-\int_0^b p(r) d_\alpha r}$, then problem (3.1) has a unique solution $x \in W_{0,b}^{\alpha,1}(I)$, and it is given by the following expression:*

$$x(t) := \int_0^b G(t,s)g(s) d_\alpha s + \frac{\lambda_0 e^{-\int_0^t p(r) d_\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r) d_\alpha r}}, \tag{3.2}$$

where

$$G(t,s) = \frac{e^{-\int_s^t p(r) d_\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r) d_\alpha r}} \begin{cases} a_0, & 0 \leq s \leq t \leq b, \\ b_0 e^{-\int_0^b p(r) d_\alpha r}, & 0 \leq t < s \leq b. \end{cases} \tag{3.3}$$

Proof. Let x be a solution of problem (3.1). Since $x \in W_{0,b}^{\alpha,1}(I)$, from Remark 2.12, we have that x is differentiable a.e. on I. Thus, Theorem 2.3 (iv), ensures that, it is a solution of the following singular differential equation:

$$t^{1-\alpha}x'(t) + p(t)x(t) = g(t), \quad \text{a.e. } t \in I, \ a_0 x(0) - b_0 x(b) = \lambda_0,$$

or, which is the same,

$$x'(t) + t^{\alpha-1}p(t)x(t) = t^{\alpha-1}g(t), \text{a.e. } t \in I, \ a_0 x(0) - b_0 x(b) = \lambda_0. \tag{3.4}$$

Now, by using that $p, g \in L_\alpha^1(I)$, we have that, for a.e. $t \in I$,

$$\frac{d}{dt}\left(x(t) e^{\int_0^t p(r)d\alpha r}\right) = e^{\int_0^t p(r)d\alpha r}\left(x'(t) + t^{\alpha-1}p(t)x(t)\right)$$
$$= e^{\int_0^t p(r)d\alpha r}t^{\alpha-1}g(t).$$

Thus, by direct integration, we have that

$$x(t) = e^{-\int_0^t p(r)d\alpha r}x(0) + \int_0^t e^{-\int_s^t p(r)d\alpha r}g(s)d\alpha s \quad \text{for all } t \in I. \tag{3.5}$$

If follows from (3.5) and the boundary condition in (3.1) that

$$x(0) = \frac{b_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}\int_0^b e^{-\int_s^b p(r)d\alpha r}g(s)d\alpha s \tag{3.6}$$
$$+ \frac{\lambda_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}.$$

Now, by substituting (3.6) into (3.5), we arrive to

$$x(t) = \frac{b_0 e^{-\int_0^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}\int_0^t e^{-\int_s^b p(r)d\alpha r}g(s)d\alpha s$$
$$+ \int_0^t e^{-\int_s^t p(r)d\alpha r}g(s)d\alpha s$$
$$+ \frac{b_0 e^{-\int_0^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}\int_t^b e^{-\int_s^b p(r)d\alpha r}g(s)d\alpha s$$
$$+ \frac{\lambda_0 e^{-\int_0^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}$$
$$= \frac{e^{-\int_s^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}\left(a_0 \int_0^t g(s)d\alpha s\right.$$
$$\left. + b_0 \int_t^b e^{-\int_0^b p(r)d\alpha r}g(s)d\alpha s\right)$$
$$+ \frac{\lambda_0 e^{-\int_0^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}} = \int_0^b G(t,s)g(s)d\alpha s$$
$$+ \frac{\lambda_0 e^{-\int_0^t p(r)d\alpha r}}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}}. \qquad \square$$

As a direct consequence, we deduce the following result:

Lemma 3.2. *The fractional Green's function G, related to the linear problem* (3.1)*, and given by the expression* (3.3)*, satisfies the following properties for every $p \in L_\alpha^1(I)$:*

(i) *$G > 0$ on $I \times I$ if and only if*

$$\frac{a_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}} > 0 \quad \text{and} \quad \frac{b_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}} > 0. \tag{3.7}$$

(ii) *$G < 0$ on $I \times I$ if and only if*

$$\frac{a_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}} < 0 \quad \text{and} \quad \frac{b_0}{a_0 - b_0 e^{-\int_0^b p(r)d\alpha r}} < 0. \tag{3.8}$$

As a direct consequence of previous result, we deduce the following expressions for the particular cases of the initial, terminal and periodic problems.

Corollary 3.3. *The initial problem*

$$\begin{cases} x^{(\alpha)}(t) + p(t)x(t) = g(t), & \text{for a.e. } t \in I, \\ x(0) = x_0, \end{cases} \tag{3.9}$$

with $p, g \in L_\alpha^1(I)$, has a unique solution $x \in W_{0,b}^{\alpha,1}(I)$, and it is given by the following expression

$$x(t) := \int_0^b G_I(t,s)g(s)d\alpha s + x_0 e^{-\int_0^t p(r)d\alpha r}, \tag{3.10}$$

where

$$G_I(t,s) = e^{-\int_s^t p(r)d\alpha r}\begin{cases} 1, & 0 \le s \le t \le b, \\ 0, & 0 \le t < s \le b. \end{cases} \tag{3.11}$$

Corollary 3.4. *The terminal problem*

$$\begin{cases} x^{(\alpha)}(t) + p(t)x(t) = g(t), & \text{for a.e. } t \in I, \\ x(b) = x_0, \end{cases} \tag{3.12}$$

with $p, g \in L_\alpha^1(I)$, has a unique solution $x \in W_{0,b}^{\alpha,1}(I)$, and it is given by the following expression

$$x(t) := \int_0^b G_T(t,s)g(s)d\alpha s + x_0 e^{-\int_t^b p(r)d\alpha r}, \tag{3.13}$$

where

$$G_T(t,s) = -e^{-\int_s^t p(r)d\alpha r}\begin{cases} 0, & 0 \le s \le t \le b, \\ 1, & 0 \le t < s \le b. \end{cases} \tag{3.14}$$

From expressions (3.10) and (3.13), it is obvious that $G_I \geq 0$ and $G_T \leq 0$ on $I \times I$. Thus, as a direct consequence of expressions (3.11) and (3.14), we deduce the following comparison result:

Lemma 3.5. *Let $x \in W_{0,b}^{\alpha,1}(I)$, then the following comparison principles hold for every $p \in L_{\alpha}^1(I)$:*

(i) *If $x^{(\alpha)}(t) + p(t)x(t) \geq 0$ a.e. $t \in I$ and $x(0) \geq 0$ then $x \geq 0$ on I.*

(ii) *If $x^{(\alpha)}(t) + p(t)x(t) \geq 0$ a.e. $t \in I$ and $x(b) \leq 0$ then $x \leq 0$ on I.*

Concerning the non homogeneous periodic problem, which follows directly by the choice of $a_0 = b_0 = 1$, as a corollary of Theorem 3.1, we deduce the following result.

Corollary 3.6. *The non homogeneous periodic problem*

$$
\begin{cases}
x^{(\alpha)}(t) + p(t)x(t) = g(t), & \text{for a.e. } t \in I, \\
x(0) - x(b) = \lambda_0,
\end{cases}
\tag{3.15}
$$

with $p, g \in L_{\alpha}^1(I)$, has a unique solution $x \in W_{0,b}^{\alpha,1}(I)$, and it is given by the following expression

$$
x(t) := \int_0^b G_P(t,s)g(s)d_\alpha s + \lambda_0 \frac{e^{-\int_0^t p(r)d_\alpha r}}{1 - e^{-\int_0^b p(r)d_\alpha r}}, \tag{3.16}
$$

where

$$
G_P(t,s) = \frac{e^{-\int_s^t p(r)d_\alpha r}}{1 - e^{-\int_0^b p(r)d_\alpha r}}
\begin{cases}
1, & 0 \leq s \leq t \leq b, \\
e^{-\int_0^b p(r)d_\alpha r}, & 0 \leq t < s \leq b.
\end{cases}
\tag{3.17}
$$

As a consequence, it is immediate to verify, from expression (3.17), that the periodic problem has a unique solution if and only if

$$
\int_0^b p(r)d_\alpha r \neq 0.
$$

Moreover the fractional Green's function G_P has the same sign of the previous integral, i.e.,

Corollary 3.7. *Let $p \in L_{\alpha}^1(I)$, then the following properties hold:*

(i) *$G_P > 0$ on $I \times I$ if and only if $\int_0^b p(r)d_\alpha r > 0$.*

(ii) *$G_P < 0$ on $I \times I$ if and only if $\int_0^b p(r)d_\alpha r < 0$.*

As a direct consequence of previous result and equality (3.16), denoting $y \succ 0$ on I as $y \geq 0$ and $y \not\equiv 0$ on I, we deduce the following comparison result.

Corollary 3.8. *Let $x \in W_{0,b}^{\alpha,1}(I)$ be such that*

$$
x^{(\alpha)}(t) + p(t)x(t) \succ 0 \text{ on } I; \text{ and } x(0) \geq x(b).
$$

Then the following comparison principles are fulfilled:

(i) *If $\int_0^b p(r)d_\alpha r > 0$ then $x > 0$ on I.*

(ii) *If $\int_0^b p(r)d_\alpha r < 0$ then $x < 0$ on I.*

4. Nonlinear Functional Boundary Conditions

In this section, we prove the existence of solutions of the nonlinear conformable fractional differential equation (1.1) coupled to nonlinear functional boundary conditions. In particular, we will consider the two following kind of functional boundary conditions:

$$
L_1(x, x(b)) = 0 \tag{4.1}
$$

and

$$
L_2(x(0), x) = 0. \tag{4.2}
$$

Here $L_1 : C(I) \times \mathbb{R} \to \mathbb{R}$ and $L_2 : \mathbb{R} \times C(I) \to \mathbb{R}$ are continuous functions that satisfy suitable monotonicity assumptions.

The used tool will be the well known method of upper and lower solutions. A solution of these problems will be a function $x \in W_{0,b}^{\alpha,1}(I)$ that satisfies equation (1.1) a.e. on I coupled to the corresponding boundary conditions (either (4.1) or (4.2) in each case).

First, we consider the problem (1.1), (4.1). To this end, we introduce the following definition of lower and upper solution related to such problem.

Definition 4.1. *Let $\gamma \in W_{0,b}^{\alpha,1}(I)$. We say that γ is a lower solution of the boundary value problem (1.1), (4.1) if*

(i) *$\gamma^{(\alpha)}(t) \geq f(t, \gamma(t))$, a.e. $t \in I$;*

(ii) *$L_1(\gamma, \gamma(b)) \geq 0$.*

Let $\delta \in W_{0,b}^{\alpha,1}(I)$. We say that δ is an upper solution of the boundary value problem (1.1), (4.1) if

(i) *$\delta^{(\alpha)}(t) \leq f(t, \delta(t))$, a.e. $t \in I$;*

(ii) *$L_1(\delta, \delta(b)) \leq 0$.*

In order to obtain existence and location results for the considered nonlinear problems, we define the sector

$$
[\gamma, \delta] = \{x \in C(I) : \gamma(t) \leq x(t) \leq \delta(t), \text{ for all } t \in I\}.
$$

Now we give the main result on the existence of solutions for the nonlinear problem (1.1), (4.1). The proof is on the basis on the one given in [9, Theorem 3.1] for two-point nonlinear boundary conditions.

Theorem 4.2. *If there exist γ and δ in $W_{0,b}^{\alpha,1}(I)$, $\gamma \leq \delta$ in I, a pair of well ordered lower and upper solutions respectively for problem* (1.1), (4.1), *with L_1 a continuous function in $[\gamma,\delta] \times [\gamma(b),\delta(b)]$ and nondecreasing in the first variable on $[\gamma,\delta]$, then problem* (1.1), (4.1) *has at least one solution $x \in [\gamma,\delta]$.*

Proof. We consider the following modified problem:

$$\begin{cases} x^{(\alpha)}(t) = f(t,\tau(t,x(t))), & for \ a.e. \ t \in I, \\ x(b) = \tau(b,x(b) + L_1(\tau(\cdot,x(\cdot)),\tau(b,x(b)))), \end{cases} \quad (4.3)$$

where τ is the truncated function, defined for any $x \in C(I)$, as follows:

$$\tau(t,x(t)) = \max\Big\{\gamma(t), \min\{x(t),\delta(t)\}\Big\}, \quad \text{for all } t \in I.$$

By the definition of function τ, it is obvious that $\gamma(b) \leq x(b) \leq \delta(b)$.

Suppose now that $x(0) < \gamma(0)$. From the continuity of both functions we know that there exists $t_0 \in (0,b]$ such that $\gamma(t_0) = x(t_0)$ with $\gamma > x$ on $[0,t_0)$. In this case, due to the linearity of the conformable α-derivative and the definition of the truncated function τ, we have that

$$(\gamma - x)^{(\alpha)}(t) \geq 0, \quad a.e. \ t \in [0,t_0], \quad (\gamma - x)(t_0) = 0.$$

So, Lemma 3.5 (*ii*) implies that $x \geq \gamma$ on $[0, t_0]$, and we arrive to a contradiction.

Analogously, we can prove that $x(0) \leq \delta(0)$.

If there exists $c \in (0,b)$ with $x(c) < \gamma(c)$, then there exists a subinterval $(t_1,t_2) \subset (0,b)$, such that $(\gamma - x)(t_1) = (\gamma - x)(t_2) = 0$, with $\gamma > x$ on (t_1,t_2).

But, arguing as before, we deduce that

$$(\gamma - x)^{(\alpha)}(t) \geq 0, \text{ a.e. } t \in [t_1,t_2].$$

Now, using Lemma 3.5, (*ii*) again, we deduce that $\gamma \leq x$ on $[t_1,t_2]$ and we attain a contradiction.

A similar argument is valid to show that $x \leq \delta$ on I.

Therefore, every solution x of problem (4.3) belongs to the sector $[\gamma,\delta]$. Let's see now that it satisfies the functional boundary condition (4.1).

Clearly, if $x(b) + L_1(\tau(\cdot,x(\cdot)),\tau(b,x(b))) < \gamma(b)$, we obtain that $x(b) = \gamma(b)$ and, in consequence, $\gamma(b) > x(b) + L_1(\tau(\cdot,x(\cdot)),\gamma(b))$.

The nondecreasing character of L_1 with respect to the first variable on the sector $[\gamma,\delta]$, and the definition of function τ, allow us to arrive at the following contradiction

$$\gamma(b) > x(b) + L_1(\gamma,\gamma(b)) \geq x(b) = \gamma(b).$$

Analogously, we can verify that

$$x(b) + L_1(\tau(\cdot,x(\cdot)),\tau(b,x(b))) \leq \delta(b),$$

and, as consequence, every solution x of the truncated problem (4.3) is a solution of (1.1), (4.1).

Now, to finalize the proof, we must ensure that the truncated problem (4.3) has a solution.

To this end, let us define the operator $\mathscr{F} : C(I) \to C(I)$ as follows:

$$(\mathscr{F}x)(t) = -\int_t^b \Big(f(s,\tau(s,x(s)))\Big)d_\alpha s$$
$$+ \tau(b,x(b) + L_1(\tau(\cdot,x(\cdot)),\tau(b,x(b)))).$$

First, notice that the solutions of problem (4.3) coincide with the fixed points of the operator $\mathscr{F}$. This property holds from equation (3.13) and the expression of the fractional Green's function G_T, related to the terminal problem (3.12), with $p \equiv 0$, which is given in (3.14).

In order to ensure that operator $\mathscr{F}$ has a fixed point, we will prove that it is compact.

We first observe that, from Definition 2.17 of a L_α^1-Carathéodory function and the definition of τ, function $f(\cdot,\tau(\cdot,x(\cdot)))$ is Lebesgue measurable on I for any continuous function x [4, Theorem 1.1], and there exists $h \in L_\alpha^1(I)$ such that

$$|f(t,\tau(t,x(t)))| \leq h(t), \quad for \ a.e. \ t \in I \text{ and all } x \in C(I).$$

The continuity of operator $\mathscr{F}$ follows from the continuous dependence with respect to x of function f, the definition of τ and the Lebesgue's Dominated convergence Theorem.

To see that $\mathscr{F}(C(I))$ is a relatively compact set on $C(I)$, consider $x \in C(I)$. Therefore

$$|\mathscr{F}(x)(t)| \leq \|h\|_{L_\alpha^1(I)} + \max\{|\gamma(b)|,|\delta(b)|\}, \quad \text{for all } t \in I,$$

and, as a consequence, $\mathscr{F}(C(I))$ is uniformly bounded on $C(I)$.

This set is also equicontinuous since for every $t_1 < t_2 \in I$,

$$|\mathscr{F}(x)(t_2) - \mathscr{F}(x)(t_1)| \leq \int_{t_1}^{t_2} |h(s)|d_\alpha s.$$

By Arzelà-Ascoli Theorem, we conclude that the set $\mathscr{F}(C(I))$ is relatively compact in $C(I)$. Hence, $\mathscr{F}$ is compact.

As a consequence, the Schauder fixed-point Theorem ensures that operator $\mathscr{F}$ has a fixed point.

From previous arguments, we conclude that such fixed point is a solution of problem (1.1), (4.1), and lies on $[\gamma,\delta]$. $\square$

Concerning the problem (1.1), (4.2), we introduce the following definition of lower and upper solution related to such problem.

Definition 4.3. *Let $\gamma \in W_{0,b}^{\alpha,1}(I)$. We say that γ is a lower solution of the boundary value problem* (1.1), (4.2) *if*

(i) $\gamma^{(\alpha)}(t) \geq f(t,\gamma(t)), \quad a.e. \ t \in I$;

(ii) $L_2(\gamma(0),\gamma) \geq 0.$

Let $\delta \in W_{0,b}^{\alpha,1}(I)$. We say that δ is an upper solution of the boundary value problem (1.1), (4.2) *if*

(i) $\delta^{(\alpha)}(t) \leq f(t, \delta(t))$,   a.e. $t \in I$;

(ii) $L_2(\delta(0), \delta) \leq 0$.

Analogously to Theorem 4.2, one can prove the following result.

Theorem 4.4. *If there exist γ and δ in $W_{0,b}^{\alpha,1}(I)$, a pair of reversed ordered lower and upper solutions respectively for problem (1.1), (4.2), such that $\gamma \geq \delta$ on I, and L_2 is a continuous function in $[\delta(0), \gamma(0)] \times [\delta, \gamma]$, nonincreasing in the second variable on $[\delta, \gamma]$, then problem (1.1), (4.2), has at least one solution $x \in [\delta, \gamma]$.*

Proof. The proof follows the same steps as Theorem 4.2. In this case, we consider the following modified problem

$$\begin{cases} x^{(\alpha)}(t) = f(t, \tau(t, x(t))), & \text{for a.e. } t \in I, \\ x(0) = \tau(0, x(0) - L_2(\tau(0, x(0)), \tau(\cdot, x(\cdot)))), \end{cases} \quad (4.4)$$

where, for any $x \in C(I)$, the function τ is defined as

$$\tau(t, x(t)) = \max \left\{ \delta(t), \min\{x(t), \gamma(t)\} \right\}.$$

$\square$

In the particular case in which the boundary conditions are defined only at the extremes of the interval, we can deduce as a direct corollary, the following result.

Corollary 4.5. *Assume that there exist γ and $\delta \in W_{0,b}^{\alpha,1}(I)$, a pair of well ordered lower and upper solutions (either $\gamma \geq \delta$ or $\gamma \leq \delta$) for problem*

$$x^{(\alpha)}(t) = f(t, x(t)), \quad \text{for a.e. } t \in I, \quad L(x(0), x(b)) = 0,$$

with L a continuous function nondecreasing in the first variable and nonincreasing in the second one on its domain of definition. Then this problem has at least one solution $x \in W_{0,b}^{\alpha,1}(I)$ lying between γ and δ.

We note that previous result can be automatically applied to the linear boundary conditions $L(x, y) = a_0 x - b_0 y - \lambda_0$, with a_0, b_0 and $\lambda_0 \in \mathbb{R}$, $a_0, b_0 \geq 0$ and $a_0 + b_0 > 0$, which includes the periodic case ($a_0 = b_0 = 1, \lambda_0 = 0$) and the initial and terminal problems.

5. Examples

In this section, we present three examples where we apply Theorems 4.2 and 4.4 to some particular cases.

Example 5.1. *Consider the linear boundary value problem:*

$$x^{(\frac{1}{3})}(t) = \frac{x^2(t)}{2} - t(1-t), \ a.e.\ t \in [0,1], \quad x(1) = \sqrt{|x(1/2)|}.$$

$$(5.1)$$

This problem is a particular case of (1.1), (4.1), with $\alpha = \frac{1}{3}$, $f(t, x) = x^2/2 - t(1-t)$ and

$$L_1(x, y) = \sqrt{|x(1/2)|} - y.$$

Obviously, function f is a $L_{1/3}^1$-Carathéodory function, and $\delta(t) = 2$, $\gamma(t) = 0$ are upper and lower solutions of the boundary-value problem (5.1), respectively with $\gamma(t) \leq \delta(t)$ for $t \in [0, 1]$. To see this, it is enough to verify the following inequalities

$$\delta^{(\frac{1}{3})}(t) = 0 \leq f(t, \delta(t)) = 2 - t(1-t), \sqrt{|\delta(1/2)|} - \delta(1) \leq 0,$$

and

$$\gamma^{(\frac{1}{3})}(t) = 0 \geq f(t, \gamma(t)) = -t(1-t), \sqrt{|\gamma(1/2)|} - \gamma(1) = 0.$$

By Theorem 4.2, problem (5.1) has a least one solution $x \in W_{0,1}^{\frac{1}{3},1}([0,1])$, such that $0 \leq x(t) \leq 2$, for all $t \in [0, 1]$.

Example 5.2. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x^{(\frac{1}{2})}(t) = t\, e^{t\, \sin^2(x(t))} & a.e.\ t \in [0,2], \\ x(0) - \sin^2(x(0)) = \frac{1}{3} \max_{t \in [0,1]} \{x(t)\}. \end{cases} \quad (5.2)$$

This problem is a particular case of (1.1), (4.2), with $\alpha = \frac{1}{2}$, $f(t, x) = t\, e^{t\, \sin^2(x)}$ and

$$L_2(x, y) = x - \sin^2(x) - \frac{1}{3} \max_{t \in [0,1]} \{y(t)\}.$$

It is clear that f is a $L_{1/2}^1$-Carathéodory function, L_2 is a continuous function in $(x, y) \in [\delta(0), \gamma(0)] \times [\delta, \gamma]$, and nonincreasing in $y \in [\delta, \gamma]$, with $\delta(t) = 0 \leq \gamma(t) = e^{t+1}$ for $t \in [0, 2]$.

The fact that δ and γ are upper and lower solutions of problem (5.2) follows from the fact that

$$\delta^{(\frac{1}{2})}(t) = 0 \leq f(t, \delta(t)) = t,\ t \in [0, 2],$$

$$\delta(0) - \sin^2(\delta(0)) - \frac{1}{3} \max_{t \in [0,1]} \{\delta(t)\} = 0$$

and

$$\gamma^{(\frac{1}{2})}(t) = \sqrt{t}e^{t+1} \geq f(t, \gamma(t)) = t\, e^{t\, \sin^2(e^{t+1})},\ t \in [0, 2],$$

$$\gamma(0) - \sin^2(\gamma(0)) - \frac{1}{3} \max_{t \in [0,1]} \{\gamma(t)\} \geq 0.$$

Theorem 4.4, implies that problem (5.2) has a least one solution $x \in W_{0,1}^{\frac{1}{2},1}([0,2])$, such that $0 \leq x(t) \leq e^{t+1}$, for all $t \in [0, 2]$.

Example 5.3. *Consider the nonlinear boundary value problem with functional boundary conditions:*

$$\begin{cases} x'(t) = \dfrac{x^3(t)+1-2t}{\sqrt{t}} & a.e.\ t \in [0,1], \\ x(1) - \cos(\frac{\pi}{2}\,x(1)) = \int_{\frac{1}{2}}^{1} x(s)ds. \end{cases} \tag{5.3}$$

This problem is a particular case of (1.1), (4.1), *with* $\alpha = 1$, $f(t,x) = \dfrac{x^3+1-2t}{\sqrt{t}}$ *and*

$$L_1(x,y) = \int_{\frac{1}{2}}^{1} x(s)ds - y + \cos(\frac{\pi}{2}\,y).$$

It is clear that f is a L^1-Carathéodory function, L_1 is a continuous function in $(x,y) \in [\gamma,\delta] \times [\gamma(1),\delta(1)]$, and nondecreasing in $x \in [\gamma,\delta]$, with $\gamma(t) = -1 \leq \delta(t) = 1$ for $t \in [0,1]$.

The fact that γ and δ are lower and upper solutions of problem (5.3) *follows from the fact that*

$$\gamma'(t) = 0 \geq f(t,\gamma(t)) = -2\sqrt{t},\ t \in [0,1],$$

$$\int_{\frac{1}{2}}^{1} \gamma(s)ds - \gamma(1) + \cos(\frac{\pi}{2}\,\gamma(1)) \geq 0$$

and

$$\delta'(t) = 0 \leq f(t,\delta(t)) = \frac{2(1-t)}{\sqrt{t}},\ t \in [0,1],$$

$$\int_{\frac{1}{2}}^{1} \delta(s)ds - \delta(1) + \cos(\frac{\pi}{2}\,\delta(1)) \leq 0.$$

Theorem 4.2, *implies that problem* (5.3) *has a least one solution $x \in W_{0,1}^{1,1}([0,1])$, such that $-1 \leq x(t) \leq 1$, for all $t \in [0,1]$.*

Acknowledgements

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