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Improved Runge-Kutta direct method with fourth derivative for solving $y''' = f(x,y)$

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Abstract

In this study, we derive explicit two-derivative improved Runge-Kutta direct methods (TDIRKD) which incorporates the fourth derivative of the solution to solve special third order ordinary differential equations. The improved Runge-Kutta direct methods are extended to these methods. TDIRKD methods which involve one evaluation of third derivative and multiple evaluations of fourth derivative per step are constructed. Order conditions for TDIRKD methods are derived up to order five. Two-stage fourth-order TDIRKD method is presented. The stability polynomial of the proposed method have been obtained. Numerical computations have been given to illustrate the accuracy and efficiency of the suggested method compared to the accessible methods in the literature.

Keywords

Special third order ordinary differential equations, RKT method, Order conditions, Runge-Kutta method , IRKD method, RKD method, TDRKT method, Stability Polynomial.

AMS Subject Classification

65L05, 65L06, 65L20.

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Contents

1. Introduction

In this paper, we are deal with the initial value problem of special third-order ordinary differential equations (ODEs) as follows:

$$y'''(x) = f(x,y) \quad y(x_0) = y_0,\ y'(x_0) = y_0',\ y''(x_0) = y_0'', \quad (1.1)$$

where $y \in \mathbb{R}^d$, $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Equation (1.1) may be used to explain mathematical models in engineering and many applied sciences. The third order system (1.1) may be solved by converting into a first-order system [4, 8, 11]. Using this way to solve Eq. (1.1), it causes to increase the computational time. Therefore, the direct integration methods for solving Eq. (1.1) are useful [1–3, 9, 10, 14–16, 20, 21].

Some studies have been done by using of higher order derivatives of the solution in the formulation of the method to increase the efficiency of the method [5, 6, 17]. Recently, some authors have presented studies on two-derivative Runge-Kutta type methods for third-order ordinary differential equations [12, 13]. Following the same idea in [12, 13], in this paper we construct two-derivative improved Runge–Kutta direct methods (TDIRKD) to solve Eq. (1.1). This method is not self-starting, that is two-step in nature. This paper is organized as follows: The formulation of the TDIRKD methods for solving (1.1) are presented in Section 2. In Section 3, we derive the order conditions for TDIRKD methods. Next, we present a two-stage explicit two-derivative improved Runge-Kutta direct method of order four for solving (1.1) in Section 4. Then, in Section 5, the stability polynomial of TDIRKD methods is investigated. Section 6 gives some numerical examples to show efficiency of our proposed methods. Finally, conclusion is given in Section 7.

2. Derivation of TDIRKD methods

The improved Runge-Kutta direct methods are extended to two-derivative improved Runge-Kutta direct methods by including the fourth-derivative of the solution for solving third-order ODEs. That is, we derive two-derivative improved Runge-Kutta direct methods using the fourth derivative informations. It is assumed that the fourth derivative of the solution is known

$$y^{(iv)}(x) = g(x,y,y') = f_x(x,y) + f_y(x,y)y'$$

and it is incorparated in the formulation of the method. We consider methods that consist of one evaluation of f (third derivative) and many evaluations of g (fourth derivative). A modified s-stage explicit improved Runge-Kutta direct method obtained by using the fourth derivative in classical improved Runge-Kutta direct methods [9, 10] is given as follows:

$$\begin{aligned} y_{n+1} &= y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2y''_n - \frac{5}{12}h^2y''_{n-1} \\ &\quad + \frac{1}{6}h^3 f(x_n,y_n) - \frac{1}{6}h^3 f(x_{n-1},y_{n-1}) \qquad (2.1) \\ &\quad + h^4 \sum_{i=1}^{s} b''_i (k_i - k_{-i}), \end{aligned}$$

$$\begin{aligned} y'_{n+1} &= y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1} + \frac{5}{12}h^2 f(x_n,y_n) \\ &\quad - \frac{5}{12}h^2 f(x_{n-1},y_{n-1}) + h^3 \sum_{i=1}^{s} b'_i (k_i - k_{-i}) \qquad (2.2) \end{aligned}$$

$$\begin{aligned} y''_{n+1} &= y''_n + \frac{3}{2}hf(x_n,y_n) - \frac{1}{2}hf(x_{n-1},y_{n-1}) \\ &\quad + h^2 \sum_{i=1}^{s} b_i (k_i - k_{-i}), \qquad (2.3) \end{aligned}$$

where

$$k_1 = g(x_n, y_n, y'_n), \qquad (2.4)$$

$$k_{-1} = g(x_{n-1}, y_{n-1}, y'_{n-1}), \qquad (2.5)$$

$$k_i = g(x_n + c_i h, Y_n^i, Y_n'^i), \qquad (2.6)$$

$$k_{-i} = g(x_{n-1} + c_i h, Y_{n-1}^i, Y_{n-1}'^i), \qquad (2.7)$$

$$\begin{aligned} Y_n^i &= y_n + hc_i y'_n + \frac{1}{2}h^2 c_i^2 y''_n + \frac{1}{6}h^3 c_i^3 f(x_n,y_n) \\ &\quad + h^4 \sum_{j=1}^{i-1} a_{ij} k_j, \qquad (2.8) \end{aligned}$$

$$\begin{aligned} Y_n'^i &= y'_n + hc_i y''_n + \frac{1}{2}h^2 c_i^2 f(x_n,y_n) \\ &\quad + h^3 \sum_{j=1}^{i-1} \hat{a}_{ij} k_j, \qquad (2.9) \end{aligned}$$

$$\begin{aligned} Y_{n-1}^i &= y_{n-1} + hc_i y'_{n-1} + \frac{1}{2}h^2 c_i^2 y''_{n-1} \\ &\quad + \frac{1}{6}h^3 c_i^3 f(x_{n-1},y_{n-1}) + h^4 \sum_{j=1}^{i-1} a_{ij} k_{-j}, \qquad (2.10) \end{aligned}$$

$$\begin{aligned} Y_{n-1}'^i &= y'_{n-1} + hc_i y''_{n-1} + \frac{1}{2}h^2 c_i^2 f(x_{n-1},y_{n-1}) \\ &\quad + h^3 \sum_{j=1}^{i-1} \hat{a}_{ij} k_{-j}. \qquad (2.11) \end{aligned}$$

The coefficients of the TDIRKD method are $b''_i, b'_i, b_i, a_{ij}, \hat{a}_{ij}$ ($a_{ij} = \hat{a}_{ij} = 0$, if $i \le j$) and $c_i (c_1 = 0)$ for i=1, ..., s and j=1, ..., s-1. TDIRKD method (2.1)-(2.11) can be expressed by the Butcher tableau as follows:

$$\begin{array}{c|ccc} c & A & \hat{A} & \\ \hline & b''^T & b'^T & b^T \end{array}$$

3. Order conditions

In this section, the order conditions for TDIRKD method are derived. There are many ways to obtain the order conditions. The Taylor series expansions are used to obtain the order conditions of TDIRKD method. System of nonlinear algebraic equations are obtained using Maple for algebraic calculations.

The order conditions up to order five for TDIRKD method are listed below:

Order three

$$\sum_{i=1}^{s} b_i = \frac{5}{12}. \qquad (3.1)$$

Order four

$$\sum_{i=2}^{s} b_i c_i = \frac{1}{6}, \qquad \sum_{i=1}^{s} b'_i = \frac{1}{6}. \qquad (3.2)$$

Order five

$$\sum_{i=2}^{s} b_i c_i^2 = \frac{31}{360}, \qquad \sum_{i=2}^{s} b'_i c_i = \frac{31}{720}, \qquad \sum_{i=1}^{s} b''_i = \frac{31}{720}. \qquad (3.3)$$

4. Derivation of explicit TDIRKD method

We present an explicit fourth-order TDIRKD method with two-stage in this section. The coefficients of the fourth-order TDIRKD method with two-stage ($s = 2$) are found by using the order conditions up to order four (3.1)-(3.2). Therefore, there are 3 nonlinear equations with 9 unknowns, which include the four unknowns a_{21}, $\hat{a}_{21}$, b''_1 and b''_2.

The parameters of TDIRKD method have been generated by utilised the following simplifying assumptions

$$\sum_{j=1}^{i-1} a_{ij} = \frac{c_i^4}{24}, \qquad \sum_{j=1}^{i-1} \hat{a}_{ij} = \frac{c_i^3}{6}, i = 2, \ldots, s. \qquad (4.1)$$

The unknowns a_{21} and $\hat{a}_{21}$ are obtained from (4.1). The free parameters can be determined by using the same idea in [7].

With Maple software, the free parameters can be chosen as $c_2 = \frac{31}{60}$, $b'_2 = \frac{1}{12}$, $b''_1 = \frac{27}{1000}$ and $b''_2 = \frac{1}{62}$. Thus, the other coefficients of the fourth-order TDIRKD method are given as follows

$$b_1 = \frac{35}{372}, \quad b_2 = \frac{10}{31}, \quad b'_1 = \frac{1}{12}.$$

The two-stage fourth-order TDIRKD method may be represented with the tableau as follows:

$$\begin{array}{c|cc|cc|cc}
0 & & & & & & \\
\frac{31}{60} & \frac{923521}{311040000} & & \frac{29791}{1296000} & & & \\
\hline
 & \frac{27}{1000} & \frac{1}{62} & \frac{1}{12} & \frac{1}{12} & \frac{35}{372} & \frac{10}{31}
\end{array}$$

We denote this method by TDIRKD4.

5. Stability Analysis

In this section, we investigate the stability properties of the TDIRKD method. This method (2.1)-(2.11) can be written as follows

$$\begin{aligned}
Y_n^1 &= y_n, \\
Y_n'^1 &= y'_n, \\
Y_{n-1}^1 &= y_{n-1}, \\
Y_{n-1}'^1 &= y'_{n-1}, \\
Y_n^i &= y_n + hc_i y'_n + \frac{1}{2}h^2 c_i^2 y''_n + \frac{1}{6}h^3 c_i^3 f(x_n, y_n) \\
&\quad + h^4 \sum_{j=1}^{i-1} a_{ij} g(x_n + c_j h, Y_n^j, Y_n'^j), \ i = 2, \ldots, s \\
Y_n'^i &= y'_n + hc_i y''_n + \frac{1}{2}h^2 c_i^2 f(x_n, y_n) \\
&\quad + h^3 \sum_{j=1}^{i-1} \hat{a}_{ij} g(x_n + c_j h, Y_n^j, Y_n'^j), \ i = 2, \ldots, s \\
Y_{n-1}^i &= y_{n-1} + hc_i y'_{n-1} + \frac{1}{2}h^2 c_i^2 y''_{n-1} \\
&\quad + \frac{1}{6}h^3 c_i^3 f(x_{n-1}, y_{n-1}) \\
&\quad + h^4 \sum_{j=1}^{i-1} a_{ij} g(x_{n-1} + c_j h, Y_{n-1}^j, Y_{n-1}'^j), \ i = 2, \ldots, s, \\
Y_{n-1}'^i &= y'_{n-1} + hc_i y''_{n-1} + \frac{1}{2}h^2 c_i^2 f(x_{n-1}, y_{n-1}) \\
&\quad + h^3 \sum_{j=1}^{i-1} \hat{a}_{ij} g(x_{n-1} + c_j h, Y_{n-1}^j, Y_{n-1}'^j), \ i = 2, \ldots, s, \\
y_{n+1} &= y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2 y''_n - \frac{5}{12}h^2 y''_{n-1} \\
&\quad + \frac{1}{6}h^3 f(x_n, y_n) - \frac{1}{6}h^3 f(x_{n-1}, y_{n-1}) \\
&\quad + h^4 \sum_{i=1}^{s} b''_i g(x_n + c_i h, Y_n^i, Y_n'^i) \\
&\quad - h^4 \sum_{i=1}^{s} b''_i g(x_{n-1} + c_i h, Y_{n-1}^i, Y_{n-1}'^i), \qquad (5.1) \\
y'_{n+1} &= y'_n + \frac{3}{2}hy''_n - \frac{1}{2}hy''_{n-1} + \frac{5}{12}h^2 f(x_n, y_n) \\
&\quad - \frac{5}{12}h^2 f(x_{n-1}, y_{n-1}), \\
&\quad + h^3 \sum_{i=1}^{s} b'_i g(x_n + c_i h, Y_n^i, Y_n'^i) \\
&\quad - h^3 \sum_{i=1}^{s} b'_i g(x_{n-1} + c_i h, Y_{n-1}^i, Y_{n-1}'^i) \\
y''_{n+1} &= y''_n + \frac{3}{2}hf(x_n, y_n) - \frac{1}{2}hf(x_{n-1}, y_{n-1}) \\
&\quad + h^2 \sum_{i=1}^{s} b_i g(x_n + c_i h, Y_n^i, Y_n'^i) \\
&\quad - h^2 \sum_{i=1}^{s} b_i g(x_{n-1} + c_i h, Y_{n-1}^i, Y_{n-1}'^i).
\end{aligned}$$

To obtain the stability polynomial of the TDIRKD is used the test equation $y''' = -\lambda^3 y$. If the TDIRKD method (5.1) is applied to the test equation, the following recursion

$$\begin{aligned}
Y'_n &= N^{-1}\left(ey'_n + hcy''_n + \frac{1}{2}h^2 c^2(-\lambda^3 y_n)\right) \\
Y'_{n-1} &= N^{-1}\left(ey'_{n-1} + hcy''_{n-1} + \frac{1}{2}h^2 c^2(-\lambda^3 y_{n-1})\right) \\
y_{n+1} &= y_n + \frac{3}{2}hy'_n - \frac{1}{2}hy'_{n-1} + \frac{5}{12}h^2 y''_n - \frac{5}{12}h^2 y''_{n-1} \\
&\quad - \frac{1}{6}z^3 y_n + \frac{1}{6}z^3 y_{n-1} - z^3 b''^T(hY'_n - hY'_{n-1}), \\
hy'_{n+1} &= hy'_n + \frac{3}{2}h^2 y''_n - \frac{1}{2}h^2 y''_{n-1} - \frac{5}{12}z^3 y_n \\
&\quad + \frac{5}{12}z^3 y_{n-1} - z^3 b'^T(hY'_n - hY'_{n-1}), \\
h^2 y''_{n+1} &= h^2 y''_n - \frac{3}{2}z^3 y_n + \frac{1}{2}z^3 y_{n-1} \\
&\quad - z^3 b^T(hY'_n - hY'_{n-1}),
\end{aligned}$$

where $z = \lambda h$, $e = (1, \ldots, 1)^T$, the matrix $\hat{A} = [\hat{a}_{i,j}]_{i,j=1}^s$ is strictly lower triangular with $\hat{a}_{ij} = 0$, if $i \leq j$, $N = I + z^3 \hat{A}$, the vectors $Y'_{n-1} = [Y_{n-1}'^1, Y_{n-1}'^2, \ldots, Y_{n-1}'^s]$, $c = [0, c_2, \ldots, c_s]^T$, $c^2 = [0, c_2^2, \ldots, c_s^2]^T$, $Y'_n = [Y_n'^1, Y_n'^2, \ldots, Y_n'^s]$, $b = [b_1, b_2, \ldots, b_s]^T$,

$b' = [b'_1, b'_2, \ldots, b'_s]^T$ and $b'' = [b''_1, b''_2, \ldots, b''_s]^T$ is obtained [9, 18, 19]. If the vectors Y'_n and Y'_{n-1} is eliminated the recursion

$$\begin{pmatrix} y_{n+1} \\ y_n \\ hy'_{n+1} \\ hy'_n \\ h^2y''_{n+1} \\ h^2y''_n \end{pmatrix} = \hat{Q}(z^3) \begin{pmatrix} y_n \\ y_{n-1} \\ hy'_n \\ hy'_{n-1} \\ h^2y''_n \\ h^2y''_{n-1} \end{pmatrix} \tag{5.2}$$

with

$$\hat{Q}(z^3) = \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & \hat{Q}_{14} & \hat{Q}_{15} & \hat{Q}_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hat{Q}_{31} & \hat{Q}_{32} & \hat{Q}_{33} & \hat{Q}_{34} & \hat{Q}_{35} & \hat{Q}_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hat{Q}_{51} & \hat{Q}_{52} & \hat{Q}_{53} & \hat{Q}_{54} & \hat{Q}_{55} & \hat{Q}_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is obtained. In matrix $\hat{Q}(z^3)$, we have the followings

$$\begin{aligned}
\hat{Q}_{11} &= 1 - \frac{1}{6}z^3 + \frac{1}{2}z^6 b''^T N^{-1} c^2 \\
\hat{Q}_{12} &= \frac{1}{6}z^3 - \frac{1}{2}z^6 b''^T N^{-1} c^2 \\
\hat{Q}_{13} &= \frac{3}{2} - z^3 b''^T N^{-1} e \\
\hat{Q}_{14} &= -\frac{1}{2} + z^3 b''^T N^{-1} e \\
\hat{Q}_{15} &= \frac{5}{12} - z^3 b''^T N^{-1} c \\
\hat{Q}_{16} &= -\frac{5}{12} + z^3 b''^T N^{-1} c \\
\hat{Q}_{31} &= -\frac{5}{12}z^3 + \frac{1}{2}z^6 b'^T N^{-1} c^2 \\
\hat{Q}_{32} &= \frac{5}{12}z^3 - \frac{1}{2}z^6 b'^T N^{-1} c^2 \\
\hat{Q}_{33} &= 1 - z^3 b'^T N^{-1} e \\
\hat{Q}_{34} &= z^3 b'^T N^{-1} e \\
\hat{Q}_{35} &= \frac{3}{2} - z^3 b'^T N^{-1} c \\
\hat{Q}_{36} &= -\frac{1}{2} + z^3 b'^T N^{-1} c \\
\hat{Q}_{51} &= -\frac{3}{2}z^3 + \frac{1}{2}z^6 b^T N^{-1} c^2 \\
\hat{Q}_{52} &= \frac{1}{2}z^3 - \frac{1}{2}z^6 b^T N^{-1} c^2 \\
\hat{Q}_{53} &= -z^3 b^T N^{-1} e \\
\hat{Q}_{54} &= z^3 b^T N^{-1} e \\
\hat{Q}_{55} &= 1 - z^3 b^T N^{-1} c \\
\hat{Q}_{56} &= z^3 b^T N^{-1} c
\end{aligned}$$

The matrix $\hat{Q}(z^3)$ in (5.2) is named as the stability matrix for the TDIRKD method (5.1). For the stability features of the IRKD method is important the roots of the stability polynomial

$$p(\mu, z^3) = det\left(\mu I - \hat{Q}(z^3)\right). \tag{5.3}$$

The region of absolute stability of the method (5.1) is the set of all z $\in \mathbb{C}$ such that all the roots $\mu_i(z)$ of the stability polynomial $p(\mu, z^3)$ are inside of the unit circle. The stability polynomial of TDIRKD4 method is given by

$$\begin{aligned}
p(\mu, z^3) = \mu^6 &+ \left(\frac{1}{2}z^3 - \frac{63271}{15552000}z^6 - 3\right)\mu^5 \\
&+ \left(3 + \frac{3}{8}z^3 + \frac{222031}{160704000}z^6\right)\mu^4 \\
&- \left(\frac{102331}{2332800000}z^9\right)\mu^4 \\
&+ \left(-1 + \frac{1}{2}z^3 - \frac{9197759}{160704000}z^6\right)\mu^3 \\
&+ \left(\frac{108922967}{289267200000}z^9\right)\mu^3 \\
&+ \left(-\frac{3}{4}z^3 - \frac{1042289}{482112000}z^6\right)\mu^2 \\
&- \left(\frac{104187847}{289267200000}z^9\right)\mu^2 \\
&+ \left(\frac{1}{2}z^3 + \frac{4949803}{80352000}z^6\right)\mu \\
&+ \left(\frac{13203523}{96422400000}z^9\right)\mu \\
&- \frac{1}{8}z^3 + \frac{3223}{6696000}z^6 - \frac{703481}{6428160000}z^9.
\end{aligned}$$

6. Numerical Experiments

In this section, the performances of our proposed method and the existing methods have been tested on four problems selected from literature. In these methods, L_∞ norm has been used for evaluating the errors. We list the methods which is used for comparison as follows:

- TDIRKD4:two-stage fourth-order TDIRKD method derived in Section 4 of this paper.
- IRKD4:three-stage fourth-order IRKD method given in [9].
- TDRKT4:two-stage fourth-order TDRKT method given in [12].
- RK4:the classical fourth-order Runge–Kutta method given in [8], p.138.

Problem 1. We take the following linear IVP

$$\begin{cases} y''' = -y, \\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1. \end{cases}$$

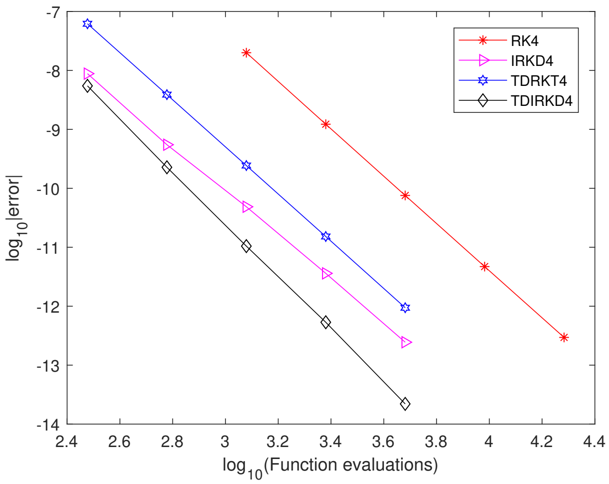

Figure 1. The graph of efficiency for Problem 1 with $h = \frac{0.05}{2^k}$, $k = 0, 1, \ldots, 4$

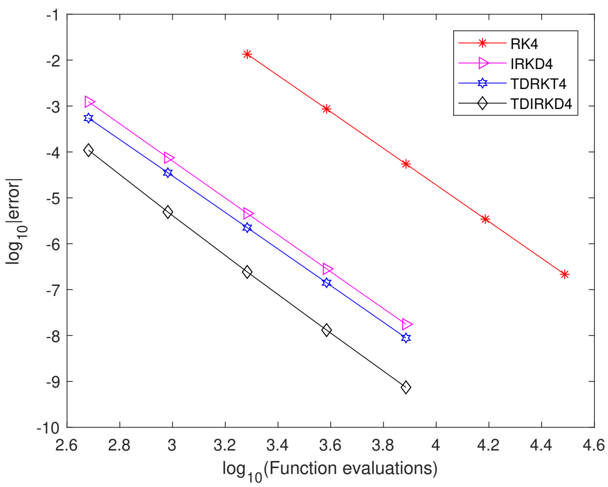

Figure 2. The graph of efficiency for Problem 2 with $h = 1/2^k$, $k = 4 \ldots, 8$

The exact solution of the problem is given by $y(x) = e^{-x}$ ([12]). Solution of the problem has been done in the interval [0,5]. The results have been shown in Fig.1.

Problem 2. We take the following linear IVP

$$\begin{cases} y''' - y = \cos(x), \\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1. \end{cases}$$

The exact solution of the problem is given by ([21])

$$y(x) = \frac{1}{2}\left(e^x - \cos(x) - \sin(x)\right).$$

Solution of the problem has been done in the interval [0,10]. The results have been shown in Fig.2.

Problem 3. We take the following linear IVP

$$\begin{cases} y''' = \left(12x - 8x^3\right) y, \\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2. \end{cases}$$

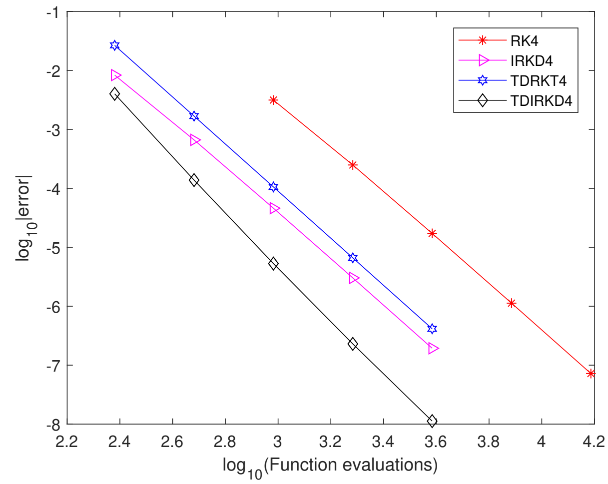

Figure 3. The graph of efficiency for Problem 3 with $h = 1/2^k$, $k = 4 \ldots, 8$

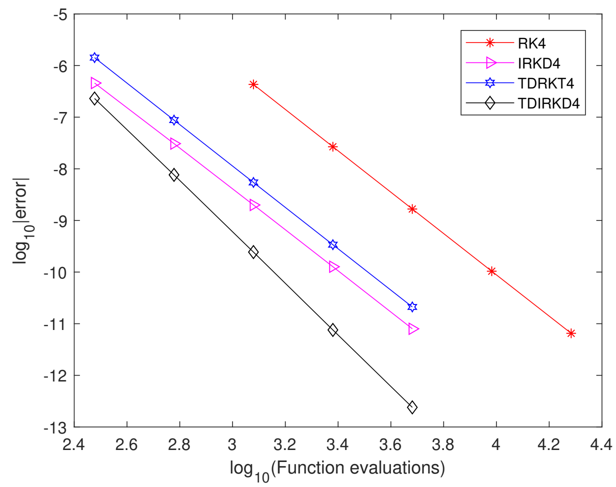

Figure 4. The graph of efficiency for Problem 4 with $h = \frac{0.05}{2^k}$, $k = 0, 1 \ldots, 4$

The exact solution of the problem is given by $y(x) = e^{-x^2}$ ([10]). Solution of the problem has been done in the interval [0,5]. The results have been shown in Fig.3.

Problem 4. We take the following nonlinear IVP

$$y''' = \frac{3}{8} y^{-5},\ y(0) = 1,\ y'(0) = \frac{1}{2},\ y''(0) = -\frac{1}{4}.$$

The exact solution of the problem is given by $y(x) = \sqrt{1+x}$ ([12]). Solution of the problem has been done in the interval [0,5]. The results have been shown in Fig.4.

In Figs. 1-4, the efficiency curves in terms of the error versus the number of function evaluations for each methods are given.

From Figs. 1-4, we can see that the TDIRKD method outperforms the classical Runge-Kutta, TDRKT and IRKD methods on both linear and nonlinear problems.

7. Conclusion

In this study, we present an explicit two-stage fourth-order two-derivative improved Runge-Kutta direct method (TDIRKD) for solving Eq. (1.1). The fourth derivative $y^{(iv)} = g(x,y,y')$ is used in the formulation of this method. This increases the efficiency of the method. We have performed experiments on four standard problems from literature. Our proposed method have been compared with the classical Runge-Kutta, TDRKT and IRKD methods. It is obvious that our proposed method is more efficient than the classical Runge-Kutta, TDRKT and IRKD methods.

In this study we have obtained the order conditions via Taylor series expansion. As a further study, we will derive the order conditions for TDIRKD based on rooted tree theory and the corresponding B-series theory and obtain higher order TDIRKD methods.

References

[1] D.O. Awoyemi, A P-stable linear multistep method for solving general third order ordinary differential equations, *International Journal of Computer Mathematics*, 80(8)(2003), 985-991.

[2] D.O. Awoyemi and O.M. Idowu, A class of hybrid collocation methods for third-order ordinary differential equations, *International Journal of Computer Mathematics*, 82(10)(2005) 1287-1293.

[3] M. Bin Suleiman, Solving nonstiff higher order ODEs directly by the direct integration method, *Applied Mathematics and Computation*, 33(3)(1989), 197-219.

[4] J.C. Butcher, Numerical methods for ordinary differential equations, *John Wiley & Sons* (2008).

[5] R.P.K. Chan and A.Y.J. Tsai, On explicit two-derivative Runge-Kutta methods, *Numerical Algorithms*, 53(2) (2010), 171-194.

[6] Z. Chen, Z. Qiu, J. Li and X. You, Two-derivative Runge-Kutta-Nyström methods for second-order ordinary differential equations, *Numerical Algorithm*s, 70(4)(2015), 897-927.

[7] J.R. Dormand, Numerical methods for differential equations: a computational approach, *CRC press*, (1996).

[8] E. Hairer, E., S.P. Nørsett and G. Wanner, Solving ordinary differential equations I: Nonstiff problems, *Springer-Verlag*, *Berlin*, (1993).

[9] K.A. Hussain, F. Ismail, N. Senu and F. Rabiei, Fourth-order improved Runge–Kutta method for directly solving special third-order ordinary differential equations, *Iranian Journal of Science and Technology, Transactions A: Science*, 41(2)(2017), 429-437.

[10] K.A. Hussain, F. Ismail, N. Senu, F. Rabiei and R. Ibrahim, Integration for special third-order ordinary differential equations using improved Runge-Kutta direct method, *MJS*, 34(2)(2015), 172-179.

[11] J.D. Lambert, Numerical methods for ordinary differential systems: the initial value problem, *John Wiley & Sons*, (1991).

[12] K.C. Lee, N. Senu, A. Ahmadian and S.N.I. Ibrahim, On Two-Derivative Runge–Kutta Type Methods for Solving $u''' = f(x,u(x))$ with Application to Thin Film Flow Problem, *Symmetry*, 12(6)(2020), 924.

[13] K.C. Lee, N. Senu, A. Ahmadian, S.N.I. Ibrahim and and D. Baleanu, Numerical study of third-order ordinary differential equations using a new class of two derivative Runge-Kutta type methods, *Alexandria Engineering Journal*, 59(4)(2020), 2449-2467.

[14] Z.A. Majid, M.B. Suleiman and N.A. Azmi, Variable step size block method for solving directly third order ordinary differential equations, *Far East Journal of Mathematical Sciences*, 41(1)(2010), 63-73.

[15] M. Mechee, N. Senu, F. Ismail, B. Nikouravan and Z. Siri, A three-stage fifth-order Runge-Kutta method for directly solving special third-order differential equation with application to thin film flow problem, *Mathematical Problems in Engineering*, 2013(2013), Article ID 795397.

[16] B.T. Olabode, B.T. and Y. Yusuph, A new block method for special third order ordinary differential equations, *Journal of Mathematics and statistics*, 5(3)(2009), 167.

[17] M. Ökten Turacı and T. Öziş, Derivation of three-derivative Runge-Kutta methods, *Numerical Algorithms*, 74(1)(2017), 247-265.

[18] M. Ökten Turacı and M. Özdemir, A sixth-order improved Runge–Kutta direct method for special third-order ordinary differential equations, *Numerical Methods for Partial Differential Equations*, 37(2)(2021), 987-997.

[19] B. Paternoster, Two step Runge-Kutta-Nyström methods for $y'' = f(x,y)$ and P-stability, *International Conference on Computational Science*, (2002), 459-466.

[20] N. Waeleh, Z.A. Majid and F. Ismail, A New Algorithm for Solving Higher Order IVPs of ODEs, *Applied Mathematical Sciences*, 5(56)(2011), 2795-2805.

[21] X. You and Z. Chen, Direct integrators of Runge–Kutta type for special third-order ordinary differential equations, *Applied Numerical Mathematics*, 74(2013), 128-150.

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