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2-Edge dominating sets and 2-Edge domination polynomials of paths

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Abstract
Let P_n be the path with n vertices and $(n-1)$ edges. Let $D_{2e}(G,i)$ be the family of 2- edge dominating sets in G with cardinality i. The polynomial $D_{2e}(G,i)=\Sigma_{i=\gamma_{2e}(G)}^{|E(G)|}d_{2e}(G,i)x^i$ is called the 2-edge domination polynomial of G. In this paper, we obtain a recursive formula for $d_{2e}(P_n,i)$. Using this recursive formula we construct 2- edge domination polynomial, $D_{2e}(P_n,x)=\Sigma_{i=\lceil\frac{n}{2}\rceil}^{n-1}d_{2e}(P_n,i)x^i$ where $d_{2e}(P_n,i)$ is the number of 2- edge dominating sets of P_n of cardinality i and obtain some properties of this polynomial.

Keywords
Path, 2-edge dominating set, 2-edge domination number and 2-edge domination polynomial.

AMS Subject Classification 05C38, 05C31.

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Contents

1. Introduction

Let $G=(V,E)$ be a simple graph of order n. For any vertex, $v\in V$, the open neighbourhood of V is the set $N(v)=\{u\in V/uv\in E\}$ and the Closed neighbourhood of V is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V$ the open nighbourhood of S is $N(S)=N[S]=N(S)\cup S$. A dominating set for a graph G is a subset D of V such that every vertex not in D is adjacent to atleast one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set of G.
An edge dominating set for a graph G is a set of $D\subseteq E$ such that every edge not in D is adjacent to atleast one edge in D. An edge dominating set is also known as a line dominating set. The edge domination number of a graph G is the minimum size of an edge dominating set in G and is denoted by $\gamma_e(G)$. A simple path is a path in which all its internal vertices have degree two and the end vertices have degree one is denoted by P_n. We use the notation $\lceil x\rceil$ for the smallest integer greater than or equal to x and $\lfloor x\rfloor$ for the largest integer less than or equal to x. Also we denote the set the $\{1,\ 2,\ 3,...n\}$ by $[n]$ throughout this paper.

2. 2-Edge Dominating Sets of Paths

In this section, we state the 2-edge domination number of path and some of its properties.

Definition 2.1. *Le G be a simple graph of order n and size m. A set $D\subseteq E$ is a 2- edge dominating set of the graph G, if every edge* e $\in$ *E-D is adjacent to atleast 2-edges in D. The 2-edge domination number $\gamma_{2e}(G)$ is the minimum cardinality among the 2-edge dominating sets of G.*

Example for 2-Edge Dominating Sets of Paths.

Let us consider P_5 as an example.

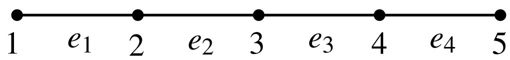

Figure 2.1

Here $E = \{e_1, e_2, e_3, e_4\}$, we take $D = \{e_1, e_2, e_4\}$, $E - D$ is $\{e_3\}$, $\{e_3\}$ is adjacent to $\{e_2\}$ and $\{e_4\}$. Therefore, the set $\{e_1, e_2, e_4\}$ is a 2-edge dominating set.

Lemma 2.2. *Let P_n, $n \geq 4$ be the path with n vertices and $n-1$ edges. Then its 2-edge domination number is $\gamma_{2e}(G) = \lceil \frac{n}{2} \rceil$.*

Lemma 2.3. *Let P_n, $n \geq 4$ be the path with $|V(P_n)| = n$ and $|E(P_n)| = n-1$. Then $d_{2e}(P_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n-1$ and $d_{2e}(P_n, i) > 0$ if $\lceil \frac{n}{2} \rceil \leq i \leq n-1$.*

Proof. If $i < \lceil \frac{n}{2} \rceil$ or $i > n-1$, then there is no 2-edge dominating set of cardinality i. Therefore, $d_{2e}(P_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n-1$. By lemma 2.2, the cardinality of the minimum 2-edge dominating set is $\lceil \frac{n}{2} \rceil$. Therefore $d_{2e}(P_n, i) > 0$ if $i \geq \lceil \frac{n}{2} \rceil$ and $i \leq n-1$. Hence we have $d_{2e}(P_n, i) = 0$ if $i < \lceil \frac{n}{2} \rceil$ or $i > n-1$ and $d_{2e}(P_n, i) > 0$ if $\lceil \frac{n}{2} \rceil \leq i \leq n-1$. □

Lemma 2.4. *Let P_n , $n \geq 4$ be the path with $|V(P_n)| = n$ and $|E(P_n)| = n-1$. Then,*
***(i)**. If $D_{2e}(P_{n-1}, i-1) = \phi$ and $D_{2e}(P_{n-3}, i-1) = \phi$ then, $D_{2e}(P_{n-2}, i-1) = \phi$.*
***(ii)**. If $D_{2e}(P_{n-1}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$ then, $D_{2e}(P_{n-2}, i-1) \neq \phi$.*
***(iii)**. If $D_{2e}(P_{n-1}, i-1) = \phi$ and $D_{2e}(P_{n-2}, i-1) = \phi$ then, $D_{2e}(P_n, i) = \phi$.*
***(iv)**. If $D_{2e}(P_{n-1}, i-1) \neq \phi$ and $D_{2e}(P_{n-2}, i-1) \neq \phi$ then, $D_{2e}(P_n, i) \neq \phi$.*

Proof. **(i)**. Since, $D_{2e}(P_{n-1}, i-1) = \phi$ and $D_{2e}(P_{n-3}, i-1) = \phi$,
by Lemma 2.3 we have

$$i-1 > n-2 \text{ or } i-1 < \lceil \tfrac{n-1}{2} \rceil \text{ and}$$

$$i-1 > n-4 \text{ or } i-1 < \lceil \tfrac{n-3}{2} \rceil.$$

Therefore, $i-1 > n-2$ or $i-1 < \lceil \frac{n-3}{2} \rceil$.
Therefore, $i-1 > n-3$ or $i-1 < \lceil \frac{n-2}{2} \rceil$ holds.
Hence, $D_{2e}(P_{n-2}, i-1) = \phi$.
(ii). Suppose $D_{2e}(P_{n-2}, i-1) = \phi$, by Lemma 2.3, we have $i-1 > n-3$ or $i-1 < \lceil \frac{n-2}{2} \rceil$.
If $i-1 > n-3$, then $i-1 > 4$. Therefore, $D_{2e}(P_{n-3}, i-1) = \phi$, which is a contradiction.
If $i-1 < \lceil \frac{n-2}{2} \rceil$, then $i-1 < \lceil \frac{n-1}{2} \rceil$. Therefore, $D_{2e}(P_{n-1}, i-1) = \phi$, which is a contradiction.
Hence, $D_{2e}(P_{n-2}, i-1) \neq \phi$.
(iii). Since $D_{2e}(P_{n-1}, i-1) = \phi$ and $D_{2e}(P_{n-2}, i-1) = \phi$, by Lemma 2.3,

$$i-1 > n-2 \text{ or } i-1 < \lceil \tfrac{n-1}{2} \rceil \text{ and}$$

$$i-1 > n-3 \text{ or } i-1 < \lceil \tfrac{n-2}{2} \rceil.$$

Therefore, $i-1 > n-2$ or $i-1 < \lceil \frac{n-2}{2} \rceil$.
Therefore, $i > n-1$ or $i < \lceil \frac{n}{2} \rceil$ holds.
Therefore, $D_{2e}(P_n, i) = \phi$.
(iv). By hyphothesis $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-2$ and $\lceil \frac{n-2}{2} \rceil \leq i-1 \leq n-3$.
Therefore, $\lceil \frac{n-2}{2} \rceil \leq i-1 \leq n-2$.
Therefore, $\lceil \frac{n}{2} \rceil \leq i \leq n-1$, holds.
Therefore, $D_{2e}(P_n, i) \neq \phi$. □

Lemma 2.5. *Let P_n , $n \geq 4$ be the path with $|V(P_n)| = n$ and $|E(P_n)| = n-1$. Suppose that $D_{2e}(P_n, i) \neq \phi$, then*
***(i)**. If $D_{2e}(P_{n-1}, i-1) = \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$ if and only if $n = 2k, i = k$.*
***(ii)**. If $D_{2e}(P_{n-2}, i-1) = \phi$, $D_{2e}(P_{n-3}, i-1) = \phi$ and $D_{2e}(P_{n-1}, i-1) \neq \phi$ if and only if $i = n-1$.*
***(iii)**. If $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) = \phi$ if and only if $i = n-2$.*
***(iv)**. If $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$ if and only if $\lceil \frac{n-1}{2} \rceil + 1 \leq i \leq n-3$.*

Proof. **(i)**. Since $D_{2e}(P_{n-1}, i-1) = \phi$, by Lemma 2.3, we get $i-1 > n-2$ or $i-1 < \lceil \frac{n-1}{2} \rceil$.
If $i-1 > n-2$, then $i > n-1$. Then by Lemma 2.3, $D_{2e}(P_n, i) = \phi$, which is a contradiction.
So $i < \lceil \frac{n-1}{2} \rceil + 1$ and since $D_{2e}(P_n, i) \neq \phi$, together $\lceil \frac{n}{2} \rceil \leq i \leq \lceil \frac{n-1}{2} \rceil + 1$, which gives $n = 2k$ and $i = k$ for some $k \in N$.
Conversely, if $n = 2k, i = k$ for some $k \in N$. Then by Lemma 2.3,
$D_{2e}(P_{n-1}, i-1) = \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$.
(ii). Assume that $D_{2e}(P_{n-2}, i-1) = \phi$, $D_{2e}(P_{n-3}, i-1) = \phi$ and $D_{2e}(P_{n-1}, i-1) \neq \phi$.
Since $D_{2e}(P_{n-2}, i-1) = \phi$ and
$D_{2e}(P_{n-3}, i-1) = \phi$, by Lemma 2.3, we have $i-1 > n-3$ or $i-1 < \lceil \frac{n-2}{2} \rceil$ and $i-1 > n-4$ or $i-1 < \lceil \frac{n-3}{2} \rceil$.
Therefore, $i-1 > n-3$ or $i-1 < \lceil \frac{n-3}{2} \rceil$.
Since $D_{2e}(P_{n-1}, i-1) \neq \phi$, we have
$\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-2$.
If $i-1 < \lceil \frac{n-3}{2} \rceil$, then $i-1 < \lceil \frac{n-1}{2} \rceil$.
Therefore by Lemma 2.3, $D_{2e}(P_{n-1}, i-1) \neq \phi$, which is a contradiction.
So we have $i-1 > n-2$. Therefore $i > n-1$........(1)
Also, since $D_{2e}(P_{n-1}, i-1) \neq \phi$, we have $i-1 \leq n-2$.
Therefore $i \leq n-1$............(2)
Combining (1) and (2), we get $i = n-1$.
Conversely, if $i = n-1$.
$D_{2e}(P_{n-2}, i-1) = D_{2e}(P_{n-2}, n-2) = \phi$.
$D_{2e}(P_{n-3}, i-1) = D_{2e}(P_{n-3}, n-2) = \phi$.
$D_{2e}(P_{n-1}, i-1) = D_{2e}(P_{n-1}, n-2) \neq \phi$.
(iii). Assume that $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) = \phi$.
Since $D_{2e}(P_{n-3}, i-1) = \phi$, by Lemma 2.3, we have $i-1 >$

$n-4$ or $i-1 < \lceil \frac{n-3}{2} \rceil$........(1)
Since $D_{2e}(P_{n-1}, i-1) \neq \phi$, we have
$\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-2$.......(2)
Suppose $i-1 < \lceil \frac{n-3}{2} \rceil$, then (2) does not hold.
Therefore our assumption is wrong.

Therefore $i-1 > n-4$.
Also since $D_{2e}(P_{n-2}, i-1) \neq \phi$
We have $\lceil \frac{n-2}{2} \rceil \leq i-1 \leq n-3$........(3)
But $i-1 > n-4$.
Therefore, $i-1 \geq n-3$........(4)

Combining (3) and (4), we get $i-1 = n-3$.
Therefore $i = n-2$. Conversely, if $i = n-2$.
Then $D_{2e}(P_{n-1}, i-1) = D_{2e}(P_{n-1}, n-3) \neq \phi$.
$D_{2e}(P_{n-2}, i-1) = D_{2e}(P_{n-2}, n-3) \neq \phi$.
$D_{2e}(P_{n-3}, i-1) = D_{2e}(P_{n-3}, n-3) = \phi$.
(iv). Assume that $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$.
Then by Lemma 2.3, $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-2$, $\lceil \frac{n-2}{2} \rceil \leq i-1 \leq n-3$ and $\lceil \frac{n-3}{2} \rceil \leq i-1 \leq n-4$.
Combining all these $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-4$ and hence $\lceil \frac{n-1}{2} \rceil + 1 \leq i \leq n-3$.

Conversely, suppose $\lceil \frac{n-1}{2} \rceil + 1 \leq i \leq n-3$.
Therefore, $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-4$.
Then $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq n-2$, $\lceil \frac{n-2}{2} \rceil \leq i-1 \leq n-3$, $\lceil \frac{n-3}{2} \rceil \leq i-1 \leq n-4$ holds.
From these we obtain $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) \neq \phi$.

Hence the theorem. □

Theorem 2.6. ***(i)***. $D_{2e}(P_{2n}, n) = \{1,3,5,7,9,...,2n-1\}$.
(ii). *If* $D_{2e}(P_{n-2}, i-1) = \phi$, $D_{2e}(P_{n-3}, i-1) = \phi$ *and* $D_{2e}(P_{n-1}, i-1) \neq \phi$, *then* $D_{2e}(P_n, i) = D_{2e}(P_n, n-1) = [n-1]$.
(iii). *If* $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ *and* $D_{2e}(P_{n-3}, i-1) = \phi$, *then*
$D_{2e}(P_n, i) = D_{2e}(P_n, n-2) = \{[n-1] - \{x\}/x \in [n-1]$ *and* $x \neq 1, n-1\}$.
(iv). *If* $D_{2e}(P_{n-1}, i-1) = \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$, *then* $D_{2e}(P_n, i) = \{X \cup \{n-1\}/X \in D_{2e}(P_{n-2}, i-1)\}$.
(v). *If* $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) = \phi$, *then* $D_{2e}(P_n, i) = \{Y \cup \{n-1\}/Y \in D_{2e}(P_{n-1}, i-1)\}$.
(vi). *If* $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$, *then* $D_{2e}(P_n, i) = \{X \cup \{n-1\} \cup Y \cup \{n-1\}\}$ *where* $X \in D_{2e}(P_{n-1}, i-1)$ *and* $Y \in D_{2e}(P_{n-2}, i-1)$.

Proof. **(i)**. For every $n \geq 6$, $D_{2e}(P_{2n}, n)$ has only one 2-edge dominating sets as $D_{2e}(P_{2n}, n) = \{1,3,5,7,9,...,2n-1\}$.
(ii). Since $D_{2e}(P_{n-2}, i-1) = \phi$, $D_{2e}(P_{n-3}, i-1) = \phi$ and $D_{2e}(P_{n-1}, i-1) \neq \phi$, by Lemma 2.5(ii), $i = n-1$. Therefore, $D_{2e}(P_n, i) = D_{2e}(P_n, n-1) = [n-1]$.
(iii). Since $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$ and $D_{2e}(P_{n-3}, i-1) = \phi$, by Lemma 2.5(iii), i = n − 2..
Therefore, $D_{2e}(P_n, i) = D_{2e}(P_n, n-2) = \{[n-1] - \{x\}/x \in [n-1]$ and $x \neq 1, n-1\}$.
(iv). Let X be 2-edge dominating set of P_{n-2} with cardinality $i-1$. All the elements of $D_{2e}(P_{n-2}, i-1)$ end with $n-3$.

Therefore $n-3 \in X$, adjoin $n-1$ with X. Hence every X of $D_{2e}(P_{n-2}, i-1)$ belongs to $D_{2e}(P_n, i)$ by adjoining $n-1$ only.
Conversely suppose $Z \in D_{2e}(P_n, i)$. Here all the elements of $D_{2e}(P_n, i)$ end with $n-1$ only. Suppose, $n-1 \in Z$ then $Z = X \cup \{n-1\}$ where X ends with $n-3$.
(v) Let Y be a 2-edge dominating set of P_{n-1} with cardinality $i-1$. All the elements of $D_{2e}(P_{n-1}, i-1)$ end with $n-2$. Therefore $n-2 \in Y$ adjoin $n-1$ with Y. Hence every Y of $D_{2e}(P_{n-1}, i-1)$ belongs to $D_{2e}(P_n, i)$ by adjoining $n-1$ only. Conversely suppose $Z \in D_{2e}(P_n, i)$. Here all the elements of $D_{2e}(P_n, i)$ ends with $n-1$ only. Suppose, $n-1 \in Z$ then $Z = Y \cup \{n-1\}$ where Y ends with $n-1$.
(vi). Construction of $D_{2e}(P_n, i)$ from $D_{2e}(P_{n-1}, i-1)$ and $D_{2e}(P_{n-2}, i-1)$. Let X be a 2-edge dominating set of P_{n-1} with cardinality $i-1$. All the elements of $D_{2e}(P_{n-1}, i-1)$ ends with $n-2$. Therefore $n-2 \in X$ adjoin $n-1$ with X.

Hence every X of $D_{2e}(P_{n-1}, i-1)$ belongs to $D_{2e}(P_n, i)$ by adjoining $n-1$ only. Let Y be a 2-edge dominating set of P_{n-2} with cardirality $i-1$. All the elements of $D_{2e}(P_{n-2}, i-1)$ ends with $n-3$. Therefore $n-3 \in Y$ adjoin $n-1$ with Y. Hence every Y of $D_{2e}(P_{n-2}, i-1)$ belongs to $D_{2e}(P_n, i)$ by adjoining $n-1$ only.

Conversely suppose $Z \in D_{2e}(P_n, i)$. Here all the elements of $D_{2e}(P_n, i)$, ends with $n-1$ only. Suppose $n-1 \in Z$, then $Z = X \cup \{n-1\} \cup Y \cup \{n-1\}$ where X ends with $n-2$, $X \in D_{2e}(P_{n-1}, i-1)$ and Y ends with $n-3$, $Y \in D_{2e}(P_{n-2}, i-1)$.
Hence the proof. □

Theorem 2.7. *If* $D_{2e}(P_n, i)$ *be the family of the 2-edge dominatirg sets of* P_n *with cardinality i, where* $i \geq \lceil \frac{n}{2} \rceil$ *then*

$$d_{2e}(P_n, i) = d_{2e}(P_{n-1}, i-1) + d_{2e}(P_{n-2}, i-1).$$

Proof. Using Theorem 2.6, we consider all the four cases given below, where $i \geq \lceil \frac{n}{2} \rceil$.
(i). If $D_{2e}(P_{n-1}, i-1) = \phi$ and $D_{2e}(P_{n-2}, i-1) = \phi$, then $D_{2e}(P_n, i) = \phi$.
(ii). If $D_{2e}(P_{n-1}, i-1) = \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$, then $D_{2e}(P_n, i) = \{X \cup \{n-1\}/X \in D_{2e}(P_{n-2}, i-1)\}$.
(iii). If $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) = \phi$, then $D_{2e}(P_n, i) = \{Y \cup \{n-1\}/Y \in D_{2e}(P_{n-1}, i-1)\}$.
(iv). If $D_{2e}(P_{n-1}, i-1) \neq \phi$, $D_{2e}(P_{n-2}, i-1) \neq \phi$, then $D_{2e}(P_n, i) = \{X \cup \{n-1\} \cup Y \cup \{n-1\}\}$ where $X \in D_{2e}(P_{n-1}, i-1)$ and $Y \in D_{2e}(P_{n-2}, i-1)$.

From the above construction in each case, we obtain

$$d_{2e}(P_n, i) = d_{2e}(P_{n-1}, i-1) + d_{2e}(P_{n-2}, i-1).$$

□

3. 2-Edge Domination Polynominals of Paths

Definition 3.1. *Let* $D_{2e}(P_n, i)$ *be the family of 2-edge dominating sets of* P_n *with cardinality i and let* $d_{2e}(P_n, i) = |D_{2e}(P_n, i)|$.

Then the 2- edge domination polynomial $D_{2e}(P_n,x)$ of P_n is defined as $D_{2e}(P_n,x)=\Sigma_{i=\gamma_{2e}(P_n)}^{n-1} d_{2e}(P_n,i)x^i$, where $\gamma_{2e}(P_n)$ is the 2-edge domination number of P_n.

Theorem 3.2. *For every $n \geq 5$,*

$$D_{2e}(P_n,x)=x[D_{2e}(P_{n-1},x)+D_{2e}(P_{n-2},x)]$$

with initial values

$$D_{2e}(P_3,x)=x^2$$

$$D_{2e}(P_4,x)=x^2+x^3.$$

Proof. We have

$$d_{2e}(P_n,i)=d_{2e}(P_{n-1},i-1)+d_{2e}(P_{n-2},i-1)$$

Therefore,

$$d_{2e}(P_n,i)x^i=d_{2e}(P_{n-1},i-1)x^i+d_{2e}(P_{n-2},i-1)x^i$$

$$\Sigma d_{2e}(P_n,i)x^i=\Sigma d_{2e}(P_{n-1},i-1)x^i+\Sigma d_{2e}(P_{n-2},i-1)x^i$$

$$\Sigma d_{2e}(P_n,i)x^i=x\Sigma d_{2e}(P_{n-1},i-1)x^{i-1}+x\Sigma d_{2e}(P_{n-2},i-1)x^{i-1}$$

$$D_{2e}(P_n,x)=xD_{2e}(P_{n-1},x)+xD_{2e}(P_{n-2},x)$$

Therefore

$$D_{2e}(P_n,x)=x[D_{2e}(P_{n-1},x)+D_{2e}(P_{n-2},x)]$$

With the initial values

$$D_{2e}(P_3,x)=x^2$$

$$D_{2e}(P_4,x)=x^2+x^3.$$

□

$d_{2e}(P_n,i)$ the number of 2-edge dominating sets of P_n witil cardinality i for $3 \leq n \leq 14$ and $2 \leq i \leq 13$ as shown in Table 1.

Table 1

i / n	2	3	4	5	6	7	8	9	10	11	12	13
3	1											
4	1	1										
5	0	2	1									
6	0	1	3	1								
7	0	0	3	4	1							
8	0	0	1	6	5	1						
9	0	0	0	4	10	6	1					
10	0	0	0	1	10	15	7	1				
11	0	0	0	0	5	20	21	8	1			
12	0	0	0	0	1	15	35	28	9	1		
13	0	0	0	0	0	6	35	56	36	10	1	
14	0	0	0	0	0	1	21	70	84	45	11	1

Theorem 3.3. *The following properties hold for the coefficients of $D_{2e}(P_n,x)$*

***(i)**. $d_{2e}(P_{2n},n)=1$, for every $n \geq 2$.*

***(ii)**. $d_{2e}(P_{2n-1},n)=n-1$, for every $n \geq 2$.*

***(iii)**. $d_{2e}(P_n,n-1)=1$, for every $n \geq 3$.*

***(iv)**. $d_{2e}(P_n,n-2)=n-3$, for every $n \geq 3$.*

***(v)**. $d_{2e}(P_n,n-3)=\frac{1}{2}[n^2-9n+20]$, for every $n \geq 6$.*

***(vi)**. $d_{2e}(P_n,n-4)=\frac{1}{6}[n^3-18n^2+107n-210]$, for every $n \geq 8$.*

***(vii)**. $d_{2e}(P_n,n-5)=\frac{1}{24}[n^4-30n^3+335n^2-1650n+3024]$, for every $n \geq 10$.*

Proof. **(i)**. Since $D_{2e}(P_{2n},n)=\{2,4,6,8,\ldots 2n\}$, we have $d_{2e}(P_{2n},n)=1$.

(ii). To prove $d_{2e}(P_{2n-1},n)=n-1$, for every $n \geq 2$, we apply induction on n.
When $n=2$, LHS: $d_{2e}(P_3,2)=1$,[From table].
RHS : $n-1=2-1=1$.
Therefore, LHS= RHS.

Now suppose that the result is true for all numbers less than $n+1$ and we prove it for n.
By Theorem 3.2, we have

$$d_{2e}(P_{2n-1},n)=d_{2e}(P_{2n-2},n-1)+d_{2e}(P_{2n-3},n-1)$$
$$=1+n-2$$
$$=n-1$$

Hence $d_{2e}(P_{2n-1},n)=n-1$, for every $n \geq 2$.

(iii). Since $D_{2e}(P_n,n-1)=\{[n-1]\}$, we have the result.

(iv). We have $D_{2e}(P_n,n-2)=\{[n-2]-\{x\} \,/\, x \in [n]$ and $x \neq 1,n\}$.
Therefore $d_{2e}(P_n,n-2)=n-3$, for every $n \geq 3$.

(v). To prove $d_{2e}(P_n,n-3)=\frac{1}{2}[n^2-9n+20]$, for every $n \geq 6$.
We apply induction on n.
When $n=6$
LHS: $d_{2e}(P_6,3)=1$ [From tabIe]

$$\text{RHS: } \frac{1}{2}[(6)^2-9(6)+20]=\frac{1}{2}[36-54+20]$$
$$=1.$$

Therefore LHS =RHS.

Now suppose that the result is true for all numbers less than n and we prove it for n.

By Theorem 3.2, we have,

$$d_{2e}(P_n,\ n-3)=d_{2e}(P_{n-1},n-4)+d_{2e}(P_{n-2},n-4)$$
$$=\frac{1}{2}[(n-1)^2-9(n-1)+20]+(n-2)-3$$
$$=\frac{1}{2}[n^2+1-2n-9n+9+20]+n-5$$
$$=\frac{n^2-9n+20}{2}$$

Hence $d_{2e}(P_n,n-3)=\frac{1}{2}[n^2-9\text{n}+20]$, for every $n \geq 6$.

(vi). To prove, $d_{2e}(P_n,n-4)=\frac{1}{6}[n^3-18n^2+107n-210]$,

for every $n \geq 8$.
We apply induction on n.
When n = 8.
LHS:
$d_{2e}(P_8, 4) = 1$[from table]

$$RHS := \frac{1}{6}[(8)^3 - 18(8)^2 + 107(8) - 210]$$

$$= \frac{1}{6}[512 - 1152 + 856 - 210]$$

$$= \frac{1}{6}[6]$$

$$= 1$$

Therefore LHS = RHS.

Now suppose that the result is true for all numbers less than n and we prove it for n.
By Theorem 3.2, we have
$d_{2e}(P_n, n-4) = d_{2e}(P_{n-1}, n-5) + d_{2e}(P_{n-2}, n-5)$

$$= \frac{1}{6}[(n-1)^3 - 18(n-1)^2 + 107(n-1) - 210]$$

$$+\frac{1}{2}[(n-2)^2 - 9(n-2) + 20]$$

$$= \frac{1}{6}[n^3 - 3n^2 + 3n - 1 - 18n^2 - 18 + 36n + 107n - 107 - 210]$$
$$+\tfrac{1}{2}[n^2 + 4 - 4n - 9n + 18 + 20]$$
$$= \frac{1}{6}[n^3 - 21n^2 + 146n - 336 + 3n^2 - 39n + 126]$$
$$= \frac{1}{6}[n^3 - 18n^2 + 107n - 210]$$

Hence $d_{2e}(P_n, n-4) = \frac{1}{6}[n^3 - 18n^2 + 107n - 210]$, for every $n \geq 8$.

(vii). To prove, $d_{2e}(P_n, n-5) = \frac{1}{24}[n^4 - 30n^3 + 335n^2 - 1650n + 3024]$, for every $n \geq 10$.
We apply induction on n.
When $n = 10$
LHS: $d_{2e}(P_{10}, 5) = 1$ [From table]
$$RHS := \frac{1}{24}[(10)^4 - 30(10)^3 + 335(10)^2 - 1650(10) + 3024]$$
$$= \frac{1}{24}x[10000 - 30000 + 33500 - 16500 + 3024]$$
$$= 1$$
Therefore LHS = RHS

Now suppose that the result is true for all numbers less than n and we prove it for n.

By Theorem 3.2, we have

$$d_{2e}(P_n, n-5) = d_{2e}(P_{n-1}, n-6) + d_{2e}(P_{n-2}, n-6)$$

$$= \frac{1}{24}[(n-1)^4 - 30(n-1)^3 + 335(n-1)^2 - 1650(n-1) + 3024]$$

$$+\frac{1}{6}[(n-2)^3 - 18(n-2)^2 + 107(n-2) - 210]$$

$$= \frac{1}{24}[n^4 - 4n^3 + 6n^2 - 4n + 1 - 30(n^3 - 3n^2 + 3n - 1) + 335(n^2 + 1 - 2n) - 1650(n-1) + 3024]$$

$$+\frac{1}{6}[n^3 - 6n^2 + 12n - 8 - 18(n^2 + 4 - 4n) + 107(n-2) + 210]$$

$$= \frac{1}{24}[n^4 - 340^3 + 431n^2 - 2414n + 5040]$$
$$+\frac{1}{6}[n^3 - 24n^2 + 191n - 504]$$

$$= \frac{1}{24}[n^4 - 34n^3 + 431n^2 - 2414n + 5040 + 4n^3 - 96n^2 + 764n - 2016]$$

$$= \frac{1}{24}[n^4 - 30n^3 + 335n^2 - 1650n + 3024]$$

Hence, $d_{2e}(P_n, n-5) = \frac{1}{24}[n^4 - 30n^3 + 335n^2 - 1650n + 3024]$, for every $n \geq 10$.

□

Theorem 3.4. ***(i)***.$\Sigma_{i=n}^{2n} d_{2e}(P_i, n) = 2\Sigma_{i=2}^{2n-2} d_{2e}(P_i, n-1)$, *for every* $n \geq 3$.
(ii). *For every* $j \geq \lceil \frac{n}{2} \rceil$,

$$d_{2e}(P_{n+1}, j+1) - d_{2e}(P_n, j+1) = d_{2e}(P_n, j) - d_{2e}(P_{n-2}, j).$$

(iii). *If* $S_n = \Sigma_{i=\lceil \frac{n}{2} \rceil}^{n} d_{2e}(P_n, j)$, *then for every* $n \geq 6$, $S_n = S_{n-1} + S_{n-2}$ *with initial values*

$$S_3 = 1, S_4 = 2, S_5 = 3, S_6 = 5, S_7 = 8.$$

Proof. **(i)**. First we prove by induction on n.
Suppose $n = 3$ then
$$\sum_{i=3}^{6} d_{2e}(P_i, 3) = 4 = 2\sum_{i=2}^{4} d_{2e}(P_i,\ 2).$$
$\sum_{i=k}^{2k} d_{2e}(P_i, k) = \sum_{i=k}^{2k} d_{2e}(P_{i-1}, k-1) + \sum_{i=k}^{2k} d_{2e}(P_{i-2}, k-1)$

$$= 2\sum_{i=k-1}^{2(k-1)} d_{2e}(P_{i-1}, k-2) + 2\sum_{i=k-1}^{2(k-1)} d_{2e}(P_{i-2}, k-2)$$

$$= 2\sum_{i=k-1}^{2(k-2)} d_{2e}(P_{i-1}, k-l)$$

Hence, $\Sigma_{i=n}^{2n} d_{2e}(P_i, n) = 2\Sigma_{i=2}^{2n-2} d_{2e}(P_i, n-1)$, for every $n \geq 3$.
(ii). By Theorem 2.7, we have
$d_{2e}(P_{n+1}, j+1) - d_{2e}(P_n, j+1)$
$= d_{2e}(P_n, j) + d_{2e}(P_{n-1}, j) - d_{2e}(P_{n-1}, j) - (d_{2e}(P_{n-2}, j)$
Therefore, $d_{2e}(P_{n+1}, j+1) - d_{2e}(P_n, j+1)$
$= d_{2e}(P_n, j) - d_{2e}(P_{n-2}, j)$
Therefore we have the result.
(iii). By Theorem 2.7, we have

$$S_n = \sum_{j=\lceil \frac{n}{2} \rceil}^{n} d_{2e}(P_n, j)$$

$$= \sum_{j=\lceil \frac{n}{2} \rceil}^{n} [d_{2e}(P_{n-1}, j-1) + d_{2e}(P_{n-2}, j-1)]$$

$$= \sum_{j=\lceil \frac{n}{2} \rceil -1}^{n-1} d_{2e}(P_{n-1},\ j-1) + \sum_{j=\lceil \frac{n}{2} \rceil -1}^{n-1} d_{2e}(P_{n-2}, j-1)$$

Hence $S_n = S_{n-1} + S_{n-2}$.

□

4. Conclusion

In this paper 2- edge domination sets of paths and 2- edge domination polynomials of paths are studied and obtained some properties. We can generalize this study to any power of path.

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