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# Nonstandard Compactification of uniform spaces

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**Abstract**

Let $(X,\Psi)$ be a uniform space. We define an equivalence relation on a superstructure ${}^*X$ of $X$. The set of equivalence classes is denoted by $\bar{X}$. We extend the uniform structure $\Psi$ of $X$ to a suitable uniform structure $\widehat{\Psi}$ on $\bar{X}$. We embed $X$ as a dense subspace of $\bar{X}$ and show that $\bar{X}$ is compact. Thus $\bar{X}$ turns out to be a uniform compactification of $X$.

**Keywords**

Standard, Nonstandard, Uniform Structure, Uniform spaces, Compactness, Compactification, Weak topology

**AMS Subject Classification**

54J05, 54E15, 54E50.

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## Contents



## 1. Introduction

Non-standard analysis is a branch of Mathematics introduced by Abraham Robinson in 1966[1]. Abraham Robinson constructed a superstructure to work in any given structure like the Euclidean spaces, topological spaces, algebraic structures (rings, fields etc.,.), graphs and so on. The basic idea is not necessarily to study the superstructure but to study the classical spaces by getting on to a higher platform, namely a superstructure, and get a microscopic view of the classical space below.

## 2. Preliminaries

We assume preliminaries and notations like $V(X), V({}^*X)$ for superstructures, as in [1] and [2]. For preliminaries on uniform spaces we refer to [3],[4],[7].

**Definition 2.1.**

Let $X$ be a set. A uniform structure on $X$ is a filter $\Psi \subseteq X \times X$ such that
(i) $\forall\, U \in \Psi, \Delta(X) \subseteq U$, where $\Delta(X) = \{(x,x) : x \in X\}$ being the diagonal of $X$.
(ii) $\forall\, U \in \Psi, U^{-1} \in \Psi$, where $U^{-1} = \{(x,y) : (y,x) \in U\}$
(iii) $\forall\, U \in \Psi, \exists\, V \subseteq U$ such that $V \circ V \subseteq U$,
where $V \circ W = \{(x,z) : (x,y) \in V \wedge (y,z) \in W\}$, for general $V, W \subseteq X \times X$.

**Definition 2.2.**

Let $X$ be a set with uniform structure $\Psi$. For $V \in \Psi, x \in X$ define $V(x) = \{y \in X : (x,y) \in V\}$. There exists a topology on $X$ such that $\forall\, x \in X$, $\{V(x) : V \in \Psi\}$ is a neighbourhood base for $x$. Henceforth $X$ with this induced topology will be referred to as the uniform space $X$.

As a common notation as in [1],[2], ${}^*X$ denotes a nonstandard extension of $X$, $V(X), V({}^*X)$ the corresponding superstructures on $X$, ${}^*X$ respectively. We assume $V({}^*X)$ is an enlargement of ${}^*X$, as defined in [2].
We now give the definition of concurrence.

**Definition 2.3.**

A binary relation $P$ is said to be concurrent on $A \subseteq domP$ if for each finite set $\{x_1, x_2 \ldots . x_n\}$ in $A$ there is a $y \in rangeP$ so that $\langle x_i, y\rangle \in P, 1 \leq i \leq n$. $P$ is concurrent if it is concurrent on $domP$.

The following proposition is from [2].

**Proposition 2.4.**

The following are equivalent.
(i) $V({}^*X)$ is an enlargement of $V(X)$.
(ii) For each concurrent relation $P \in V(X)$ there is an element $b \in range\ {}^*P$ so that $\langle {}^*x, b\rangle \in {}^*P$ for all $x \in domP$

## 3. Main Results

Let $(X,\Psi)$ be a uniform space.
In ${}^*X$ define $x' \sim y'$ if ${}^*f(x') \simeq {}^*f(y')\ \ \forall f \in C(X,R)$
Here $C(X,R)$ is the space of bounded continuous real-valued functions on $X$.
Clearly $\sim$ is an equivalence relation on $X$.
Let $\bar{X}$ be the set of equivalence classes. We denote the equivalence class of $x' \in {}^*X$ by $[x']$.
First we make the following observation.

**Proposition 3.1.**
For $x \in X$, $[x] = m(x)$, the monad of $x$.

*Proof.*
We recall $m(x) = \cap\, {}^*G$ where $G$ is a neighbourhood of $x$.
Let $y \in [x]$
Suppose $y \notin m(x)$
Then $y \notin {}^*G$ for some neighbourhood $G$ of $x$ in $X$.
By complete regularity of a uniform space,
$\exists\, f \in C(X,R)\,,\ f : X \to [0,1]$ such that $f(x) = 0,\ f(X-G) = \{1\}$
Then ${}^*f(x) = 0,\ {}^*f(y) = 1$ since $y \in {}^*(X-G)$
Therefore ${}^*f(x)$ and ${}^*f(y)$ are not infinitely close to each other, contradicting $x \sim y$
Therefore $y \in m(x)$
Therefore $[x] \subseteq m(x)$

Conversely let $y \in m(x)$
Then $y \simeq x$
Therefore ${}^*f(y) \simeq f(x)\ \ \forall\ f \in C(X,R)$, by continuity of $f$
Therefore $y \sim x$
That is, $y \in [x]$
Therefore $m(x) \subseteq [x]$
Hence $[x] = m(x)$ □

Next we have the following.

**Proposition 3.2.**
The map $\varphi : X \to \bar{X}$ defined by $\varphi(x) = [x]$ is one-one.

*Proof.*
Let $[x] = [y]$ for $x,y \in X$
Then $x \sim y$
Therefore ${}^*f(x) \simeq {}^*f(y)\ \ \forall f \in C(X,R)$
That is, $f(x) = f(y)\ \ \forall f \in C(X,R)$
Therefor $x = y$, by complete regularity of $X$
Therefore $\varphi$ is one-one. □

For $U \in \Psi$, let $\widehat{U} \subseteq \bar{X} \times \bar{X}$ be defined by
$\widehat{U} = \{([x'],[y']) : (x',y') \in {}^*U\}$
Let $\widehat{\Psi} = \left\{E \subseteq \bar{X} \times \bar{X} : E \supseteq \hat{U}\ \mathit{for\ some}\ U \in \Psi\right\}$
That is, $\widehat{\Psi}$ is the collection of all supersets of the $\widehat{U}$'s, $U \in \Psi$

We make the following fundamental observation.

**Proposition 3.3.**
$\left(\bar{X},\widehat{\Psi}\right)$ is a uniform space.

*Proof.*
We shall verify the conditions one by one.
(i) Since $\phi \notin \Psi$, we get $\phi \notin \widehat{\Psi}$

Now let $V,W \in \Psi$
Then $([x],[y]) \in \widehat{V} \cap \widehat{W} \Leftrightarrow ([x],[y]) \in \widehat{V}$ and $([x],[y]) \in \widehat{W}$
$\Leftrightarrow (x,y) \in {}^*V$ and $(x,y) \in {}^*W$
$\Leftrightarrow (x,y) \in {}^*V \cap {}^*W = {}^*(V \cap W)$
$\Leftrightarrow ([x],[y]) \in \widehat{V \cap W}$
Therefore $\widehat{V} \cap \widehat{W} = \widehat{V \cap W}$

Let $E,F \in \widehat{\Psi}$
Then $\widehat{V} \subseteq E, \widehat{W} \subseteq F$ for some $V,W \in \Psi$
Therefore $\widehat{V \cap W} = \widehat{V} \cap \widehat{W} \subseteq E \cap F$ and $V \cap W \in \Psi$
Therefore $E \cap F \in \widehat{\Psi}$

Next let $E \in \widehat{\Psi}$ and $E \subseteq F$
$\exists\, V \in \Psi$ such that $\widehat{V} \subseteq E$
Therefore $\widehat{V} \subseteq F$
$F \in \widehat{\Psi}$
Therefore $\widehat{\Psi}$ is a filter of subsets of $\bar{X} \times \bar{X}$

(ii) Let $E \in \widehat{\Psi}$
Then $\widehat{V} \subseteq E$ for some $V \in \Psi$
$\forall\, x \in X,\ (x,x) \in V$
By Transfer $\forall\, x \in {}^*X,\ (x,x) \in {}^*V$
Therefore $\forall\, x \in {}^*X,\ ([x],[x]) \in \widehat{V} \subseteq E$
Therefore $\forall\ [x] \in \bar{X}, ([x],[x]) \in E$

(iii) For $V \in \Psi; x,y \in {}^*X$;
$([y],[x]) \in \widehat{V}^{-1} \Leftrightarrow ([x],[y]) \in \widehat{V} \Leftrightarrow (x,y) \in {}^*V$
Now $(x,y) \in V \Leftrightarrow (y,x) \in V^{-1}$ for $x,y \in X$
Therefore, by Transfer $(x,y) \in {}^*V \Leftrightarrow (y,x) \in {}^*\left(V^{-1}\right) \Leftrightarrow ([y],[x]) \in \widehat{V^{-1}}$
Thus $([y],[x]) \in \left(\widehat{V}\right)^{-1} \Leftrightarrow ([y],[x]) \in \widehat{V^{-1}}$
Therefore $\left(\widehat{V}\right)^{-1} = \widehat{V^{-1}}$
Let $\widehat{V} \in \widehat{\Psi}$, where $V \in \Psi$
Then $V^{-1} \in \Psi$
Therefore $\widehat{V^{-1}} \in \widehat{\Psi}$
That is, $\left(\widehat{V}\right)^{-1} \in \widehat{\Psi}$
Thus $\widehat{V} \in \widehat{\Psi} \Rightarrow \left(\widehat{V}\right)^{-1} \in \widehat{\Psi}$
If $E \in \widehat{\Psi}$, then $\widehat{V} \subseteq E$ for some $V \in \Psi$
Therefore $\left(\widehat{V}\right)^{-1} \subseteq E^{-1}$
As already shown, $\left(\widehat{V}\right)^{-1} \in \widehat{\Psi}$
Therefore $E^{-1} \in \widehat{\Psi}\ \forall\, E \in \widehat{\Psi}$

(iv) Let $\widehat{U} \in \widehat{\Psi}$, where $U \in \Psi$
$\exists\, V \in \Psi$ such that $V \subseteq U$ and $V \circ V \subseteq U$
Claim : $\widehat{V} \circ \widehat{V} \subseteq \widehat{U}$
Let $([x],[y]) \in \widehat{V} \circ \widehat{V}$, where $x,y \in {}^*X$

Then $([x],[z]) \in \widehat{V}$ and $([z],[y]) \in \widehat{V}$ for some $z \in {}^*X$
Therefore $(x,z) \in {}^*V$ and $(z,y) \in {}^*V$
Now $(x,z) \in V$ and $(z,y) \in V \Rightarrow (x,y) \in (V \circ V)$
By Transfer, $(x,z) \in {}^*V$ and $(z,y) \in {}^*V \Rightarrow (x,y) \in {}^*(V \circ V)$
$\Rightarrow (x,y) \in {}^*U$
$\Rightarrow ([x],[y]) \in \widehat{U}$
Therefore $\widehat{V} \circ \widehat{V} \subseteq \widehat{U}$, proving our claim.

If $E \in \widehat{\Psi}$, then $\widehat{U} \subseteq E$ for some $U \in \Psi$
By what we have proved, $\exists\ V \in \Psi$ such that $V \subseteq U$ and $\widehat{V} \circ \widehat{V} \subseteq \widehat{U} \subseteq E$
Now $V \subseteq U \Rightarrow \widehat{V} \subseteq \widehat{U} \subseteq E$
Thus $\forall\ E \in \widehat{\Psi}$, $\exists\ \widehat{V} \in \widehat{\Psi}$ such that $\widehat{V} \subseteq E$ and $\widehat{V} \circ \widehat{V} \subseteq E$
Therefore $\left(\bar{X}, \widehat{\Psi}\right)$ is a uniform space. □

For $f \in C(X,R)$, define $\bar{f}$ on $\bar{X}$ by $\bar{f}([x]) = st\ {}^*f(x)$, where $x \in {}^*X$.
Since $f$ is bounded, ${}^*f(x)$ is a finite real number and hence $st\ {}^*f(x)$ exists.
Also if $[x] = [y]$, then $x \sim y$; so ${}^*f(x) \simeq {}^*f(y)$.
Hence $\bar{f}([x]) = \bar{f}([y])$ showing that $\bar{f}$ is a well-defined map.

Next we have the striking result.

**Proposition 3.4.**
The uniform topology $\Im$, generated by the uniform structure $\widehat{\Psi}$, on $\bar{X}$, is the same as the weak topology $\omega$ induced by the $\bar{f}$'s on $\bar{X}$, where each $f \in C(X,R)$

*Proof.*
Let $\widehat{U}([x])$ be a basic open set in $(\bar{X}, \Im)$, where $U \in \Psi$.
Let $V \in \Psi$ be such that $V \subseteq U$ and $V \circ V \subseteq U$
$\exists\, W \in \Psi$ such that $W \subseteq V$, $\bar{W}(x) \subseteq V(x)$, by regularity of $X$.
$\exists\ f \in C(X,R)$ such that $f : X \to [0,1],\ f \equiv 1$ in $\bar{W}(x)$ and $f \equiv 0$ in $(V(x))^c$, by complete regularity of $f$.
Now $f^{-1}\left(\left(\frac{1}{2},1\right]\right)$ is an open set containing $V(x)$, by continuity of $f$.
Claim : $\left(\bar{f}\right)^{-1}\left(\left(\frac{1}{2},1\right]\right) \subseteq \widehat{U}([x])$
Let $[y] \in \left(\bar{f}\right)^{-1}\left(\left(\frac{1}{2},1\right]\right)$
Therefore $\bar{f}([y]) \in \left(\frac{1}{2},1\right]$
That is, $st\ {}^*f(y) \in \left(\frac{1}{2},1\right]$
Therefore $\frac{1}{2} < {}^*f(y) \leq 1$
Now we need to prove $[y] \in \widehat{U}([x])$
Equivalently, $([x],[y]) \in \widehat{U}$
That is to prove $(x,y) \in {}^*U$
Since $f \equiv 0$ in $(V(x))^c$, ${}^*f \equiv 0$ in $({}^*(V(x)))^c$
Since $\frac{1}{2} < {}^*f(y) \leq 1$, we get $y \in {}^*V(x)$
Therefore $y \in {}^*U(x)$, since $V \subseteq U$
Therefore $(x,y) \in {}^*U$ proving the claim.
Hence $\Im \subseteq \omega$

Conversely let $G = \cap_{i=1}^{n} \left\{[z] \in \bar{X} : \left|\bar{f}_i([z]) - \bar{f}_i([y])\right| < \varepsilon\right\}$ be a typical $\omega$-basic neighbourhood of $[y] \in \bar{X}$, where $y \in {}^*X$
Now $G = \cap_{i=1}^{n} \{[z] \in \bar{X} : |st\ {}^*f_i(z) - st\ {}^*f_i(y)| < \varepsilon\}$
$= \cap_{i=1}^{n} \{[z] \in \bar{X} : |{}^*f_i(z) - st\ {}^*f_i(y)| < \varepsilon\}$
$= \cap_{i=1}^{n} \widehat{U}_i$, where $U_i = \{z \in X : |f_i(z) - st\ {}^*f_i(y)| < \varepsilon\}$ is open in $\Im$, by continuity of each $f_i$
Therefore $G \in \Im$
Therefore $\omega \subseteq \Im$
Hence $\omega = \Im$ □

**Theorem 3.5.**
$\left(\bar{X}, \widehat{\Psi}\right)$ is a compactification of $(X, \Psi)$.

*Proof.*
We intend to show that $\varphi : X \to \bar{X}$, defined by $\varphi(x) = [x]$, imbeds $X$ as a dense subspace in $\bar{X}$ and that $\bar{X}$ is compact.
We have already seen that $\varphi$ is one-one, by Proposition 1.2
Let $\widehat{V}[x]$ be a basic neighbourhood of $[x]$ in $\bar{X}$, where $x \in X,\ V \in \Psi$
Now $y \in V(x) \Rightarrow (x,y) \in V$
$\Rightarrow (x,y) \in {}^*V$
$\Rightarrow ([x],[y]) \in \widehat{V}$
$\Rightarrow \varphi(y) = [y] \in \widehat{V}[x]$
Therefore $\varphi$ is continuous.
To prove $\varphi$ is an open map into $\varphi(X)$, let $V(x)$ be a basic neighbourhood of $x \in X$, where $V \in \Psi$
By complete regularity of $X$, fix $f \in C(X,R)$ such that $f : X \to [0,1],\ f(x) = 0$ and $f(V(x)^c) = \{1\}$
Now $z \in {}^*X,\ |{}^*f(z)| < \frac{1}{2} \Rightarrow z \notin ({}^*V(x))^c \Rightarrow z \in {}^*V(x)$———(1)
Let $[z] \in \bar{X}$ with $z \in X$, $\left|\bar{f}([z])\right| < \frac{1}{2}$
Then $|f(z)| < \frac{1}{2}$
Therefore $z \in V(x)$, by (1).
Therefore $[z] \in \varphi(V(x))$———(2)
Also $\left\{[z] \in \bar{X} : z \in X,\ \left|\bar{f}([z]) - \bar{f}([x])\right| < \frac{1}{2}\right\}$ is a neighbourhood of $[x] = \varphi(x)$ in $\varphi(X)$
This neighbourhood $= \left\{[z] \in \bar{X} : z \in X,\ \left|\bar{f}([z])\right| < \frac{1}{2}\right\}$, since $f(x) = 0$
$\subseteq \varphi(V(x))$, by (2)
Therefore $\varphi(V(x))$ is open in $\varphi(X)$
Therefore $\varphi$ is an open map.
Therefore $\varphi : X \to \bar{X}$ is a homeomorphism of $X$ onto $\varphi(X)$
Next we show $\varphi(X)$ is dense in $\bar{X}$.
Let $[y] \in \bar{X} - \varphi(X)$
We take a basic neighbourhood of $[y]$ given by
$G = \cap_{i=1}^{n} \left\{[z] \in \bar{X} : \left|\bar{f}_i([z]) - \bar{f}_i[y]\right| < \varepsilon\right\}$
Now $\cap_{i=1}^{n} \left\{x \in {}^*X : \left|{}^*f_i(x) - \bar{f}_i([y])\right| < \varepsilon\right\} \neq \phi$, since it contains $y$.
$\cap_{i=1}^{n} \left\{x \in X : \left|f_i(x) - \bar{f}_i([y])\right| < \varepsilon\right\} \neq \phi$, by Downward Transfer.
That is, $\cap_{i=1}^{n} \left\{x \in X : \left|\bar{f}_i([x]) - \bar{f}_i([y])\right| < \varepsilon\right\} \neq \phi$
Therefore $\exists\ [x] \in G$ for some $x \in X$
Therefore $\varphi(X)$ is dense in $\bar{X}$
Finally we show $\bar{X}$ is compact and complete the proof.
For each $[y] \in \bar{X}$, we associate a map $T([y])$ from $C(X,R)$ to $R$ defined by
$T[y](f) = \bar{f}([y]) = st\ {}^*f(y)$
Let $A$ be the range of $T$.
Claim : $T$ is a one-one mapping of $\bar{X}$ onto $A$.
$[y_1] \neq [y_2] \Rightarrow {}^*f(y_1)$ and ${}^*f(y_2)$ are not infinitely close to

each other for some $f \in C(X,R)$

$\Rightarrow st\ {}^*f(y_1) \neq st\ {}^*f(y_2)$

$\Rightarrow \bar{f}([y_1]) \neq \bar{f}([y_2])$

$\Rightarrow T([y_1])(f) \neq T([y_2])(f)$

Therefore $T([y_1]) \neq T([y_2])$

Therefore $T$ is one-one, establishing the claim.

Define a topology on $A$ by declaring $U$ open in $A$ if $T^{-1}(U)$ is open in $\bar{X}$.

By definition, $T$ is a homeomorphism of $\bar{X}$ onto $A$.

To show $\bar{X}$ is compact, all we need to show is that $A$ is compact.

A basic neighbourhood of $\alpha \in A$ is of the form $G = \cap_{i=1}^{n} \{\beta \in A : |\alpha(f_i) - \beta(f_i)| < \varepsilon\}$,

where $\varepsilon > 0$ and $f_1, f_2, \ldots f_n \in C(X,R)$

Since $X$ is dense in $\bar{X}, \exists\, x \in X$ such that $T[x] \in G$

That is, $|\alpha(f_i) - T[x](f_i)| < \varepsilon$ for $i = 1,2,\ldots.n$

That is, $\alpha(f_i) - f_i(x) < \varepsilon$ for $i = 1,2,\ldots.n$

That is, $\forall\ \alpha \in A, \forall\ f_1,\ldots f_n \in C(X,R),\ \forall\ \varepsilon > 0$ in R,

$\exists\, x \in X$ such that $|\alpha(f_i) - f_i(x)| < \varepsilon$ for $i = 1,2,\ldots.n$

By concurrence, $\exists\ x \in\ {}^*X$ such that $\forall\ \alpha \in\ {}^*A,\ \forall\ f \in C(X,R),\ \forall\ \varepsilon > 0\ \ in\ {}^*R$,

$|\alpha({}^*f) -\ {}^*f(x)| < \varepsilon$

Taking $\varepsilon > 0$ as a positive infinitesimal, we get the following :

$\exists\, x \in\ {}^*X$ such that $\forall\ \alpha \in\ {}^*A,\ \forall\ f \in C(X,R),\ \alpha({}^*f) \simeq\ {}^*f(x)$——(3)

Now let $\gamma \in\ {}^*A$

To show $A$ is compact, we need to show that $\gamma$ is near some $\delta \in A$

By (3), $\exists\, x \in\ {}^*X$ such that $\forall\ f \in C(X,R),\ \gamma({}^*f) \simeq\ {}^*f(x)$

Take $\delta = T([x])$

Then $\forall\ f \in C(X,R),\ \delta(f) = T[x](f) = st\ {}^*f(x) \simeq\ {}^*f(x) \simeq \gamma({}^*f)$

Therefore $\gamma \simeq \delta \in A$

Therefore $A$ is compact.

This completes the proof. □

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