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On a subclass meromorphic functions with positive coefficients

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Abstract

In this paper we introduce and study a new subclass of meromorphically convex functions with positive coefficients defined by a differential operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the class $\sigma_p(\alpha)$.

Keywords

Meromorphic, convex, starlike, coefficient estimates.

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Contents

1. Introduction

Let $\sum^*$ be denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m,\ \ (a_m\geq 0) \tag{1.1}$$

which are analytic in the punctured unit disc $E=\{z: z\in C \text{ and } 0<|z|<1\}$. Let $g(z)\in\sum^*$ be given by

$$g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_m z^m,\ \ (b_m\geq 0). \tag{1.2}$$

Then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_m b_m z^m=(g*f)(z). \tag{1.3}$$

A function $f\in\sum^*$ is meromorphic starlike of order α $(0\leq\alpha<1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad z\in C. \tag{1.4}$$

The class of all such functions is denoted by $\sum_s^*(\alpha)$.

A function $f\in\sum^*$ is meromorphically convex of order $\alpha(0\leq\alpha<1)$ if

$$-Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad z\in E. \tag{1.5}$$

The class of all such functions is denoted by $\sum_k^*(\alpha)$. The classes $\sum_s^*(\alpha)$ and $\sum_k^*(\alpha)$ were introduced and studied by Pommerenke [11], Clunie [3], Royster [15] and others.

For functions $f(z)\in\sum^*$, we define a linear operator D^n

by the following form

$$
\begin{aligned}
D^0 f(z) &= f(z) \\
D^1 f(z) &= \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \cdots \quad = \frac{(z^2 f(z))'}{z} \\
D^2 f(z) &= D(D^1 f(z)) \\
&\vdots \\
D^n f(z) &= D(D^{n-1} f(z)) \\
&= \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m \\
&= \frac{(z^2 D^{n-1} f(z))'}{z}, \quad \text{for } n = 1,2,\cdots \qquad (1.6)
\end{aligned}
$$

Now, we define a new subclass $\sigma_p(\alpha)$ of Σ^*.

Definition 1.1. *For $-1 \le \alpha < 1$, we let $\sigma_p(\alpha)$ be the subclass of Σ^* consisting of the form (1.1) and satisfying the analytic criterion*

$$Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - \alpha\right\} > \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right|, \qquad (1.7)$$

$D^n f(z)$ is given by (**??**)

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\sigma_p(\alpha)$.

2. Coefficient inequality

In this section we obtain the coefficient bounds of function $f(z)$ for the class $\sigma_p(\alpha)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\sigma_p(\alpha)$ if*

$$\sum_{m=1}^{\infty} (m+2)^n (2m+3-\alpha)|a_m| \le (1-\alpha), \quad -1 \le \alpha < 1. \qquad (2.1)$$

Proof. It sufficient to show that $\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| - Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right\} \le (1-\alpha)$.

Now

$$
\begin{aligned}
&\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| - Re\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right\} \\
&\le 2\left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| \\
&\le \frac{2\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m||z^m|}{\frac{1}{|z|} - \sum_{m=1}^{\infty} (m+2)^n |a_m||z^m|}
\end{aligned}
$$

Letting $z \to 1$ along the real axis, we obtain

$$\le \frac{2\sum_{m=1}^{\infty} (m+2)^n (m+1)|a_m|}{1 - \sum_{m=1}^{\infty} (m+2)^n |a_m|}.$$

The above expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty} (m+2)^n (2m+3-\alpha)|a_m| \le (1-\alpha).$$

Hence the theorem is completed □

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $\sigma_p(\alpha)$. Then*

$$a_m \le \frac{(1-\alpha)}{\sum_{m=1}^{\infty} (m+2)^n (2m+3-\alpha)}, \quad m \ge 1. \qquad (2.2)$$

Equality holds for the function of the form

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{(m+2)^n (2m+3-\alpha)} z^m \qquad (2.3)$$

3. Distortion Theorems

In this section we obtain the sharp for the Distortion theorems of the form (1.1) .

Theorem 3.1. *Let the function $f(z)$ defined by (1.1) be in the class $\sigma_p(\alpha)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r} - \frac{(1-\alpha)}{3^n(5-\alpha)} r \le |f(z)| \le \frac{1}{r} + \frac{(1-\alpha)}{3^n(5-\alpha)} r \qquad (3.1)$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n(5-\alpha)} z, \ at \ z = r, ir. \qquad (3.2)$$

Proof. Suppose $f(z)$ is in $\sigma_p(\alpha)$. In view of Theorem 2.1, we have
$3^n(5-\alpha) \sum_{m=1}^{\infty} a_m \le \sum_{m=1}^{\infty} (m+2)^n (2m+3-\alpha) \le (1-\alpha)$
which evidently yields $\sum_{m=1}^{\infty} a_m \le \frac{1-\alpha}{3^n(5-\alpha)}$.

Consequently, we obtain

$$|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \leq \left|\frac{1}{z}\right| + \sum_{m=1}^{\infty} a_m |z|^m$$
$$\leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m$$
$$\leq \frac{1}{r} + \frac{(1-\alpha)}{3^n(5-\alpha)} r.$$

$$\text{Also, } |f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m$$
$$\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m$$
$$\geq \frac{1}{r} - \frac{(1-\alpha)}{3^n(5-\alpha)} r.$$

Hence the results (3.1) follow. □

Theorem 3.2. *Let the function $f(z)$ defined by (1.1) be in the class $\sigma_p(\alpha)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{(1-\alpha)}{3^n(5-\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n(5-\alpha)}.$$

The result is sharp, the extremal function being of the form (3.2)

Proof. From Theorem 2.1, we have

$$3^n(5-\alpha)\sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} (m+2)^n(2m+3-\alpha)$$
$$\leq (1-\alpha)$$

which evidently yields $\sum\limits_{m=1}^{\infty} m a_m \leq \frac{1-\alpha}{3^n(5-\alpha)}$.
Consequently, we obtain

$$|f'(z)| \leq \left|\frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}\right|$$
$$\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m$$
$$\leq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n(5-\alpha)}.$$

$$\text{Also, } |f'(z)| \geq \left|\frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}\right|$$
$$\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$
$$\geq \frac{1}{r^2} + \frac{(1-\alpha)}{3^n(5-\alpha)}.$$

This completes the proof. □

4. Class preserving integral operators

In this section we consider the class preserving integral operator of the form (1.1) .

Theorem 4.1. *Let the function $f(z)$ defined by (1.1) be in the class $\sigma_p(\alpha)$. Then*

$$f(z) = cz^{-c-1}\int_0^z t^c f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{c}{c+m+1}\, a_m z^m, \;\; c > 0 \tag{4.1}$$

belongs to the class $\sigma[\delta(\alpha,n,c)]$, where

$$\delta(\alpha,n,c) = \frac{3^n(5-\alpha)(c+2) - (1-\alpha)c}{3^n(5+\alpha)(c+2) - (1-\alpha)c} \tag{4.2}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n(5-\alpha)} z$.

Proof. Suppose

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

is in $\sigma_p(\alpha)$. we have
$f(z) = cz^{-c-1}\int\limits_0^z t^c f(t)dt = \frac{1}{z} + \sum\limits_{m=1}^{\infty} \frac{c}{c+m+1}\, a_m z^m, \;\; c > 0.$
It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta}\,\frac{c a_m}{m+c+1} \leq 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_p(\alpha)$, we have

$$\frac{\sum\limits_{m=1}^{\infty} (m+2)^n(2m+3-\alpha)|a_m|}{1-\alpha} \leq 1. \tag{4.4}$$

Thus (4.3) will be satisfied if
$\frac{(m+\delta)c}{(1-\delta)(m+c+1)} \leq \frac{(m+2)^n(2m+m-\alpha)}{1-\alpha}$, for each m or

$$\delta \leq \frac{(m+2)^n(2m+3-\alpha)(c+m+1) - mc(1-\alpha)}{(m+2)^n 2m+3-\alpha)(c+m+1) + (1-\alpha)} = G(m) \tag{4.5}$$

Then $G(m+1) - G(m) > 0$, for each m.
Hence $G(m)$ is an increasing function of m.
Since $G(1) = \frac{3^n(5-\alpha)(c+2) - c(1-\alpha)}{3^n(5-\alpha)(c+2) + c(1-\alpha)}$.
The result follows. □

5. Convex linear combinations and convolution properties

In this section we obtain the radius meromorphically convex of order δ, convex linear combinations and convolution properties for functions in the class $\sigma_p(\alpha)$.

Theorem 5.1. *If the function* $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m$ *is in* $\sigma_p(\alpha)$ *then* $f(z)$ *is meromorphically convex of order* $\delta(0\leq\delta<1)$ *in* $|z|<r=r(\alpha,\delta)$ *where*
$r(\alpha,\delta)=\inf_{n\geq 1}\left\{\frac{(1-\delta)(m+2)^n(2m+3-\alpha)}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}$.
The result is sharp.

Proof. Let $f(z)$ be in $\sigma_p(\alpha)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty}(m+2)^n(2m+3-\alpha)|a_m|\leq(1-\alpha). \tag{5.1}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right|\leq(1-\delta)$ for $|z|<r=r(\alpha,\delta)$, where $r(\alpha,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum_{m=1}^{\infty}m(m+1)a_mz^{m-1}}{\frac{-1}{z^2}+\sum_{m=1}^{\infty}ma_mz^{m-1}}\right|$$
$$\leq\frac{\sum_{m=1}^{\infty}m(m+1)a_m|z|^{m+1}}{1-\sum_{m=1}^{\infty}ma_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1}\leq 1. \tag{5.2}$$

By (5.1), it follows that (5.2) is true if
$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1}\leq\frac{(m+2)^n(2m+3-\alpha)}{1-\alpha}, m\geq 1$
or

$$|z|\leq\left\{\frac{(1-\delta)(m+2)^n(2m+3-\alpha)}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{5.3}$$

Setting $|z|=r(\alpha,\delta)$ in (5.3), the result follows.
The result is sharp for the function.

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{(m+2)^n(2m+3-\alpha)}z^m,\ \ m\geq 1.$$

□

Theorem 5.2. *Let* $f_0(z)=\frac{1}{z}$ *and*
$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{(m+2)^n(2m+3-\alpha)}z^m,\ \ m\geq 1.$
Then $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m$ *is in the class* $\sigma_p(\alpha)$ *if and only if it can be expressed in the form*

$$f(z)=\lambda_0f_0(z)+\sum_{m=1}^{\infty}\lambda_mf_m(z),$$

where $\lambda_0\geq 0,\lambda_m\geq 0,m\geq 1$ *and* $\lambda_0+\sum_{m=1}^{\infty}\lambda_m=1$.

Proof. Let $f(z)=\lambda_0f_0(z)+\sum_{m=1}^{\infty}\lambda_mf_m(z)$ with $\lambda_0\geq 0$, $\lambda_m\geq 0,m\geq 1$ and $\lambda_0+\sum_{m=1}^{\infty}\lambda_m=1$. Then

$$f(z)=\lambda_0f_0(z)+\sum_{m=1}^{\infty}\lambda_mf_m(z)$$
$$=\frac{1}{z}+\sum_{m=1}^{\infty}\lambda_m\frac{(1-\alpha)}{(m+2)^n(2m+3-\alpha)}z^m$$

$$\text{Since }\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}\lambda_m\frac{(1-\alpha)}{(m+2)^n(2m+3-\alpha)}$$
$$=\sum_{m=1}^{\infty}\lambda_m=1-\lambda_0\leq 1.$$

By Theorem 2.1, $f(z)$ is in the class $\sigma_p(\alpha)$.

Conversely suppose that the function $f(z)$ is in the class $\sigma_p(\alpha)$, since
$a_m\leq\frac{(1-\alpha)}{(m+2)^n(2m+3-\alpha)}z^m, m\geq 1.$
$\lambda_m=\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}a_m$ and $\lambda_0=1-\sum_{m=1}^{\infty}\lambda_m$. It follows that $f(z)=\lambda_0f_0(z)+\sum_{m=1}^{\infty}\lambda_mf_m(z)$.
This completes the proof of the theorem. □

For the functions $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m$ and $g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_mz^m$ belongs to $\sum_p$, we denoted by $(f*g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as
$(f*g)(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mb_mz^m$

Theorem 5.3. *If the function* $f(z)=\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m$ *and* $g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_mz^m$ *are in the class* $\sigma_p(\alpha)$ *then* $(f*g)(z)$ *is in the class* $\sigma_p(\alpha)$

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_p(\alpha)$. By Theorem 2.1, we have $\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}a_m\leq 1$
and $\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}b_m\leq 1$.
Since $f(z)$ and $g(z)$ are regular are in E, so is $(f*g)(z)$. Further more

$$\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}a_mb_m$$
$$\leq\sum_{m=1}^{\infty}\left\{\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}\right\}^2a_mb_m$$
$$\leq\left(\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}a_m\right)$$
$$(\times)\left(\sum_{m=1}^{\infty}\frac{(m+2)^n(2m+3-\alpha)}{(1-\alpha)}b_m\right).$$
$$\leq 1.$$

Hence, by Theorem 2.1, $(f*g)(z)$ is in the class $\sigma_p(\alpha)$. □

6. Neighborhoods for the class $\sigma_p(\alpha,\gamma)$

In this section we define the $\delta-$neighborhood of a function $f(z)$ and establish a relation between $\delta-$neighborhood and $\sigma_p(\alpha,\gamma)$ class of a function.

Definition 6.1. *A function $f \in \sum_p$ is said to in the class $\sigma_p(\alpha,\gamma)$ if there exists a function $g \in \sigma_p(\alpha)$ such that*

$$\left|\frac{f(z)}{g(z)} - 1\right| < (1-\gamma), \quad z \in E,\ 0 \le \gamma < 1. \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4] and Ruschweyh [16]. We defined the $\delta-$neighborhood of a function $f \in \sum_p$ by

$$N_\delta(f) = \{g \in \sum_p \mid g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \sum_{m=1}^{\infty} m|a_m - b_m| \le \delta\} \tag{6.2}$$

Theorem 6.2. *If $g \in \sigma_p(\alpha)$ and*

$$\gamma = 1 - \frac{\delta(5-\alpha)}{4} \tag{6.3}$$

then $N_\delta(g) \subset \sigma_p(\alpha,\gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \le \delta \tag{6.4}$$

which implies the coefficient of inequality $\sum_{m=1}^{\infty} |a_m - b_m| \le \delta,\ \ m \in \mathbb{N}$.

Since $g \in \sigma_p(\alpha)$, we have $\sum_{m=1}^{\infty} b_m = \frac{1-\alpha}{5-\alpha}$.

So that $\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{m=1}^{\infty} |a_m - b_m|}{1 - \sum_{m=1}^{\infty} b_m} < \frac{\delta(5-\alpha)}{4} = 1-\gamma,$

provided γ is given by (6.3).

Hence, by Definition, $f \in \sigma_p(\alpha,\gamma)$ for γ given by (6.3), which completes the proof of theorem. □

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