Median vertex vague labeling of vague graphs

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Abstract

Vague graph is a pair $G = (A, B)$ where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that $t_B(xy) \leq min(t_A(x), t_A(y))$ and $f_B(xy) \geq max(f_A(x), f_A(y))$ for each edge $xy \subseteq E$. A is called as the vague vertex set of G and B as the vague edge set of G. The main purpose of this paper is to introduce Median Vertex Vague Labeling of vague graphs. In addition, an application of Median Vertex Vague Labeling in traffic flow network is discussed in this paper.

Keywords

Vague graph, Path Vague Graph, Median Vertex Vague Labeling, t-Vertex Vague Labeling.

AMS Subject Classification

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1. Introduction

Majority of our conventional devices for formal demonstrating, thinking, and registering are crisp, deterministic, and exact in character. In dual logic, for example, an announcement can be valid or bogus and nothing in the middle. In set hypothesis, a component can either have a place in a set or not; and in optimization, an answer is either possible or not. Apart from this, numerous genuine circumstances are all 4the time questionable or vague in various manners. Fuzziness can be found in numerous areas of day today life, in which human decision, assessment and choices are significant [1].

In 1993, Gau and Buehrer presented the idea of vague set theory as a generalization of Zadeh fuzzy set theory [2] by replacing the value of an element in a set with a subinterval [0, 1]. To be specific, true- membership value and false-membership value are the boundary values of the membership degree. Vague set theory becomes a promising tool to deal with inaccurate, unsure or ambiguous information [2]. Numerous specialists have applied this theory to diverse circumstances, for example, fuzzy control, decision-making, etc. Ramakrishna introduced vague Graph idea [3]. Akram et al. [4] presented certain types of vague graph such as neighbourly irregular vague graphs, neighbourly total irregular vague graphs, highly irregular vague graphs and highly total irregular vague graphs. In that, some properties related with these vague graphs are examined. Furthermore, under necessary and sufficient condition, neighbourly irregular and highly irregular vague graphs are equivalent [3]. In addition, Borzooei and Rashmanlou defined domination, degree of vertices, homomorphism and isomorphism in vague graphs [5, 6]. A graph Labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. From the meaning of Graph Labeling, in this paper, the new definition for Labeling the vertices and edges of vague graph is proposed. In addition, Vague Labeling for some isomorphic vague graphs has been discussed.

2. Preliminaries

In this section, some of the basic definitions of graphs are discussed. Throughout this paper, $G^* = (V, E)$ means a non-trivial, finite connected and undirected graph without loops or multiple edges.

Definition 2.1. [7] *A fuzzy graph $G = (\alpha, \mu)$ is a pair of functions $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ with $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ for all $u,v \in V$, where V is a finite non-empty set and $\wedge$ denotes minimum.*

Definition 2.2. [6,8] *A vague set A in a universe of discourse U is characterized by a truth membership function t_A and a false membership function f_A as follows :*
$t_A: U \to [0,1]$, $f_A: U \to [0,1]$ and $t_A(u) + f_A(u) \le 1$, where $t_A(u)$ is a lower bound on the grade of membership of u derived from the evidence for u, and $f_A(u)$ is a lower bound on the grade of membership of negation of u derived from the evidence against u.
The degree of membership of u in the vague set A is characterized by the interval $[t_A(u), 1 - f_A(u)]$. The interval $[t_A(u), 1 - f_A(u)]$ is called the vague value of u in A and is denoted by $V_A(u)$. Zero vague value and unit vague value are denoted by $0 = [0,0]$ and $1 = [1,1]$ respectively. The imprecision membership value of each vertex / edge is given by, $1 - f_A(u) - t_A(u)$

Definition 2.3. [9] *Let $G^* = (V,E)$ be a crisp graph. A pair $G = (A,B)$ is called a vague graph on a crisp graph $G^* = (V,E)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that $t_B(xy) \le min(t_A(x), t_A(y))$ and $f_B(xy) \ge max(f_A(x), f_A(y))$ for each edge $xy \in E$. A is called the vague vertex set of G and B as the vague edge set of G, respectively.*
A vague graph G is said to be strong, if $t_B(xy) = t_A(x) \wedge t_A(y)$ and $f_B(xy) = f_A(x) \vee f_B(y)$ for all $xy \in E$.
A vague graph G is said to be complete, if $t_B(xy) = t_A(x) \wedge t_A(y)$ and $f_B(xy) = f_A(x) \vee f_B(y)$ for all $x,y \in V$.

Definition 2.4. [9] *A path ρ in a vague graph $G = (A,B)$ is a sequence of distinct nodes $v_0, v_1, \cdots v_k$ such that,*
$(t_B(v_{i-1}v_i), f_B(v_{i-1}v_i)) > 0$, $i = 1,2, \cdots k$ where k is the length of the path.

3. Labeling of Vague Graphs

Labeling means assigning numbers to the vertices/edges of a graph. If the domain set is vertices, then it is considered as Vertex Labeling and if the domain set is edge, then it is considered as Edge Labeling. When both vertex and edge are taken into consideration, it is named as Total Labeling [10]. In 1966, A. Rosa introduced a new graph Labeling called $\beta-$ Labeling in which the vertices are labelled with distinct numbers chosen from 0 to m, where m is the number of edges, such that each edge is labelled with the absolute difference of the labels of its end vertices and it is unique in the graph [11]. A few years later, S. W. Golomb renamed $\beta-$ Labeling as graceful Labeling as it is known today [12]. As a continuation, Rosa also defined $\alpha-$ Labeling of a graph as graceful Labeling f with an additional property, that there exists a

constant $k(0 \le k < q)$, called the characteristic of f, such that for every two vertices x,y of G with $f(x) < f(y)$, it holds that $f(x) \le k < f(y)$, then f is called $\alpha-$ Labeling (or $\alpha-$ valuation). Hence, the classical $\alpha-$ Labeling defined by Rosa is for simple graphs [13]. Later on, Nagoor Gani defined fuzzy Labeling using fuzzy numbers for fuzzy graph[14] .

In simple graph, each vertex/ edge can be identified using its labels, in fuzzy graph, each vertex/ edge can be identified using its membership value, and in vague graph, each vertex/ edge can be identified using true and false membership value of that vertex/edge. Hence to label the vertices of a vague graph, it is necessary to consider the true and false membership values of each vertex/edge.

In [15] vague set can be transformed into a fuzzy set using Median Membership function. By considering this fact, vague graph is labelled using Median membership value of each vertex, which is named as Median Vertex Vague Labeling (MVVL)

Definition 3.1. *Let $G^* = (V,E)$ be a graph. A pair $G = (A,B)$ is a vague graph on G^* where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$, the Median Membership value of each vertex/edge of a vague graph G is as follows,*
$\mu_{A^M}(v_i) = t_A(v_i) + \frac{1}{2}(1 - f_A(v_i) - t_A(v_i))$

Definition 3.2. *Let $G^* = (V,E)$ be a graph with n vertices. A pair $G = (A,B)$ is a vague graph on G^*. Define a set $S = 1,2 \cdots n$ and a Labeling function $L: V(G) \to S$ such that, $L(u) < L(v)$, if $\mu_{A^M}(u) \ge \mu_{A^M}(v)$, for all $u,v \in V(G)$ and $u \ne v$, where $\mu_{A^M}(v_i)$ denotes Median Membership value of each vertex, then L is called Median Vertex Vague Labeling. Graph that admits Median Vertex Vague Labeling is said to be Median Vertex Vague Labelled graph or MVVL graph.*
In the above definition, if vertex is replaced by edge, then it is said to be Median Edge Vague Labeling. Graph that admits Median Edge Vague Labeling is said to be Median Edge Vague Labelled Graph or MEVL graph.

Definition 3.3. *Median Vertex Vague Labeling of a Vague graph $G = (A,B)$ is said to be $t-$ Vertex Vague Labeling if for every edge $uv \in E(G)$, with $L(u) < L(v)$, it holds that $t_A(u) \ge t_A(v)$. Graph that admits $t-$ Vertex Vague Labeling is said to be $t-$ Vertex Vague Labelled graph or $t-VVL$ graph. Similarly, Median Edge Vague Labeling of a Vague graph $G = (A,B)$ is said to be $t-$ Edge Vague Labeling if for every pair of adjacent edges $uv, vw \in E(G)$, with $L(uv) < L(vw)$, it holds that $t_B(uv) > t_B(vw)$. Graph that admits $t-$ Edge Vague Labeling is said to be $t-$ Edge Vague Labelled graph or $t-EVL$ graph.*

Theorem 3.1. *If $G = (A,B)$ be a vague graph with n vertices and $t_A(v_i), f_A(v_i)$ are constant functions, then there exists $n!$ isomorphic MVVL graph.*

Proof.
Let $G = (A,B)$ be a vague graph with n vertices, say $v_1, v_2, \cdots v_n$ on $G^* = (V,E)$. Since $t_A(v_i)$ and $f_A(v_i)$ are constant functions and from the definition of Median Vertex Vague Labeling of G^* ,$\mu(v_i)$ remains same for $v_1, v_2, \cdots, v_n$. Hence, these n vertices can be labelled in n! different ways. Thus there exists, n! Median Vertex Vague Labeling. Since the graph that admits Median Vertex Vague Labeling is a MVVL graph, there exists n! isomorphic MVVL graph.

Corollary 3.2. *If $G = (A,B)$ be a vague graph with n vertices and with distinct median membership values, then there exists exactly one MVVL– Graph.*

Theorem 3.3. *A Complete Vague graph $G = (A,B)$ with n vertices and constant imprecision membership value has $(n-1)! \times (n-2)! \cdots 2! \times 1!$ isomorphic MEVL graph.*
Proof.
Let $G = (A,B)$ be a vague graph with n vertices. Arrange the vertices of vague graph in such a way that $t(v_1) < t(v_2) \cdots < t(v_n)$. Since G is a complete vague graph, each edge $v_1 v_i \in E(G)$ has $t(v_1 v_i) = t(v_1)$,for all $i = 2, 3, \cdots n$. Hence, there exist $n-1$ edges adjacent to v_1 and receive same true membership value.
Similarly, the edge $v_2 v_i \in E(G)$ has $t(v_2 v_i) = t(v_2)$,for all $i = 3, \cdots n$. Hence there exist $n-2$ edges adjacent to v_2 and receive same true membership value.
By proceeding in the above way, there are $(n-1)$ edges with the true membership value $t(v_1)$, $(n-2)$ edges with the true membership value $t(v_2)$ and $(n-(n-1))$ edges have the true membership value $t(v_{(n-1)})$.
Further, it is clear that, if the imprecision membership value of all vertices is constant, it is constant for edges too. Each Median Edge Vague Labeling depends only on the true membership value of each vertex. Hence, there exists $(n-1)! \times (n-2)! \cdots 2! \times 1!$ isomorphic MEVL graph.

Theorem 3.4. *If $G = (A,B)$ be a vague graph with n vertices and having m vertices $(m < n)$ with equal median membership values, then there exists at most m! isomorphic MVVL graph.*
Proof.
Let $G = (A,B)$ be a vague graph with n vertices on $G^* = (V,E)$. And let $v_1, v_2, \cdots, v_m$ be the vertices having same median membership values. Thus, these m vertices can be labelled in m! ways, with unique Labeling of remaining $n-m$ vertices. Hence there exists m! different Median Vertex Vague Labeling and so as MVVL graph.

Theorem 3.5. *Let $G = (A,B)$ be a path vague graph on $G^* = (V,E)$. If t_A is a constant function, and $f_A(v_i) < f_A(v_{i+1})$ for all $i = 1, 2, \cdots k-1, k \geq 3 [V_{k+1} = V_1]$, then the Median Vertex Vague Labeling is done in such a way that, $L: V(G) \to S$ such that $L(V_i) = i$.*

Proof.
Let $G = (A,B)$ be a path vague graph on $G^* = (V,E)$. From the definition of Median Vertex Vague Labeling on G^* ,

$$\mu_{A^M}(v_i) = t_A(v_i) + \frac{1}{2}(1 - f_A(v_i) - t_A(v_i)) \quad (3.1)$$

Since,$t_A(v_i) = C$,for all i , and $f_A(v_i) < f_A(v_{i+1})$ for all $i = 1, 2, \cdots k-1$, Equation (1) results in, $\mu(v_1) \geq \mu(v_2) \geq \mu(v_3) \geq \cdots \mu(v_k)$.
Hence, $L(V_i) = i$.

Theorem 3.6. *Median Vertex Vague Labelled graph with constant t_A is a t− Vertex Vague labelled graph. Proof.*
Let G be a Median Vertex Vague Labelled graph. Since G has a constant t_A, that is, $t_A(v_i) = C$, for all i , then t– Vertex Vague Labeling is same as of Median vertex Vague Labeling of G^*. Thus, Median Vertex Vague Labelled graph with constant t_A is a t− Vertex Vague labelled Graph.

Theorem 3.7. *A Vague graph $G = (A,B)$ is a t− Vertex Vague Labelled graph if the Imprecision Membership value of each vertex is constant.*
Proof.
Let G be a vague graph. Since G has constant Imprecision Membership value, that is, $(1 - f_A(v_i) - t_A(v_i)) = C$, for all i , then Median Vertex Vague Labeling depends only on the true membership value of each vertices, and so it is a t− Vertex Vague Labelled graph.

Corollary 3.8. *Every t− VVL graph is a Median Vertex Vague labelled graph; but, its converse fails, that is, every MVVL graph need not be a t− VVL graph.*

4. Application of Vague Labeling in Traffic Network

Here, vague graph associated with the traffic flow for 4 different regions of a town is discussed.

4.1 Formation of Vague Graph
Consider a town with 4 different regions, where most of the people live and/or work. These regions a,b,c,d will be named as the vertices of a vague graph. The roads between any two regions are named as the edges of the vague graph. The true and false membership values of vertex/edges in a vague graph can be calculated as the amount of traffic in that region and in between the regions (roads). The traffic on both sides of a given road is quite close. This is because, if an individual leaves his/her home for work or shopping on a day, he/she will probably return to his/her home using the same route on the same day, thus contributing to the traffic on both sides of the road.

The traffic is calculated in such a way that, on a particular day, the number of vehicles crossing a region a is calculated and the amount of traffic in a particular time, say 1 hour is calculated by

(Number of vehicles passed in particular time) / (Total number of vehicles passed on that whole day) $\times$ 100.

It is noted for some consecutive days at that particular time. By taking this survey in to account, it is noted that, in a particular region say a, 25% to 40% of the total traffic will drive or park anywhere in a. Hence, the true and false membership values of that particular region are 0.25 and 0.60 respectively, where 0.25 denotes traffic and 0.60 denotes no traffic in region a at that particular time. Here, 0.15 is considered as the vague region, which corresponds to two categories (traffic / no traffic). Similarly, the true and false membership values of the remaining regions can be found as :

$a(0.25, 0.60)$, $b(0.15, 0.70)$, $c(0.30, 0.40)$, $d(0.10, 0.70)$. Now, the edge (a, b) represents the act of "entering the region b through the road connecting region a and region b". The amount of traffic passing through any side of the road never exceeds the traffic at any end. That is, if the traffic in region a is 25% to 40% and region b is 15% to 30%, then the road connecting a and b will have 14% to 29% traffic. Then, the true and false membership values of edge connecting region a and b are given as 0.14 and 0.71. Thus, the above situation represents the vague graph as shown in Fig 1.
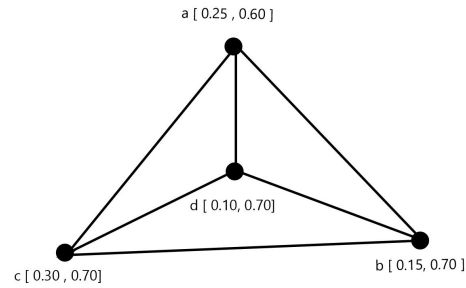

Figure 1. Vague Graph

as,
$L(a)$–1, $L(b)$–3, $L(c)$–2, $L(d)$–4
Here, the vague graph G is a MVVL graph. But it is not a t-VVL graph.

From the above labeling, it is concluded that, region a has more traffic compared to all other regions. Hence, in the above example, if true membership value alone is considered, region c will record high traffic. By taking vague region into account, it is concluded that region a will have high traffic. Similar concept can be applied for the edges to find the road that records more traffic.

Traffic	a	b	c	d
a	0% No such road	14 – 29 %	20– 30%	5-20%
b	14-29%	0% No such road	10-20%	4 -20%
c	20 – 30%	10-20%	0% No such road	5-10%
d	5-20%	4 -20%	5-10%	0% No such road

Table 1. Percentage of traffic flow

From the given table 1, the true and false membership values of the edges are given as :
(a, b) - $e_1 : t_B(e_1) = 0.14, f_B(e_1) = 0.71$
(a, c) - $e_2 : t_B(e_2) = 0.2, f_B(e_2) = 0.7$
(a, d) - $e_3 : t_B(e_3) = 0.05, f_B(e_3) = 0.8$
(b, c) - $e_4 : t_B(e_4) = 0.1, f_B(e_4) = 0.8$
(b, d) - $e_5 : t_B(e_5) = 0.04, f_B(e_5) = 0.8$
(c, d) - $e_6 : t_B(e_6) = 0.05, f_B(e_6) = 0.9$

Median Membership Value of each vertices of Vague graph in *Fig*–1 is given as,
$\mu_{A^M}(a) = 0.325, \mu_{A^M}(b) = 0.225, \mu_{A^M}(c) = 0.30, \mu_{A^M}(d) = 0.2$.
Thus, Median Vertex Vague Labeling of Vague graph is given

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