Arc domination in fuzzy graphs using strong arcs

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Abstract

In this paper, the concept of strong arc domination number is introduced by using membership values of strong arcs in fuzzy graphs. The strong arc domination number γ_s^l of classes of fuzzy graphs such as complete fuzzy graph and complete bipartite fuzzy graph, fuzzy cycle and fuzzy tree is determined. Also the relationship between the strong arc domination number and strong arc covering number of fuzzy graphs is discussed and bounds are obtained for the strong arc domination number of fuzzy graphs.

Keywords

Fuzzy graph, Strong arcs, Weight of arcs, Strong domination, fuzzy trees, fuzzy cycles.

AMS Subject Classification

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1. Introduction

Fuzzy graphs were introduced by Rosenfeld, who has described the fuzzy analogue of several graph theoretic concepts like paths, cycles, trees and connectedness [18]. Bhutani and Rosenfeld have introduced the concept of strong arcs [2]. The work on fuzzy graphs was also done by Mordeson, Pradip, Talebi, and Yeh [11, 17, 27, 28]. It was during 1850s, a study of dominating sets in graphs started purely as a problem in the game of chess. Chess enthusiasts in Europe considered the problem of determining the minimum number of queens that can be placed on a chess board so that all the squares are either attacked by a queen or occupied by a queen. The concept of domination in graphs was introduced by Ore and Berge in 1962, the domination number and independent domination number are introduced by Cockayne and Hedetniemi [6]. The concept of edge domination in graphs was introduced by Mitchell and Hedetniemi[19]. Somasundaram and Somasundaram discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graph [21, 22]. Nagoorgani, Chandrasekharan and Prasanna defined domination and arc domination in fuzzy graphs using strong arcs [13, 14].According to them a dominating set of a fuzzy graph $G : (V, \sigma, \mu)$ is a set D of nodes of G such that every node in $V - D$ has at least one strong neighbor in D. An arc e_i dominates e_j if e_i is a strong arc in G and adjacent to e_j. A subset D of $E(G)$ is said to be an arc dominating set of G if for every $e_j \in E(G) - D$ there exists $e_i \in D$ such that e_i dominates e_j. Also they defined arc domination number of G as the minimum number of arcs in any D . In this paper, it is defined the arc domination number of fuzzy graph using the weights of strong arcs so as to minimize this parameter further.

2. Preliminaries

We summarize briefly some basic definitions in fuzzy graphs which are presented in [1, 2, 4, 12, 13, 18, 21, 23].

A fuzzy graph is denoted by $G : (V, \sigma, \mu)$ where V is a node set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ. i.e., $\mu(x, y) \leq \sigma(x) \bigwedge \sigma(y)$ for all $x, y \in V$. We call σ the fuzzy node set of G and μ the fuzzy arc set of G, respectively.

We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e.,$\mu(x,x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x,y) = \mu(y,x)$, for all (x,y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H : (\tau, \nu)$ is said to be a partial fuzzy subgraph of $G : (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$, a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u,v) = \mu(u,v)$ for all $(u,v) \in \nu^*$. A fuzzy subgraph $H : (\tau, \nu)$ spans the fuzzy graph $G : (V, \sigma, \mu)$ if $\tau = \sigma$. The fuzzy graph $H : (P, \tau, \nu)$ is called an induced fuzzy subgraph of $G : (V, \sigma, \mu)$ induced by P if $P \subseteq V$ and $\tau(u) = \sigma(u)$ for all $u \in P$ and $\nu(u,v) = \mu(u,v)$ for all $u,v \in P$. We shall use the notation $<P>$ to denote the fuzzy subgraph induced by P. $G : (V, \sigma, \mu)$ is called trivial if $|\sigma^*| = 1$.

In a fuzzy graph $G : (V, \sigma, \mu)$, a path P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x,y)$.

A fuzzy graph $G : (\sigma, \mu)$ is connected if for every x,y in σ^*, $CONN_G(x,y) > 0$.

An arc (u,v) of a fuzzy graph is called an effective arc (M-strong arc) if $\mu(u,v) = \sigma(u) \bigwedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$.

A fuzzy graph G is said to be complete if $\mu(u,v) = \sigma(u) \bigwedge \sigma(v)$, for all $u,v \in \sigma^*$ and is denoted by K_σ.

The order p and size q of a fuzzy graph $G : (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in V \times V} \mu(x,y)$.

Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the scalar cardinality of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|$. Let p denotes the scalar cardinality of V, also called the order of G.

An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. Depending on $CONN_G(x,y)$ of an arc (x,y) in a fuzzy graph G, Sunil Mathew and Sunitha [23] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x,y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x,y). An arc (x,y) in G is $\alpha-$ strong if $\mu(x,y) > CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\beta-$ strong if $\mu(x,y) = CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\delta-$ arc if $\mu(x,y) < CONN_{G-(x,y)}(x,y)$. Thus an arc (x,y) is a strong arc if it is either $\alpha-$ strong or $\beta-$ strong. A path P is called strong path if P contains only strong arcs.

A fuzzy graph G is said to be bipartite [21] if the node set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u,v) = \sigma(u) \bigwedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a complete bipartite graph and is denoted by K_{σ_1, σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

A connected fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy tree if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree [spanning tree], where for all arcs (x,y) not in F there exists a path from x to y in F whose strength is more than $\mu(x,y)$ [18]. Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G.

A maximum spanning tree of a connected fuzzy graph $G : (V, \sigma, \mu)$ is a fuzzy spanning subgraph $T : (\sigma, \nu)$, such that T is a tree, and for which $\sum_{u \neq v} \nu(u,v)$ is maximum. A node which is not an endnode of T is called an internal node of T [12].

A node u is said to be isolated if $\mu(u,v) = 0$ for all $v \neq u$.

3. Strong Arc Domination in Fuzzy Graphs

The concept of domination in graphs was introduced by Ore and Berge in 1962 and further studied by Cockayne and Hedetniemi [6]. The concept of edge domination in graphs was introduced by Mitchell and Hedetniemi[19].

It is referred to [5, 19] for the terminology of domination and edge domination in crisp graphs .

For a vertex v of a graph $G : (V, E)$, recall that a neighbor of v is a vertex adjacent to v in G. Also the neighborhood $N(v)$ of v is the set of neighbors of v. The closed neighborhood $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A vertex v in a graph G is said to dominate itself and each of its neighbors, that is, v dominates the vertices in $N[v]$. A set S of vertices of G is a dominating set of G if every vertex of $V(G) - S$ is adjacent to some vertex in S. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G and is denoted by $\gamma(G)$. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma^l(G)$ of G is the minimum cardinality taken over all edge dominating sets G.

These ideas are extended to fuzzy graphs using strong arcs as follows.

Nagoorgani, Chandrasekharan and Prasanna [13, 14] introduced the concept of domination and arc domination in fuzzy graphs using strong arcs. These concepts have motivated researchers to reformulate some of the concepts in domination and arc domination more effectively. The studies in [14] is the main motivation of this paper and it is modified the definition of arc domination number of a fuzzy graph. This modification is required due to the fact that the parameter 'arc(edge) domination number' defined by Nagoor gani and Prasanna is inclined more towards graphs than to fuzzy graphs. Using the new definition of arc domination number it is reduced

the value of old arc domination number and extracted classic results in a fuzzy graph.

According to Nagoor gani a node v in a fuzzy graph G is said to strongly dominate itself and each of its strong neighbors, that is, v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set of G if every node of $V(G) - D$ is a strong neighbor of some node in D. They defined a minimum strong dominating set in a fuzzy graph G as a strong dominating set with minimum number of nodes. A minimum strong dominating set as a strong dominating set of minimum scalar cardinality. The scalar cardinality of a minimum strong dominating set is called the strong domination number of G [13, 15].

In [14] Nagoorgani defined an arc e_i dominates e_j if e_i is a strong arc in G and adjacent to e_j. A subset D of $E(G)$ is said to be an arc dominating set of G if for every $e_j \in E(G) - D$ there exists $e_i \in D$ such that e_i dominates e_j. A minimum strong arc dominating set as a strong arc dominating set of minimum cardinality and the cardinality of a minimum strong arc dominating set is called the strong arc domination number of G.

The concept of strong arc domination in fuzzy graphs has applications to several fields. Strong arc domination arises in traffic problems in networks. In such applications, the membership values of strong arcs in fuzzy graph give more optimum results for strong arc domination number than using minimum number of arcs in a strong arc dominating set. Hence it is modified the definition of strong arc domination number using membership values of strong arcs and extracted some interesting results using the new definition.

Definition 3.1. *Let $G : (V, \sigma, \mu)$ be a non trivial fuzzy graph. The fuzzy weight of a strong arc dominating set D is defined as $W(D) = \sum_{(u,v) \in D} \mu(u, v)$. The strong arc domination number of a fuzzy graph G is defined as the minimum fuzzy weight of strong arc dominating sets of G and it is denoted by $\gamma_s^l(G)$ or simply γ_s^l. A minimum strong arc dominating set in a fuzzy graph G is a strong arc dominating set of minimum fuzzy weight.*

Let $\gamma_s^l(\overline{G})$ or $\overline{\gamma_s}^l$ denote the strong arc domination number of the complement of a fuzzy graph G.

Remark 3.2. *In crisp graph, the edge set E is a strong arc dominating set. But in fuzzy graph, the edge set E need not be a strong arc dominating set of G. Let D be the set of all strong arcs of G. Then the set D is a strong arc dominating set of G.*

Remark 3.3. *If all the nodes are isolated, then G has no strong arc dominating set.*

Example 3.4. *Consider the fuzzy graph in Figure 1. In this fuzzy graph (a,c) and (e,d) are $\delta-$ arcs and all others are strong arcs. Hence $D = \{(b,c), (f,d)\}$ is a minimum strong arc dominating set and strong arc domination number is $\gamma_s^l(G) = 0.3 + 0.5 = 0.8$.*
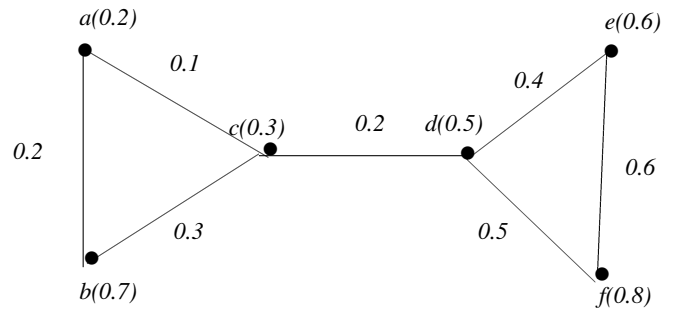

Figure 1: Illustration of strong arc domination in fuzzy graphs

Now, we determine γ_{s^l} of complete fuzzy graph, complete bipartite fuzzy graph and fuzzy cycle.

Proposition 3.5. *If $G : (V, \sigma, \mu)$ is a complete fuzzy graph, then,*

$$\gamma_{s^l}(G) = \bigwedge \{W(D) : D \text{ is a strong arc dominating set in}$$
$$G \text{ with } |D| \geq \lfloor \frac{n}{2} \rfloor\},$$

where n is the number of nodes in G.

Proof. Since $G : (V, \sigma, \mu)$ is a complete fuzzy graph, all arcs are strong, [2] and each node is adjacent to all other nodes. Also, the number of arcs in a strong arc dominating set of both G and G^* are the same because each arc in both graphs is strong. Now, the strong arc domination number of G^* is $\lfloor \frac{n}{2} \rfloor$, [19]. Hence the minimum number of arcs in a strong arc dominating set of G is $\lfloor \frac{n}{2} \rfloor$. Hence the result follows. □

Remark 3.6. *Since every complete fuzzy graph contains at most one $\alpha-$ strong arc, [23], every strong arc dominating set in a complete fuzzy graph G contains only $\beta-$ strong arcs or contains at most one $\alpha-$ strong arc and other arcs are $\beta-$ strong.*

Proposition 3.7. *For a complete bipartite fuzzy graph K_{σ_1, σ_2} with partite sets V_1 and V_2,*

$$\gamma_{s^l}(K_{\sigma_1, \sigma_2}) = \bigwedge \{W(D) : D \text{ is a strong arc dominating set in}$$
$$K_{\sigma_1, \sigma_2} \text{ with } |D| \geq \bigwedge \{|V_1|, |V_2|\}\}.$$

Proof. In K_{σ_1, σ_2}, all arcs are strong, [21]. Also each node in V_1 is adjacent with all nodes in V_2 and vice-versa. Also, the number of arcs in a strong arc dominating set of both K_{σ_1, σ_2} and K_{σ_1, σ_2}^* are the same because each arc in both graphs is strong. Now, the strong arc domination number of K_{σ_1, σ_2}^* is $\bigwedge \{|V_1|, \{|V_2|\}, [19]$. Hence the minimum number of arcs in a strong arc dominating set of K_{σ_1, σ_2} is $\bigwedge \{|V_1|, \{|V_2|\}$. Hence the result follows. □

Proposition 3.8. *If $G : (V, \sigma, \mu)$ is a fuzzy cycle such that G^* is a cycle. Then*

$$\gamma_{s^l}(G) = \bigwedge \{W(D) : D \text{ is a strong arc dominating set in}$$
$$G \text{ with } |D| \geq \lceil \frac{n}{3} \rceil\}$$

Proof. In a fuzzy cycle $G : (V, \sigma, \mu)$, every arc is strong, [2]. Also, the number of arcs in a strong arc dominating set of both G and G^* are the same because each arc in both graphs is strong. Now, the strong arc domination number of G^* is $\lceil \frac{n}{3} \rceil$, [19]. Hence the minimum number of arcs in a strong arc dominating set of G is $\lceil \frac{n}{3} \rceil$. Hence the result follows. $\square$

Proposition 3.9. *If* $G : (V, \sigma, \mu)$ *is a fuzzy graph such that* G^* *is a star with* $n \geq 2$ *nodes. Then* $\gamma_{sl}(G) = \mu(u, v)$ *where* $\mu(u, v)$ *is the weight of a weakest arc in G.*

Proof. Since $G : (V, \sigma, \mu)$ is a fuzzy graph such that G^* is a star, every arc is strong and the weakest arc say (u, v) forms the minimum strong arc dominatiing set of G. Hence the result follows.

Recall the definition of the strong node cover, strong arc cover and matching of a fuzzy graph defined by Manjusha and Sunitha [10]. $\square$

Definition 3.10. *Let* $G : (V, \sigma, \mu)$ *be a fuzzy graph. A node and an incident strong arc are said to strong cover each other. A strong node cover in a fuzzy graph G is a set D of nodes that strong cover all strong arcs of G. The fuzzy weight of a strong node cover D is defined as* $W(D) = \sum_{u \in D} \mu(u, v)$, *where* $\mu(u, v)$ *is the minimum of the membership values (weights) of strong arcs incident on u. The strong node covering number of a fuzzy graph G is defined as the minimum fuzzy weight of strong node covers of G and it is denoted by* $\alpha_{S_o}(G)$ *or simply* α_{S_o}. *A minimum strong node cover in a fuzzy graph G is a strong node cover of minimum fuzzy weight.*

Definition 3.11. *Let* $G : (V, \sigma, \mu)$ *be a fuzzy graph without isolated nodes. A strong arc cover of G is a set X of strong arcs of G that strong covers all nodes of G. The fuzzy weight of a strong arc cover X is defined as* $W(X) = \sum_{(u,v) \in X} \mu(u, v)$. *The strong arc covering number of a fuzzy graph G is defined as the minimum fuzzy weight of strong arc covers of G and it is denoted by* $\alpha_{S_1}(G)$ *or simply* α_{S_1}. *A minimum strong arc cover of a fuzzy graph G is a strong arc cover of minimum fuzzy weight.*

Definition 3.12. [16] *Let* $G : (V, \sigma, \mu)$ *be a fuzzy graph. A set M of strong arcs in G such that no two arcs in M have a common node is called a strong independent set of arcs or a strong matching in G.*

Definition 3.13. *Let M be a strong matching in a fuzzy graph* $G : (V, \sigma, \mu)$. *If* $e = (u, v) \in M$, *then we say that M strongly matches u to v. The fuzzy weight of a strong matching is defined as* $W(M) = \sum_{(u,v) \in M} \mu(u, v)$. *The strong arc independence number or strong matching number of a fuzzy graph G is defined as the maximum fuzzy weight of strong matchings of G and it is denoted by* $\beta_{S_1}(G)$ *or simply* β_{S_1}. *A maximum strong matching in a fuzzy graph G is a strong matching of maximum fuzzy weight.*

Remark 3.14. *From the definition of strong node cover it is clear that in any fuzzy graph* $G : (V, \sigma, \mu)$, *the set of all nodes*

incident to each arc in any strong arc dominating set D is a strong node cover of G.

Theorem 3.15. *For any non trivial fuzzy graph* $G : (V, \sigma, \mu)$,

$$\gamma_{sl}(G) \leq \alpha_{S_1}(G)$$
$$\gamma_{sl}(G) \leq \beta_{S_1}(G)$$

Proof. The proof follows directly from the definition. $\square$

Definition 3.16. [11, 12] *Union of two fuzzy graphs: Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *be two fuzzy graphs with* $G_1^* : (V_1, E_1)$ *and* $G_2^* : (V_2, E_2)$ *with* $V_1 \cap V_2 = \phi$ *and let* $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ *be the union of* G_1^* *and* G_2^*.
Then the union of two fuzzy graphs G_1 *and* G_2 *is a fuzzy graph* $G : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ *defined by*

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 \setminus V_2 \\ \sigma_2(u) & \text{if } u \in V_2 \setminus V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in E_1 \setminus E_2 \\ \mu_2(u, v) & \text{if } (u, v) \in E_2 \setminus E_1 \end{cases}$$

Definition 3.17. [11, 12] *Join of two fuzzy graphs: Consider the join* $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ *of graphs where* E' *is the set of all arcs joining the nodes of* V_1 *and* V_2 *where we assume that* $V_1 \cap V_2 = \phi$. *Then the join of two fuzzy graphs* G_1 *and* G_2 *is a fuzzy graph* $G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ *defined by*

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u), \ u \in V_1 \cup V_2$$

and

$$(\mu_1 + \mu_2)(u, v) = \begin{cases} (\mu_1 \cup \mu_2)(u, v) & \text{if } (u, v) \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) & \text{if } (u, v) \in E' \end{cases}$$

Theorem 3.18. *For any fuzzy graph* $G : (V, \sigma, \mu)$, $\gamma_s^l(K_\sigma + G) = \gamma_s^l(K_\sigma) + \gamma_s^l(G)$.

Proof. For any fuzzy graph G, any arc in K_σ is adjacent to all other arcs in K_σ and G and note that all such arcs are strong arcs. Hence any strong arc dominating set of $(K_\sigma + G)$ is a union of arcs in K_σ and in G. Hence the result. $\square$

4. Minimal strong arc domination in fuzzy graphs

In this section it is discussed minimal strong arc dominating sets and some properties.

Definition 4.1. [14] *A strong arc dominating set D of a fuzzy graph* $G : (V, \sigma, \mu)$ *is called a minimal strong arc dominating set if no proper subset of D is a strong arc dominating set of G.*

Remark 4.2. [14] *Every minimum strong arc dominating set is minimal but not conversely.*

Example 4.3. *Consider the fuzzy graph in Figure 2.*
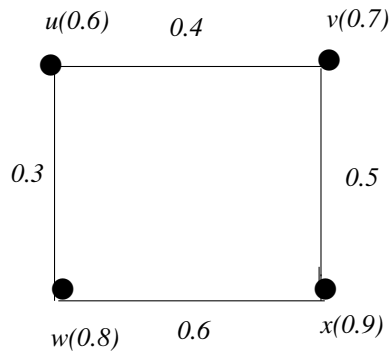

Figure 4: Illustration of minimal strong arc domination

In the fuzzy graph of Figure 4, $(u,v),(v,x),(x,w)$ are strong arcs and (u,w) is a $\delta-$ arc. $D = \{(u,v),(w,x)\}$ is a minimal strong arc dominating set but not minimum strong arc dominating set since the set $\{(u,v),(v,x)\}$ forms a minimum strong arc dominating set with $\gamma_s^I(G) = 0.9$, but $W(D) = 1$.

Note that in a complete fuzzy graph the minimum and minimal strong arc dominating sets are same. Hence the following theorems are obvious.

Theorem 4.4. *Every non trivial complete fuzzy graph G has a strong arc dominating set D whose complement $E \setminus D$ is also a strong arc dominating set.*

Note that in a complete bipartite fuzzy graph the the set of arcs with cardinality $\bigwedge\{|V_1|,\{|V_2|\}$ forms a minimal strong arc dominating set. Hence the following theorems are obvious.

Theorem 4.5. *Every non trivial complete bipartite fuzzy graph G has a strong arc dominating set D whose complement $E \setminus D$ is also a strong arc dominating set.*

Theorem 4.6. *Let G be a complete bipartite fuzzy graph. If D is a minimal strong arc dominating set then $E \setminus D$ is a strong arc dominating set.*

Remark 4.7. *Theorems 4.4 to 4.6 are not true in general fuzzy graphs as seen in the following example.*

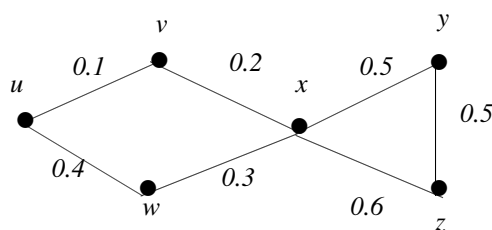**Example 4.8.** *Consider the fuzzy graph given in Figure 5.*

Figure 5: Example of a strong arc dominating set D

such that $V \setminus D$ is not a strong arc dominating set

In this fuzzy graph all node weights are taken as 1. $D = \{(v,x),(u,w),(y,z)\}$ is a minimal strong arc dominating set. But $E \setminus D = \{(u,v),(x,w),(x,z),(x,y)\}$ is not a strong arc dominating set.

5. Strong arc Domination in Fuzzy Trees

Note that in the definition of a fuzzy tree, F is the unique maximum spanning tree (MST) of G [26].

An arc is called a fuzzy bridge of a fuzzy graph $G : (V, \sigma, \mu)$ if its removal reduces the strength of connectedness between some pair of nodes in G [18].

Similarly a fuzzy cut node w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w [18].

A node z is called a fuzzy end node if it has exactly one strong neighbor in G [3].

A non trivial fuzzy tree G contains at least two fuzzy end nodes and every node in G is either a fuzzy cut node or a fuzzy end node [3].

In a fuzzy tree G an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G [2, 26]. Note that these strong arcs are α-strong and there are no β-strong arcs in a fuzzy tree [23]. Also note that in a fuzzy tree G an arc (x,y) is α-strong if and only if (x,y) is a fuzzy bridge of G [23].

Theorem 5.1. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, each arc of a strong arc dominating set is an $\alpha-$strong arc (fuzzy bridge) of G.*

Proof. Let D be a strong arc dominating set of G. Let $(u,v) \in D$. Then (u,v) is a strong arc. Then (u,v) is an arc of the unique MST F of G [2, 26]. Hence (u,v) is an $\alpha-$strong arc or a fuzzy bridge of G [18]. Since (u,v) is arbitrary, this is true for every arc of the strong arc dominating set of G. This completes the proof. □

Proposition 5.2. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, no arc of a strong arc dominating set is a $\beta-$strong arc.*

Proof. Note that a fuzzy graph is a fuzzy tree if and only if it has no $\beta-$strong arcs [23]. Hence the proposition. □

Theorem 5.3. *In a non trivial fuzzy tree $G : (V, \sigma, \mu)$, each node incident on a strong arc of any strong arc dominating set of G is either a fuzzy cut node or a fuzzy end node.*

Proof. Let D be any strong arc dominating set of a non trivial fuzzy tree $G : (V, \sigma, \mu)$. Then all arcs in D are strong. Since G is a fuzzy tree, every node of G is either a fuzzy cut node or a fuzzy end node [3]. Hence every node incident on each strong arc D is either a fuzzy cut node or a fuzzy end node. Since D is arbitrary, this is true for every strong arc dominating set of G. This completes the proof. □

Proposition 5.4. *In a fuzzy tree $G : (V, \sigma, \mu)$, every arc of G is strongly arc dominated by an $\alpha-$ strong arc.*

Proof. Note that in G all strong arcs are $\alpha-$ strong arcs. Hence the proposition. □

Remark 5.5. *The set of all fuzzy end nodes need not be the end nodes of arcs in a strong arc dominating set in a non trivial fuzzy tree $G : (V,\sigma,\mu)$ except K_2.*

Theorem 5.6. *In a fuzzy tree $G : (V,\sigma,\mu)$, each arc of every strong arc dominating set is contained in the unique maximum spanning tree of G.*

Proof. In a fuzzy tree G an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G [2, 26].Hence F contains all the strong arcs of G. In particular, F contains all arcs of every strong arc dominating set of G. This completes the proof. □

Theorem 5.7. *In a non trivial fuzzy tree $G : (V,\sigma,\mu)$, $\gamma_s^J(G) \leq W(F)$ where F is the associated unique maximum spanning tree of G.*

Proof. In a fuzzy tree $G : (V,\sigma,\mu)$, each arc of every strong arc dominating set is contained in the unique maximum spanning tree of G [Theorem 5.6].Also note that $W(F)$ is the sum of the weights of the strong arcs of F. Hence clearly $\gamma_s^J(G) \leq W(F)$. □

6. Strong arc Domination in Complement of Fuzzy Graphs

Sunitha and Vijayakumar [25] has defined the present notion of complement of a fuzzy graph. Sandeep and Sunitha have studied the connectivity concepts in a fuzzy graph and its complement [20]. The complement of a fuzzy graph G, denoted by $\overline{G}$ or G^c is defined to be $\overline{G} = (V,\sigma,\overline{\mu})$ where $\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y)$ for all $x,y \in V$ [25]. Bhutani has defined the isomorphism between fuzzy graphs [1]. Consider the fuzzy graphs $G_1 : (V_1,\sigma_1,\mu_1)$ and $G_2 : (V_2,\sigma_2,\mu_2)$ with $\sigma_1^* = V_1$ and $\sigma_2^* = V_2$. An isomorphism [1] between two fuzzy graphs G_1 and G_2 is a bijective map $h : V_1 \rightarrow V_2$ that satisfies

$$\sigma_1(u) = \sigma_2(h(u)) \text{ for all } u \in V_1.$$

$\mu_1(u,v) = \mu_2(h(u),h(v))$ for all $u,v \in V_1$ and we write $G_1 \approx G_2$. A fuzzy graph G is self complementary [25] if $G \approx \overline{G}$.

Theorem 6.1. *If G is a non trivial fuzzy graph with no M-strong arcs then G and G^c contains at least one strong arc dominating set.*

Proof. If G is a fuzzy graph with no $M-$strong arcs then G^c is also non trivial and contains strong arcs. Hence both G and G^c contains at least one strong arc dominating set. □

Remark 6.2. *There are fuzzy graphs which contain $M-$strong arcs such that G and G^c contain strong arc dominating set [Example 6.3].*

Example 6.3. *Consider the fuzzy graph in Figure 6. Here (v,w) is the only $M - strong$ arc in G and G and G^c are connected. In G, $D = \{(u,v)\}$ is a strong arc dominating set and in G^c, $D = \{(u,w)\}$ is a strong arc dominating set.*
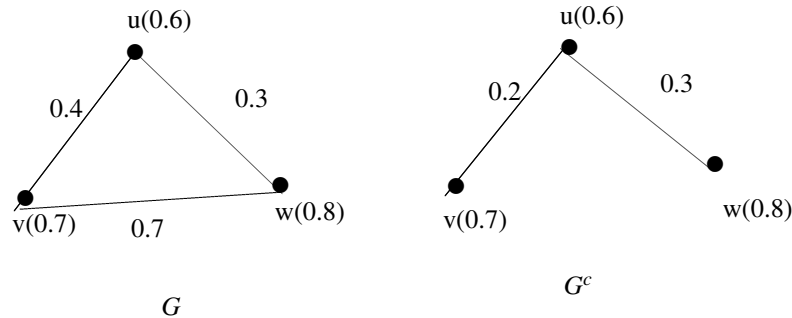

Figure 6: Illustraton of a strong arc dominating set in a fuzzy graph and its complement

Theorem 6.4. *If $G : (V,\sigma,\mu)$ is a non trivial self complementary fuzzy graph except K_σ,then each of G and G^c contain at least one strong arc dominating set.*

Proof. Since G is self complementary G is isomorphic to G^c. Also G has at least one strong arc dominating set. Hence G^c also has at least one strong arc dominating set since G is isomorphic to G^c. Hence the result. □

Corollary 6.5. *If G is a fuzzy graph such that $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v))$ for all $u,v \in \sigma^*$ then each of G and G^c contain at least one strong arc dominating set.*

Proof. Since $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v))$ for all $u,v \in \sigma^*$, G is self complementary [25]. Hence the result follows. □

7. Practical Application

Let G be a graph which represents the rail road network of a particular state. Let the vertices denote the cities and the edges denote the rail roads connecting the cities. The membership functions σ and μ on the vertex set and the edge set of G can be constructed from the statistical data that represents the number of trains passing through various cities and the number of trains passing through various rail roads during a busy hour. Now the term 'busy' is vague in nature. It depends on availability of railway line, time of journey, seasonal demands, special requirements of passengers etc. Thus we get a fuzzy graph model. In this fuzzy graph a strong arc dominating set D can be interpreted as a set of rail roads in which traffic is heavier than the other rail roads not in D.

Let G be a graph which represents the computer network of a particular server. Let the vertices denote the users and the edges denote the network path connecting the users. The membership functions σ and μ on the vertex set and the edge set of G can be constructed from the statistical data that represents the number of data passing through various users and the number of data passing through various network path during a busy hour. Now the term 'busy' is vague in nature. As mentioned earlier we get a fuzzy graph model. In this fuzzy graph a strong arc dominating set D can be interpreted as the set of network paths in which traffic is heavier than the other paths not in D.

8. Conclusion

The concept of arc domination in graph is very rich both in theoretical developments and applications. More than thirty domination parameters have been investigated by different authors, and in this paper it is introduced the concept of strong arc domination number and minimal strong arc domination in fuzzy graphs. The strong arc domination number of classes of fuzzy graphs is obtained. It is found some bounds for the strong arc domination number of fuzzy graphs and studied the same in fuzzy trees and fuzzy cycles. Also it is studied the concept of strong arc domination in complement of fuzzy graphs.

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