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On equivalent conditions of quasi quaternion normal bimatrices

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Abstract

In this paper we introduced the concept of quasi quaternion normal bimatrices and some theorems are derived. We obtain the sum and product of quasi quaternion with normal bimatrix.

Keywords

Normal matrix ,Quasi matrix, bimatrix, Quaternion bimatrix.

AMS Subject Classification

15A09, 15B05,15A99, 15B99.

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1. Introduction

A list of conditions on an $n \times n$ matrix A, equivalent to its being normal, published twenty years ago by Grone, Johnson, Sa, and Wolkowicz has proved to be very useful [1] it contains 70 conditions, each equivalent to the original definition of normality. Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model [7].

A square complex matrix $A_B \in C_{n\times n,}$ is called normal if $A_B A_B^{\circ} = A_B^{\circ} A_B$ where $A_B^* = \bar{A}_B^T$ denotes the conjugate transpose of A[5]. There are many equivalent conditions in the literature for a square matrix to be normal[1].Our purpose to present a list of condition on $n \times n$ quatemion bimatrices A each of which is equivalent to A being normal. We define A to be quatemion quasi normal[2], [7], if and only if

$$A_B A_B^{c\tau} = A_B^T A_B^c$$
$$A_1 A_1^{cT} \cup A_2 A_2^{cT} = A_1^T A_1^c \cup A_2^T A_2^c$$

Though many conditions we have listed are similar, the list could be expanded much further by including variations on the statement of commutatively, etc.

Also, we have refrained from going beyond characterizations of the quasi normal of a single bimatrix and not included results about sums or products of quaternion quasi normality bimatrices etc.

The condition of quasi normality is a strong one, but as it includes the Hermitian, Unitary and Skew Hermitian bimatrices, it is an important one which often appears as the appropriate level of generality in highly algebraic work and for numerical results dealing with perturbation analysis.

2. Preliminaries and Definitions

Definition 2.1. *A matrix $A \in C_{mn}$ is said to be hermitian if $A = A^*$. That is, $a_{ij} = \overline{a_{ji}}, i, j = 1, 2 \dots .n$.*

Definition 2.2. *A matrix $A \in C_{nxn}$ is said to be skew hermitian if $A = -A^*$. That is $_{th}\, a_{ij} = -\overline{a_{ji}}$, $i, j = 1, 2 \dots .n$*

Definition 2.3. *A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A$. That is, $a_{ij}\overline{a_{n-j+1i}} = \overline{a_{jn-i+1}} a_{ij}$; $i, j = 1, 2 \dots n$*

Definition 2.4. *A matrix $A \in C_{n\wedge n}$ is said to be unitary if $AA^* = A^*A = I$.*

Definition 2.5. *A bimatrix A_B is defined as the union of two square or rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$, where*

$A_1 \neq A_2$ with

$$A_1 = \begin{bmatrix} a^1_{11} & a^1_{12} & \ldots\ldots & a^1_{1n} \\ a^1_{21} & a^1_{22} & \ldots\ldots & a^1_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a^1_{m1} & a^1_{m2} & \cdots & a^1_{mn} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a^2_{11} & a^2_{12} & \ldots\ldots & a^2_{1n} \\ a^2_{21.} & a^2_{22} & \ldots\ldots & a^2_{2n.} \\ \vdots & \vdots & & \vdots \\ a^2_{m1} & a^2_{m2} & \cdots & a^2_{mn} \end{bmatrix}$$

'$\cup$'is just the notational convenience (symbol) only.

Definition 2.6. *A quasi normal bimatrix A is defined to be a quaternion quasi normal bimatrix such that,*

$$A_B A_B^{CT} = A_B^T A_B^C$$
$$\Rightarrow (A_1 \cup A_2)\left(A_1^{CT} \cup A_2^{CT}\right) = \left(A_1^T \cup A_2^T\right)\left(A_1^C \cup A_2^C\right)$$
$$A_1 A_1^{CT} \cup A_2 A_2^{CT} = A_1^T A_1^C \cup A_2^T A_2^C$$

Definition 2.7. *A square bimatrix A_B is normal if it commutes with its conjugate transpose $A_B^* A_B = A_B A_B^*$ if $A_B = A_B^T$ is real, then $A_B^T A_B = A_B A_B^T$ obviously unitary bimatrices $\left(A_B^* = A_B^{-1}\right)$, Hermitian bimatries $(A_B^* = A_B)$, and skew-Hermitian bimatrices ($A_B{}^* = -A_B$) are all normal. But there exist normal bimatrices not belong to any of these.*

Definition 2.8. *The bimatrix is called non-negative if all entries of are non-negative. The non-negative bimatrix is called quasi positive if there is a natural number such that all entries of are positive.*

3. Some theorems and examples

Theorem 3.1. *If A_B is quaternion quasinormal then A_B^C is quaternion quasinormal bimatrix for any conjugate.*

Proof. Given A_B is quaternion quasi normal. We have to prove A_B^C is quaternion quasi normal.

$$A_B^C\left(A_B^C\right)^{CT} = A_B^C\left(A_B^{CT}\right)^c$$
$$(A_1^c \cup A_2^c)(A_1^c \cup A_2^c)^{cT} = (A_1^c \cup A_2^c)^T (A_1^c \cup A_2^c)^c$$
$$A_1^C A_1^T \cup A_2^C A_2^T = A_1^{CT} A_1 \cup A_2^{CT} A_2$$

Hence A_B^C is quaternion quasinormal. □

Theorem 3.2. *If A_B is quaternion quasi normal then A_B^{CT} is quaternion quasi normal.*

Proof. Given A_B is quaternion quasinormal. We have to prove A_B^{CT} is quaternion quasi normal.

$$A_B^{CT}\left(A_B^{CT}\right)^{CT} = \left(A_B^{CT}\right)^T \left(A_B^{CT}\right)^C$$
$$\left(A_1^{CT} \cup A_2^{CT}\right)\left(A_1^{CT} \cup A_2^{CT}\right)^{cT} = \left(A_1^{CT} \cup A_2^{CT}\right)^T \left(A_1^{CT} \cup A_2^{CT}\right)^c$$
$$A_1 A_1^{CT} \cup A_2 A_2^{CT} = A_1^T A_1^C \cup A_2^T A_2^C$$

Hence, A_B^{CT} is quaternion quasinormal. □

Theorem 3.3. *If A_B is quaternion quasi normal then A_B^T is quaternion quasi normal.*

Proof. Given A_B is quaternion quasi normal.
We have to prove: A_B^T is quaternion quasi normal.

$$A_B^T\left(A_B^T\right)^{CT} = \left(A_B^T\right)^T \left(A_B^T\right)^c$$
$$\left(A_1^T \cup A_2^T\right)\left(A_1^T \cup A_2^T\right)^{cT} = \left(A_1^T \cup A_2^T\right)^T \left(A_1^T \cup A_2^T\right)^c$$
$$A_1^T A_1^C \cup A_2^T A_2^C = A_1 A_1^{CT} \cup A_2 A_2^{CT}$$
$$A_1 A_1^{CT} \cup A_2 A_2^{CT} = A_1^T A_1^C \cup A_2^T A_2^C$$

Hence A_B^T is quaternion quasinormal. □

Theorem 3.4. *If the sum if quaternion quasi normal bimatrices A_B and B_B are quaternion quasi normal then,*

$$A_B B_B^{cT} + B_B A_B^{CT} = \left(B_B^{cT} A_B + A_B^{CT} B_B\right)^T.$$

Proof.

$$A_B B_B^{CT} + B_B A_B^{CT} = \left(B_B^{CT} A_B + A_B^{CT} B_B\right)^T$$
$$\left(A_1 B_1^{cT} \cup A_2 B_2^{CT}\right) + \left(B_1 A_1^{CT} \cup B_2 A_2^{cT}\right)$$
$$= \left(B_1^{cT} A_1 \cup B_2^{CT} A_2\right)^T + \left(A_1^{CT} B_1 \cup A_2^{CT} B_2\right)^T$$
$$\left(A_1 B_1^{CT} + B_1 A_1^{CT}\right) \cup \left(A_2 B_2^{CT} + B_2 A_2^{CT}\right)$$
$$= \left(B_1^C A_1^T + A_1^C B_1^T\right) \cup \left(B_2^C A_2^T + A_2^C B_2^T\right)$$
$$\therefore A_B B_B^{CT} + B_B A_B^{CT} = \left(B_B^{CT} A_B + A_B^{CT} B_B\right)^T$$

□

Theorem 3.5. *If A_B and B_B are quaternion quasi normal bimatrices then the product of $A_B B_B$ is also a quaternion quasinormal.*

Proof. Given A_B and B_B are quaternion quasi normal bimatrices.

$$(A_B B_B)(A_B B_B)^{cT} = (A_B B_B)^T (A_B B_B)^c$$
$$\Rightarrow (A_1 B_1 \cup A_2 B_2)(A_1 B_1 \cup A_2 B_2)^{cT}$$
$$= (A_1 B_1 \cup A_2 B_2)^T (A_1 B_1 \cup A_2 B_2)^c$$
$$\Rightarrow A_1 B_1 A_1^{CT} B_1^{cT} \cup A_2 B_2 A_2^{CT} B_2^{CT} = A_1^T B_1^T A_1^C B_1^C \cup A_2^T B_2^T A_2^C B_2^c$$

$\therefore A_B B_B$ is also a quaternion quasi normal. □

Theorem 3.6. *If A_B is quaternion quasi normal bimatrices then A_B^{-1} is quaternion quasi normal for invertible A_B.*

Proof. Let A_B is quaternion quasi normal bimatrices then to prove A_B^{-1} is quaternion quasi normal for invertible A_B.
W.K.T, $A_B A_B^{CT} = A_B^T A_B^C$. Taking inverse on both sides,

$$A_B^{-1}\left(A_B^{-1}\right)^{CT} = \left(A_B^{-1}\right)^T \left(A_B^{-1}\right)^C$$
$$\Rightarrow \left(A_1^{-1} \cup A_2^{-1}\right)\left(A_1^{-1} \cup A_2^{-1}\right)^{cT} = \left(A_1^{-1} \cup A_2^{-1}\right)^T \left(A_1^{-1} \cup A_2^{-1}\right)^c$$
$$A_1^{-1} A_1^{-1^{CT}} \cup A_2^{-1} A_2^{-1^{CT}} = A_1^{-1^T} A_1^{-1^C} \cup A_2^{-1^T} A_2^{-1^C}$$

$\therefore A_B^{-1}$ is quaternion quasi normal for invertible A_B. □

Theorem 3.7. *If A_B is quaternion quasi normal bimatrices then $P(A_B)$ is quaternion quasi normal bimatrices for any polynomial of degree*

Proof. Let

$$P(A_B) = \alpha_0 + \alpha_1 A_B + \alpha_2 A_B^2 + \dots\dots\dots + \alpha_n A_B^n$$

$$P\left(A_B A_B^{CT}\right) = \alpha_0 + \alpha_1\left(A_B A_B^{CT}\right) + \alpha_2\left(A_B A_B^{CT}\right)^2 + \dots + \alpha_n\left(A_B A_B^{CT}\right)^n$$

$$\Rightarrow P\left(A_B A_B^{CT}\right) = P\left(A_B^T A_B^C\right)$$

$$P\left(A_1 A_2 \cup A_1^{CT} A_2^{CT}\right) = P\left(A_1^T A_2^T \cup A_1^C A_2^C\right)$$

$$P\left(A_1 A_1^{cT}\right) \cup P\left(A_2 A_2^{CT}\right) = P\left(A_1^T A_2^C\right) \cup P\left(A_2^T A_2^C\right)$$

$\therefore P(A_B)$ is quaternion quasinormal bimatrices. □

Theorem 3.8. *If a quaternion bimatrices $A_B \in H_{m\times n}$ is defined as double representation of the form $A_B = A_{0B} + A_{1B} j$ where A_{0B} and A_{1B} are normal then A_B^T and A_B^{CT} are also a quaternion quasi normal.*

Proof. $A_B = A_{0B} + A_{1B} j$
To prove: A_B^T is quaternion quasi normal.

$$A_B^T\left(A_B^T\right)^{CT} = \left(A_B^T\right)^T\left(A_B^T\right)^C$$

$$\left(A_1^T \cup A_2^T\right)\left(A_1^T \cup A_2^T\right)^{CT} = \left(A_1^T \cup A_2^T\right)^T\left(A_1^T \cup A_2^T\right)^C$$

$$A_1^T A_1^C \cup A_2^T A_2^C = A_1 A_1^{CT} \cup A_2 A_2^{CT}$$

$$\Rightarrow A_B^T = A_{0B}^T + A_{1B}^T j$$

$$A_B^T\left(A_B^T\right)^{CT} = \left(A_{0B}^T + A_{1B}^T j\right)\left(A_{0B}^C - A_{1B}^C j\right) = A_{0B} A_0^{CT} - A_{1B} A_1^{CT} j$$

Since, $A_0 A_0^{CT} = A_{0B}^T A_{0B}^C$ and $A_{1B} A_1^{CT} = A_{1B}^T A_{1B}^C$

$$\left(A_B^T\right)\left(A_B^T\right)^C = \left(A_{0B}^T + A_{1:B}^T j\right)^T\left(A_{0B}^T + A_{1B}^T j\right)^C = \left(A_B^T\right)^T\left(A_B^T\right)^C$$

Next to prove A_B^{cT} is quaternion quasi normal

$$A_B^{CT}\left(A_B^{CT}\right)^{CT} = \left(A_B^{CT}\right)^T\left(A_B^{CT}\right)^c$$

Let us consider $A_B = A_{0B} + A_{1B} j$

$$\Rightarrow A_B^{CT}\left(A_B^{CT}\right)^{CT} = \left(A_0^{CT} - A_{1B}^{CT} j\right)\left(A_{0B} + A_{1B} j\right) = A_{0B}^T A_{0B}^C - A_{1B}^T A_{1B}^C j$$

By definition,

$$A_B A_B^{CT} = A_B^{CT} A_B = A_B^T A_B^C \quad = \left(A_{0B}^C - A_{1B}^C j\right.$$

$$= A_B^{CT}\Big)^T\left(A_B^{CT}\right)^C$$

$$A_B^{CT}\left(A_B^{CT}\right)^{CT} = \left(A_B^{CT}\right)^T\left(A_B^{CT}\right)^C$$

Hence proved. □

4. Conclusion

In this paper, some equivalent conditions for a bimatrix to be quasi quaternion normal matrices are discussed. This concept reflects the quasi quaternion normality arises in many ways. In this list some equivalent conditions need an additional requirement of non-singularity or distinct eigen bivalues.

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