Malaya Journal of Matematik, Vol. 9, No. 1, 782-788, 2021

https://doi.org/10.26637/MJM0901/0138

Hamiltonian fuzzy labeling and Hamiltonian fuzzy magic labeling of graphs

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Abstract

In this paper, we introduce two fuzzy labeling concepts such as hamiltonian fuzzy labeling and hamiltonian fuzzy magic labeling of graphs. Using this definition, we find hamiltonian fuzzy labeling and hamiltonian fuzzy magic labeling of some graphs.

Keywords

Hamiltonian fuzzy labeling, Hamiltonian fuzzy magic labeling, umbrella graph, octopus graph, comb graph, hurdle graph.

AMS Subject Classification

05C72, 05C78.

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Article History: Received **14** January **2021**; Accepted **01** March **2021**

Contents

1. Introduction

The first definition of fuzzy graph was introduced by Kaufmann in 1973, based on Zadeh's fuzzy relations in 1971[11]. A more elaborate deinition is due to Azriel Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graph in 1975[12].A fuzzy graph is the generalisation of the crisp graph. Therefore it is natural that many properties are similar to crisp graph and also it deviates at many places. In crisp graph, A bijection $f : V \cup E \longrightarrow N$ that assigns to each vertex and/or edge if $G = (V,E)$, a unique natural number is called a labeling.

The concept of magic labeling in crisp graph was motivated by the notion of magic squares in number theory. The concept of Hamiltonian labeling was introduced by Willem Renzema and Ping Zhang in 2009[9]. The concept of Fuzzy labeling was introduced by Nagoor Gani and Rajalaxmi in 2012[7].Also, the concept of Fuzzy magic labeling was introduced by Nagoor Gani and Rajalaxmi in 2014[1]. In this paper, we introduce two fuzzy labeling concepts such as hamiltonian fuzzy labeling and hamiltonian fuzzy magic labeling of graphs.

A **fuzzy graph** $G = (V,\sigma,\mu)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma : V \longrightarrow [0,1]$ and $\mu : E \longrightarrow [0,1]$ such that for all $x,y \in V, \mu(x,y) \leq \sigma(x) \wedge \sigma(y)$.Let $G = (V,\mu,\rho)$ be a fuzzy graph. Then the **order** of G is defined to be $O(G) = \sum_{v \in V} \mu(v)$.

A **path** P in a fuzzy graph $G = (\sigma,\mu)$ is a sequence of distinct vertices $x_0,x_1,x_2,....,x_n$ such that $\mu(x_{i-1}x_i) > 0; i = 1,2,3,...,n$. Here n is called the **length** of the path. The **strength** of P is defined to be $\wedge_{i=n}^{n} \mu(x_{i-1}x_i)$.In words, the strength of a path is defined to be the weight of the weakest edge.The **strength of connectedness** between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x,y)$.

A fuzzy graph $G = (V,\sigma,\mu)$ is **connected** if for every $x,y \in V, CONN_G(x,y) > 0$. An **arc** of a fuzzy graph $G = (V,\sigma,\mu)$ is called **strong** if its weight is at least as great as the strength of the connectedness of its end nodes when it is deleted. An $x-y$ path P is called **strong path** if P contains only strong arcs. The length of a longest strong $u-v$ path between two nodes u and v in a connected fuzzy graph G

is called **fuzzy detour** $g-$**distance** from u to v, denoted by $D_g(u,v)$.

A **Hamiltonian labeling** of a connected graph G of order n is an assignment $c : V(G) \longrightarrow N$ of labels to the vertices of G such that $|c(u)-c(v)|+D(u,v) \geq n$, for every two distinct vertices u and v of G. A graph $G=(\sigma,\mu)$ is said to be a **fuzzy labeling graph**, if $\sigma : V \longrightarrow [0,1]$ and $\mu : E \longrightarrow [0,1]$ is a bijective such that the membership value of edges and vertices are distinct and $\mu(u,v) < \sigma(u) \wedge \sigma(v)$ for all $u,v \in V$. A fuzzy labeling graph is said to be a **fuzzy magic graph** if $\sigma(u)+\mu(u,v)+\sigma(v)$ has a same value for all $u,v \in V$ which is denoted as $m_0(G)$.

A **diamond graph** is a graph on 4 vertices with exactly one pair of non-adjacent vertices. A **butterfly** graph is a planar undirected graph with 5 vertices and 6 edges. A **Umbrella graph** $U(m,n)$ is the graph obtained by joining a path P_n with the central vertex of a fan F_m. An **Octopus graph** $O_n, n \geq 2$ can be constructed by joining a fan graph $F_n, n \geq 2$ to a star graph $K_{1,n}$ by with a common vertex, where n is any positive integer. i.e.$O_n = F_n + K_{1,n}$. A graph obtained from a path P_n by attaching a pendant edge to every internal vertices of the path is called **Hurdle graph** with $n-2$ hurdles and is denoted by Hd_n. Let P_n be a path with n vertices. The **comb graph** is defined as $P_n \bigodot K_1$. It has $2n$ vertices and $2n-1$ edges.

2. Hamiltonian Fuzzy Labeling of Graphs

Definition 2.1. *A fuzzy labeling graph is said to be a* ***Hamiltonian fuzzy labeling graph*** *if* $|\sigma(u)-\sigma(v)|+D_g(u,v) \geq \sum_{u \in V} \sigma(u) \forall\, u,v \in V$, *where* $D_g(u,v)$ *is the fuzzy detour* $g-$ *distance from u to v and* $\sum_{u \in V} \sigma(u)$ *is the order of fuzzy graph.*

Example 2.2. *The following figure 1 shows that diamond graph is a hamiltonian fuzzy labeling graph.*

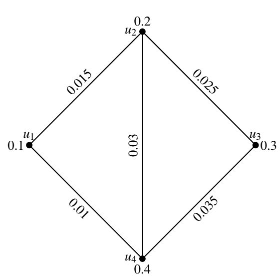

Figure 1. Hamiltonian fuzzy labeling of diamond graph

Theorem 2.3. *The Umbrella graph* $U(m,n), m \geq 3, n \geq 1$ *is a Hamiltonian fuzzy labeling graph.*

Proof. Let $V\big(U(m,n)\big) = \{u_i, v_j : 1 \leq i \leq m+1, 1 \leq j \leq n\}$ be the vertex set of the umbrella graph $U(m,n)$. Let $E\big(U(m,n)\big) = \{u_i u_{i+1}, u_k u_{m+1}, u_{m+1} v_1, v_j v_{j+1} : 1 \leq i \leq m-1, 1 \leq k \leq m, 1 \leq j \leq n-1\}$ be the edge set of the umbrella graph $U(m,n)$. The umbrella graph $U(m,n)$ has $m+n+1$ vertices and $m+n+2$ edges.

Define a labeling $\sigma : V\big(U(m,n)\big) \longrightarrow [0,1]$ by

$$\sigma(u_i) = i \times 10^{-(m+n-2)}, 1 \leq i \leq m-1,$$
$$\sigma(v_j) = (j+m+1) \times 10^{-(m+n-2)}, 1 \leq j \leq n.$$

Define a labeling $\mu : E\big(U(m,n)\big) \longrightarrow [0,1]$ by

$$\mu(u_i u_{i+1}) = \left(\frac{2i+1}{2}\right) \times 10^{-(m+n-1)}, 1 \leq i \leq m-1,$$
$$\mu(u_k u_{m+1}) = k \times 10^{-(m+n)}, 1 \leq k \leq m-1,$$
$$\mu(u_{m+1} v_1) = \left(\frac{2m+3}{2}\right) \times 10^{-(m+n)},$$
$$\mu(v_j v_{j+1}) = \left(\frac{2(j+m)+3}{2}\right) \times 10^{-(m+n)}, 1 \leq j \leq n-1.$$

Then the order of the fuzzy umbrella graph

$$U(m,n) = \sum_{u \in V\left(U(m,n)\right)} \sigma(u) = \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

Clearly, the membership values of all vertices and all edges are distinct. Therefore, it is clear that $\mu(uv) < \sigma(u) \wedge \sigma(v)\ \forall\, u,v \in V\big(U(m,n)\big)$.

Note that, all arcs except $(u_k u_{m+1}, 1 \leq k \leq m-1)$ are strong arcs. Then the fuzzy detour g-distance between two vertices are

$$D_g(u_i,u_j) = j-i, 1 \leq i < j \leq m,$$
$$D_g(v_i,v_j) = j-i, 1 \leq i < j \leq n,$$
$$D_g(u_k,u_{m+1}) = m+1-k, 1 \leq k \leq n, m \geq 3,$$
$$D_g(u_{m+1},v_i) = i, 1 \leq i \leq n, m \geq 3,$$
$$D_g(u_i,v_j) = m+j-i+1, 1 \leq i \leq m, 1 \leq j \leq n.$$

Case 1: $D_g(u_i,u_j) = j-i,\ 1 \leq i < j \leq m$.
Let $u_i, u_j \in V\big(U(m,n)\big)$.
Then $|\sigma(u_i)-\sigma(u_j)|+D_g(u_i,u_j)$

$$= |i \times 10^{-(m+n-2)} - j \times 10^{-(m+n-2)}| + j - i.$$
$$= |(i-j) \times 10^{-(m+n-2)}| + j - i.$$
$$\geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

Case 2: $D_g(v_i,v_j) = j-i,\ 1 \leq i < j \leq n$.
Let $v_i, v_j \in V\big(U(m,n)\big)$.
Then $|\sigma(v_i)-\sigma(v_j)|+D_g(v_i,v_j)$

$$= |(i+m+1) \times 10^{-(m+n-2)} - (j+m+1) \times 10^{-(m+n-2)}| + j - i.$$
$$= |(i-j) \times 10^{-(m+n-2)}| + j - i.$$
$$\geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

Case 3: $D_g(u_k, u_{m+1}) = m+1-k, 1 \leq k \leq n, m \geq 3$.
Let $u_k, u_{m+1} \in V\big(U(m,n)\big)$.
Then $|\sigma(u_k) - \sigma(u_{m+1})| + D_g(u_k, u_{m+1})$

$$= |k \times 10^{-(m+n-2)} - (m+1) \times 10^{-(m+n-2)}| + m+1-k.$$
$$= |(k-m-1) \times 10^{-(m+n-2)}| + m+1-k.$$
$$\geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

Case 4: $D_g(u_{m+1}, v_i) = i, 1 \leq i \leq n, m \geq 3$.
Let $u_{m+1}, v_i \in V\big(U(m,n)\big)$.
Then $|\sigma(u_{m+1}) - \sigma(v_i)| + D_g(u_{m+1}, v_i)$

$$= |(m+1) \times 10^{-(m+n-2)} - (j+m+1) \times 10^{-(m+n-2)}| + i.$$
$$= |-j \times 10^{-(m+n-2)}| + i.$$
$$\geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

Case 5: $D_g(u_i, v_j) = m+j-i+1, 1 \leq i \leq m, 1 \leq j \leq n$.
Let $u_i, v_j \in V\big(U(m,n)\big)$.
Then $|\sigma(u_i) - \sigma(v_j)| + D_g(u_i, v_j)$

$$= |i \times 10^{-(m+n-2)} - (j+m+1) \times 10^{-(m+n-2)}|$$
$$+ m+j-i+1.$$
$$= |(i-j-m-1) \times 10^{-(m+n-2)}| + j-i+m+1.$$
$$\geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)}.$$

In each case, $|\sigma(u) - \sigma(v)| + D_g(u,v) \geq \left(\frac{(m+n+1)(m+n+2)}{2}\right) \times 10^{-(m+n-2)} \ \forall\, u, v \in V\big(U(m,n)\big)$.

Hence, the umbrella graph $U(m,n), m \geq 3, n \geq 1$ is a Hamiltonian fuzzy labeling graph. □

Example 2.4. *The following figure 2 shows that Umbrella graph $U(6,3)$ is a hamiltonian fuzzy labeling graph.*

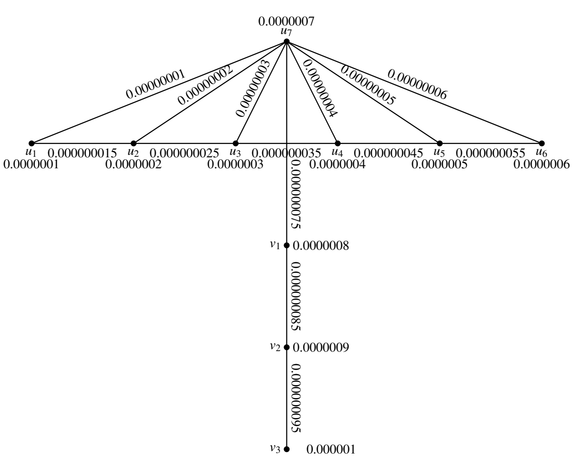

Figure 2. Hamiltonian fuzzy labeling of umbrella graph $U(6,3)$

Theorem 2.5. *The Octopus Graph $O_n, n \geq 2$ is a Hamiltonian fuzzy labeling graph.*

Proof. Let $V(O_n) = \{u_i, v_j : 1 \leq i \leq n+1, 1 \leq j \leq n\}$ be the vertex set of the octopus graph O_n. Let $E(O_n) = \{u_i u_{i+1}, u_1 u_{n+1}, u_{n+1} v_j : 1 \leq i, j \leq n\}$ be the edge set of the octopus graph O_n. The octopus graph O_n has $2n+1$ vertices and $3n-1$ edges.

Define a labeling $\sigma : V(O_n) \longrightarrow [0,1]$ by

$$\sigma(u_i) = (n+i) \times 10^{-n}, 1 \leq i \leq n+1,$$
$$\sigma(v_j) = j \times 10^{-n}, 1 \leq j \leq n.$$

Define a labeling $\mu : E(O_n) \longrightarrow [0,1]$ by

$$\mu(u_i u_{i+1}) = \left(\frac{2(n+i)+1}{2}\right) \times 10^{-(n+1)}, 1 \leq i \leq n,$$
$$\mu(u_{n+1} u_i) = (n+i) \times 10^{-(n+1)}, 1 \leq i \leq n,$$
$$\mu(u_{n+1} v_j) = j \times 10^{-(n+1)}, 1 \leq j \leq n.$$

Then the order of the fuzzy octopus graph

$$O_n = \sum_{u \in V(O_n)} \sigma(u) = \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Clearly, the membership values of all vertices and all edges are distinct. Therefore, it is clear that $\mu(uv) < \sigma(u) \wedge \sigma(v) \ \forall\, u, v \in V(O_n)$.

Note that, all arcs except $(u_{n+1} u_i, 1 \leq i \leq n-1)$ are strong arcs. Then the fuzzy detour $g-$distance between two vertices are

$$D_g(u_i, u_j) = j-i, 1 \leq i < j \leq n+1,$$
$$D_g(v_i, v_j) = 2, 1 \leq i, j \leq n,$$
$$D_g(u_i, v_j) = n+2-i, 1 \leq i, j \leq n,$$
$$D_g(u_{n+1}, u_i) = n+1-i, 1 \leq i \leq n,$$
$$D_g(u_{n+1}, v_i) = 1, 1 \leq i \leq n.$$

Case 1: $D_g(u_i, u_j) = j-i, 1 \leq i < j \leq n+1$.
Let $u_i, u_j \in V(O_n)$. Then $|\sigma(u_i) - \sigma(u_j)| + D_g(u_i, u_j)$

$$= |(n+i) \times 10^{-n} - (n+j) \times 10^{-n}| + j-i.$$
$$= |(i-j) \times 10^{-n}| + j-i.$$
$$\geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Case 2: $D_g(v_i, v_j) = 2, 1 \leq i, j \leq n$.
Let $v_i, v_j \in V(O_n)$. Then $|\sigma(v_i) - \sigma(v_j)| + D_g(v_i, v_j)$

$$= |i \times 10^{-n} - j \times 10^{-n}| + j-i.$$
$$= |(i-j) \times 10^{-n}| + j-i.$$
$$\geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Case 3: $D_g(u_i,v_j) = n+2-i, 1 \leq i,j \leq n$.
Let $u_i, v_j \in V(O_n)$. Then $|\sigma(u_i)-\sigma(v_j)| + D_g(u_i,v_j)$

$$= |(n+i)\times 10^{-n} - j \times 10^{-n}| + n+2-i.$$
$$= |(n+i-j)\times 10^{-n}| + n+2-i.$$
$$\geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Case 4: $D_g(u_{n+1},u_i) = n+1-i, 1 \leq i \leq n$.
Let $u_{n+1}, u_i \in V(O_n)$.
Then $|\sigma(u_{n+1})-\sigma(u_i)| + D_g(u_{n+1},u_i)$

$$= |(2n+1)\times 10^{-n} - (n+i) \times 10^{-n}| + n+1-i.$$
$$= |(n+1-i)\times 10^{-n}| + n+1-i.$$
$$\geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Case 5: $D_g(u_{n+1},v_i) = 1, 1 \leq i \leq n$.
Let $u_{n+1}, v_i \in V(O_n)$.
Then $|\sigma(u_{n+1})-\sigma(v_i)| + D_g(u_{n+1},v_i)$

$$= |(2n+1)\times 10^{-n} - i \times 10^{-n}| + 1.$$
$$= |(2n+1-i)\times 10^{-n}| + 1.$$
$$\geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n}.$$

Therefore, it is clear that $|\sigma(u)-\sigma(v)| + D_g(u,v) \geq \left(\frac{(2n+1)(2n+2)}{2}\right) \times 10^{-n} \ \forall\, u,v \in V(O_n)$.
Hence, the octopus graph $O_n, n \geq 2$ is a Hamiltonian fuzzy labeling graph. □

Example 2.6. *The following figure 3 shows that octopus graph O_5 is a hamiltonian fuzzy labeling graph.*

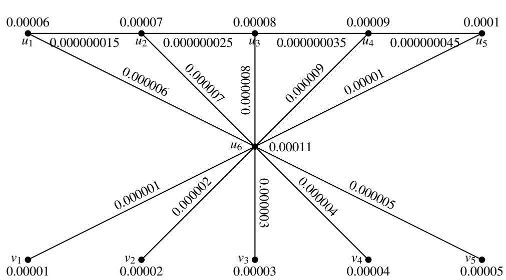

Figure 3. Hamiltonian fuzzy labeling of octopus graph O_5.

3. Hamiltonian Fuzzy Magic Labeling of Graphs

Definition 3.1. *A fuzzy magic labeling graph is said to be a Hamiltonian fuzzy magic labeling graph if $|\sigma(u)-\sigma(v)| + D_g(u,v) \geq \sum_{u \in V} \sigma(u) \ \forall\, u,v \in V$, where $D_g(u,v)$ is the fuzzy detour $g-$distance from u to v and is the order of fuzzy graph.*

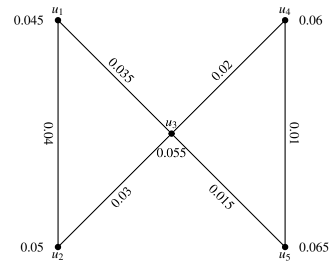

Figure 4. Hamiltonian fuzzy magic labeling of butterfly graph

Example 3.2. *The following figure 4 shows that butterfly graph is a hamiltonian fuzzy magic labeling graph.*

Theorem 3.3. *The Comb graph $P_n \odot K_1, n \geq 2$ is a Hamiltonian fuzzy magic labeling graph.*

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of the comb graph $P_n \odot K_1$. Let $E(P_n \odot K_1) = \{u_iu_{i+1}, u_jv_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ be the edge set of the comb graph $P_n \odot K_1$. The comb graph $P_n \odot K_1$ has $2n$ vertices and $2n-1$ edges.

Define a labeling $\sigma : V(P_n \odot K_1) \longrightarrow [0,1]$ by

$$\sigma(u_i) = \left(\frac{2(2n+i)-3}{2}\right) \times 10^{-n}, 1 \leq i \leq n,$$
$$\sigma(v_i) = (2n+i-1) \times 10^{-n}, 1 \leq i \leq n.$$

Define a labeling $\mu : E(P_n \odot K_1) \longrightarrow [0,1]$ by

$$\mu(u_iu_{i+1}) = \left(\frac{4(n-i)+1}{2}\right) \times 10^{-n}, 1 \leq i \leq n-1,$$
$$\mu(u_iv_i) = \big(2(n-i)+1\big) \times 10^{-n}, 1 \leq i \leq n.$$

Then the order of the fuzzy comb graph

$$P_n \odot K_1 = \sum_{u \in V(P_n \odot K_1)} \sigma(u) = \left(\frac{10n^2-3n}{2}\right) \times 10^{-n}.$$

Clearly, the membership values of all vertices and all edges are distinct.Therefore, it is clear that $\mu(uv) < \sigma(u) \wedge \sigma(v) \ \forall\, u,v \in V(P_n \odot K_1)$.

Let $u_i, u_{i+1} \in V(P_n \odot K_1)$ and $u_iu_{i+1} \in E(P_n \odot K_1)$. Then the fuzzy magic constant

$$m_0(P_n \odot K_1) = \sigma(u_i) + \mu(u_iu_{i+1}) + \sigma(u_{i+1})$$
$$= \left(\frac{2(2n+i)-3}{2}\right) \times 10^{-n} + \left(\frac{4(n-i)+1}{2}\right)$$
$$\times 10^{-n} + \left(\frac{2(2n+i+1)-3}{2}\right) \times 10^{-n}.$$
$$= \left(\frac{3(4n-1)}{2}\right) \times 10^{-n}.$$

Note that, all arcs are strong arcs. Then the fuzzy detour $g-$distance between two vertices are

$$D_g(u_i,u_j) = j-i, 1 \leq i < j \leq n,$$
$$D_g(v_i,v_j) = j-i+2, 1 \leq i < j \leq n,$$
$$D_g(u_i,v_j) = D_g(v_i,u_j) = j-i+1, 1 \leq i \leq j \leq n.$$

Case 1: $D_g(u_i,u_j) = j-i, 1 \leq i < j \leq n$.
Let $u_i,u_j \in V(P_n \odot K_1)$.
Then $|\sigma(u_i)-\sigma(u_j)| + D_g(u_i,u_j)$

$$= \left|\left(\frac{2(2n+i)-3}{2}\right) \times 10^{-n} - \left(\frac{2(2n+j)-3}{2}\right) \times 10^{-n}\right| + j-i.$$
$$= \left|\left(\frac{2(i-j)}{2}\right) \times 10^{-n}\right| + j-i.$$
$$\geq \left(\frac{10n^2-3n}{2}\right) \times 10^{-n}.$$

Case 2: $D_g(u_i,u_j) = j-i, 1 \leq i < j \leq n$.
Let $v_i,v_j \in V(P_n \odot K_1)$. Then $|\sigma(v_i)-\sigma(v_j)| + D_g(v_i,v_j)$

$$= |(2n+i-1) \times 10^{-n} - (2n+j-1) \times 10^{-n}| + j-i+2.$$
$$= |(i-j) \times 10^{-n}| + j-i+2.$$
$$\geq \left(\frac{10n^2-3n}{2}\right) \times 10^{-n}.$$

Case 3: $D_g(u_i,v_j) = D_g(v_i,u_j) = j-i+1, 1 \leq i \leq m \leq j \leq n$.
Let $u_i,v_j \in V(P_n \odot K_1)$. Then $|\sigma(u_i)-\sigma(v_j)| + D_g(u_i,v_j)$

$$= \left|\left(\frac{2(2n+i)-3)}{2}\right) \times 10^{-n} - (2n+j-1) \times 10^{-n}\right| + j-i+1.$$
$$= \left|\left(\frac{2(i-j)-1}{2}\right) \times 10^{-n}\right| + j-i+1.$$
$$\geq \left(\frac{10n^2-3n}{2}\right) \times 10^{-n}.$$

In each case, $|\sigma(u)-\sigma(v)| + D_g(u,v) \geq \left(\frac{10n^2-3n}{2}\right) \times 10^{-n} \; \forall \, u,v \in V(P_n \odot K_1)$.

Hence, the comb graph $P_n \odot K_1, n \geq 2$ is a Hamiltonian fuzzy magic labeling graph. □

Example 3.4. *The following figure 5 shows that comb graph $P_5 \odot K_1$ is a hamiltonian fuzzy magic labeling graph.*

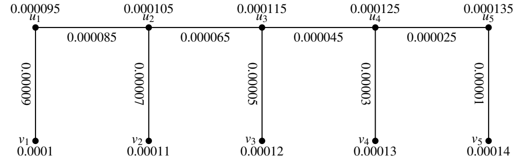

Figure 5. Hamiltonian fuzzy magic labeling of $P_5 \odot K_1$

Theorem 3.5. *The Hurdle graph $Hd_n, n \geq 3$ is a Hamiltonian fuzzy magic labeling graph.*

Proof. Let $V(Hd_n) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ be the vertex set of the hurdle graph Hd_n. Let $E(Hd_n) = \{u_iu_{i+1}, u_{j+1}v_j : 1 \leq i \leq n-1, 1 \leq j \leq n-2\}$ be the edge set of the hurdle graph Hd_n. The hurdle graph Hd_n has $2n-2$ vertices and $2n-3$ edges.

Define a labeling $\sigma : V(Hd_n) \longrightarrow [0,1]$by

$$\sigma(u_i) = \left(\frac{2(2n+i)-7}{2}\right) \times 10^{-(n-1)}, 1 \leq i \leq n,$$
$$\sigma(v_j) = \big(2n+j-3\big) \times 10^{-(n-1)}, 1 \leq j \leq n-2.$$

Define a labeling $\mu : E(Hd_n) \longrightarrow [0,1]$ by

$$\mu(u_iu_{i+1}) = \left(\frac{4(n-i)-3}{2}\right) \times 10^{-(n-1)}, 1 \leq i \leq n-1,$$
$$\mu(u_{j+1}v_j) = \big(2(n-j-1)\big) \times 10^{-(n-1)}, 1 \leq j \leq n-2.$$

Then the order of the fuzzy hurdle graph

$$Hd_n = \sum_{u \in V(Hd_n)} \sigma(u) = \left(\frac{10n^2-23n+14}{2}\right) \times 10^{-(n-1)}.$$

Clearly, the membership values of all vertices and all edges are distinct. Therefore, it is clear that $\mu(uv) < \sigma(u) \wedge \sigma(v) \; \forall \, u,v \in V(Hd_n)$.

Let $u_i,u_{i+1} \in V(Hd_n)$ and $u_iu_{i+1} \in E(Hd_n)$.
Then the fuzzy magic constant

$$m_0(Hd_n) = \sigma(u_i) + \mu(u_iu_{i+1}) + \sigma(u_{i+1})$$
$$= \left(\frac{2(2n+i)-7}{2}\right) \times 10^{-(n-1)} + \left(\frac{4(n-i)-3}{2}\right) \times 10^{-(n-1)} + \left(\frac{2(2n+i+1)-7}{2}\right) 10^{-(n-1)}$$
$$= \left(\frac{3(4n-5)}{2}\right) \times 10^{-(n-1)}.$$

Note that, all arcs are strong arcs. Then the fuzzy detour $g-$distance between two vertices are

$$D_g(u_i,u_j) = j-i, 1 \leq i < j \leq n,$$
$$D_g(v_i,v_j) = j-i+2, 1 \leq i < j \leq n-2,$$
$$D_g(u_i,v_i) = 2, 1 \leq i \leq n-2,$$
$$D_g(u_{i+1},v_i) = 1, 1 \leq i \leq n-2,$$
$$D_g(u_i,v_j) = j+i-2, 2 \leq i \leq n-1, 1 \leq j \leq n-2,$$
$$D_g(v_i,u_j) = j+i-2, 1 \leq i \leq n-2, 2 \leq j \leq n-1,$$
$$D_g(u_1,v_i) = i+1, 1 \leq i \leq n-2,$$
$$D_g(u_n,v_i) = n-i, 1 \leq i \leq n-2.$$

Case 1: $D_g(u_i,u_j) = j-i, 1 \leq i < j \leq n$.

Let $u_i, u_j \in V(Hd_n)$. Then $|\sigma(u_i) - \sigma(u_j)| + D_g(u_i, u_j)$

$$= \left| \left(\frac{2(2n+i)-7}{2} \right) \times 10^{-(n-1)} - \left(\frac{2(2n+j)-7}{2} \right) \times 10^{-(n-1)} \right| + j - i.$$

$$= \left| \left(\frac{2(i-j)}{2} \right) \times 10^{-(n-1)} \right| + j - i.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 2: $D_g(v_i, v_j) = j - i + 2, 1 \leq i < j \leq n-2.$

Let $v_i, v_j \in V(Hd_n)$. Then $|\sigma(v_i) - \sigma(v_j)| + D_g(v_i, v_j)$

$$= |(2n+i-3) \times 10^{-(n-1)} - (2n+j-3) \times 10^{-(n-1)}| + j - i + 2$$

$$= |(i-j) \times 10^{-(n-1)}| + j - i + 2$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 3: $D_g(u_i, v_i) = 2, 1 \leq i \leq n-2.$

Let $u_i, v_i \in V(Hd_n)$. Then $|\sigma(u_i) - \sigma(v_i)| + D_g(u_i, v_i)$

$$= \left| \left(\frac{2(2n+i)-7}{2} \right) \times 10^{-(n-1)} - (2n+i-3) \times 10^{-(n-1)} \right| + 2.$$

$$= \left| \left(\frac{-1}{2} \right) \times 10^{-(n-1)} \right| + 2.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 4: $D_g(u_{i+1}, v_i) = 1, 1 \leq i \leq n-2.$

Let $u_{i+1}, v_i \in V(Hd_n)$.

Then $|\sigma(u_{i+1}) - \sigma(v_i)| + D_g(u_{i+1}, v_i)$

$$= \left| \left(\frac{2(2n+i+1)-7}{2} \right) \times 10^{-(n-1)} - (2n+i-3) \times 10^{-(n-1)} \right| + 1.$$

$$= \left| \left(\frac{1}{2} \right) \times 10^{-(n-1)} \right| + 1.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 5: $D_g(u_i, v_j) = j + i - 2, 2 \leq i \leq n-1, 1 \leq j \leq n-2.$

Let $u_i, v_j \in V(Hd_n)$. Then $|\sigma(u_i) - \sigma(v_j)| + D_g(u_i, v_j)$

$$= \left| \left(\frac{2(2n+i)-7}{2} \right) \times 10^{-(n-1)} - (2n+j-3) \times 10^{-(n-1)} \right| + j + i - 2.$$

$$= \left| \left(\frac{2(i-j)-1}{2} \right) \times 10^{-(n-1)} \right| + j + i - 2.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 6: $D_g(v_i, u_j) = j + i - 2, 1 \leq i \leq n-2, 2 \leq j \leq n-1.$

Let $v_i, u_j \in V(Hd_n)$. Then $|\sigma(v_i) - \sigma(u_j)| + D_g(v_i, u_j)$

$$= \left| (2n+i-3) \times 10^{-(n-1)} - \left(\frac{2(2n+j)-7}{2} \right) \times 10^{-(n-1)} \right| + j + i - 2.$$

$$= \left| \left(\frac{2(i-j)+1}{2} \right) \times 10^{-(n-1)} \right| + j + i - 2.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 7: $D_g(u_1, v_i) = i + 1, 1 \leq i \leq n-2.$

Let $u_1, v_i \in V(Hd_n)$. Then $|\sigma(u_1) - \sigma(v_i)| + D_g(u_1, v_i)$

$$= \left| \left(\frac{2(2n+1)-7}{2} \right) \times 10^{-(n-1)} - (2n+i-3) \times 10^{-(n-1)} \right| + 1 + i.$$

$$= \left| \left(\frac{-2i+1}{2} \right) \times 10^{-(n-1)} \right| + 1 + i.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

Case 8: $D_g(u_n, v_i) = n - i, 1 \leq i \leq n-2.$

Let $u_n, v_i \in V(Hd_n)$. Then $|\sigma(u_n) - \sigma(v_i)| + D_g(u_n, v_i)$

$$= \left| \left(\frac{6n-7}{2} \right) \times 10^{-(n-1)} - (2n+j-3) \times 10^{-(n-1)} \right) \right| + n - i.$$

$$= \left| \left(\frac{2(n-j)-1}{2} \right) \times 10^{-(n-1)} \right| + n - i.$$

$$\geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)}.$$

In each case, $|\sigma(u) - \sigma(v)| + D_g(u, v) \geq \left(\frac{10n^2 - 23n + 14}{2} \right) \times 10^{-(n-1)} \; \forall \, u, v \in V(Hd_n).$

Hence, the hurdle graph $Hd_n, n \geq 3$ is a Hamiltonian fuzzy magic labeling graph. □

Example 3.6. *The following figure 6 shows that hurdle graph Hd_5 is a hamiltonian fuzzy magic labeling graph.*

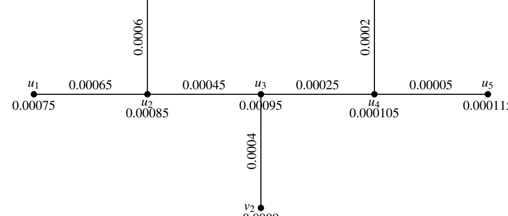

Figure 6. Hamiltonian fuzzy magic labeling of hurdle graph Hd_5

4. Conclusion

In this paper, we introduce two fuzzy labeling concepts such as hamiltonian fuzzy labeling and hamiltonian fuzzy magic labeling of graphs.Also we find some hamiltonian fuzzy labeling and hamiltonian fuzzy magic labeling graphs.

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ISSN(P):2319 − 3786
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