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Solving the problems related to shortest distance using He's variational iteration method and theory of variational problems with moving boundaries

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Abstract
In this paper we considered two variational problems related to shortest distance with moving boundaries. These problems are solved by applying the variational iteration method. It is observed that at the first iteration the exact solution is reached by using the variational iteration method.

Keywords
Variational method, moving boundaries, transversality condition.

AMS Subject Classification
65F10.

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Contents

1. Introduction

The problems in engineering, science, economics and mathematical modeling yield differential equations. For example the problem of harmonic motion yields a differential equation $\frac{d^2x}{dt^2}+\omega^2x=0$. For the physical interpretation we need its solution. The analytical solution for every differential equation may not be possible with the existing methods in the theory of ordinary differential equations. In such cases we employ the numerical methods of solving the differential equations. These methods have some drawbacks. The variational iteration method is developed by J.H.He [2]. It was shown by many researchers [1, 4, 6, 7] that the variational iteration method is more powerful and effective than existing techniques such as the Adomain method and perturbation method. E. Rama, K. Somaiah and K. Sambaiah [5] studied variational iteration method for solving various types of problems. The successive approximations of an initial value problem or boundary value problem found by this method are rapidly converges to the exact solution. We considered two variational problems related to shortest distance with moving boundaries. Variational iteration method is used to obtain the solution of these problems. It is observed that at the first iteration the exact solution is reached by applying the variational iteration method.

2. Description of He's Variational Iterative Method

We consider the following general nonlinear system

$$L[y(x)]+N[y(x)]=g(x), \tag{2.1}$$

where L is linear operator, N is a non-linear operator respectively and $g(x)$ is known continuous function. According to the variational iteration method the correction functional of (2.1) is given by

$$y^{(n+1)}(x)=y^{(n)}(x)+\int_0^x \lambda(x,t)\left[L[y^{(n)}(t)] + N\left[\tilde{y}^{(n)}-g(t)\right]\right]dt, \tag{2.2.}$$

where $\lambda(x,t)$ is a general Lagrange multiplier which can be identified optimally via variational theory and $\widetilde{y}^{(n)}(x)$ is a restricted variation which means $\delta\widetilde{y}^{(n)}=0$.

Further, $y^{(n)}(x)$ is the n^{th} approximate solution. The successive approximations $y^{(n+1)}(x)$ will be found by applying the obtained general Lagrange multiplier and properly chosen initial approximation $y^{(0)}(x)$. For the k^{th} order ordinary differential equation

$$\frac{d^k y}{dx^k}+f\left(y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^{k-1}y}{dx^{k-1}}\right)=0. \tag{2.3}$$

Ji-Huan He and XuHong Wu [3] shown that the general Lagrange multiplier is

$$\lambda(x,t)=\frac{(-1)^k}{(k-1)!}(t-x)^{k-1} \tag{2.4}$$

and

$$y^{(n+1)}(x)=y^{(n)}(x)+\frac{(-1)^k}{(k-1)!}\int_0^x (t-x)^{k-1} (\times)\left\{\frac{d^k}{dt^k}y^{(n)}(t)+f(y^{(n)}(t),\frac{d}{dt}y^{(n)}(t), y_n''(t),\ldots,\frac{d^{k-1}}{dt^{k-1}}y^{(n)}(t))\right\}dt \tag{2.5}$$

The solution of (2.1) is obtained as $y(x)=\underset{n\to\infty}{Lt}\, y^{(n)}$.

3. Transversality conditions

Let $v(y(x))=\int_{x_0}^{x_1}F(x,y,y')\,dx$ be a variational problem and x_1 is moving on the curve $y=\phi(x)$. The transversality condition is the relation between the directional coefficients of $\phi'(x)$ and the derivative of the extremal of $v(y(x))$ at (x_1,y_1).

If $v(y(x),z(x))=\int_{x_0}^{x_1}F(x,y,z,y',z')\,dx$ be a variational problem and $P(x_1,y_1,z_1)$ is moving on the surface $z=\phi(x,y)$. The transversality condition is the relation between the directional coefficients of the extremal at P and directional coefficients of normal to the surface at the point P.

The proofs of the following theorems 3.1 and 3.2 see [1]. Using these theorems we derive the transversality conditions for the problems related to shortest distance.

Theorem 3.1. *If* $v(y(x))=\int_{x_0}^{x_1}F(x,y,y')\,dx, y(x_0)=y_0$ *and* (x_1,y_1) *is moving on the curve* $g=f(x)$ *then the transversality condition is*

$$\left[F+(g'-y')F_{y'}\right|_{x=x_1}=0 \tag{3.1}$$

Particular case: If $v(y(x))=\int_{x_0}^{x_1}\sqrt{1+y'^2}dx, y(x_0)=y_0$ and (x_1,y_1) is moving on the curve $y=g(x)$ then the Transversality condition is $y'=-\frac{1}{g'}$ at $x=x_1$. Proof: Considering $F=\sqrt{1+y'^2}$ and using Theorem3.1 the transversality condition (3.1) becomes $\left[\sqrt{1+y'^2}+(g'-y')\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_1}=0.$

i.e. $\left[\sqrt{1+y'^2}-\frac{y'^2}{\sqrt{1+y'^2}}+g'\frac{y'}{\sqrt{1+y'^2}}\right]_{x=x_1}=0$

$\Rightarrow\left[\frac{1}{\sqrt{1+y'^2}}\{1+g'y'\}\right]_{x=x_1}=0.$

Since $\sqrt{1+y'^2}\neq 0$ we get $[1+g'y']_{x=x_1}=0$.

Hence,

$$y'=-\frac{1}{g'}\text{ at }x=x_1. \tag{3.2}$$

Theorem 3.2. *If* $V(y(x),z(x))=\int_{x_0}^{x_1}F(x,y,z,y',z')\,dx, y(x_0)=y_0$ *and* $z(x_0)=z_0$ *and the point* (x_1,y_1,z_1) *moving on the surface* $z=g(x,y)$ *then the transversality conditions are* $\left[F-y'F_{y'}+(g_x-z')F_{z'}\right]_{x=x_1}=0$ *and*

$$\left[F_{y'}+g_xF_{z'}\right|_{x=x_1}=0 \tag{3.4}$$

Particular case: The transversality conditions for the functional $V(y(x),z(x))=\int_{x_0}^{x_1}\sqrt{1+y'^2+z'^2}dx, y(x_0)=y_0, z(x_0)=z_0$ *and* (x_1,y_1,Z_1) *is moving on the surface* $z=g(x,y)$ *are* $\frac{1}{g_x}=\frac{y'}{g_y}=\frac{z}{-1}$ *at* $x=x_1$

Proof. Here $F=\sqrt{1+y'^2+z'^2}$. The transversality conditions (3.3) and (3.4) become

$\left[\sqrt{1+y'^2+z'^2}-\frac{y'^2}{\sqrt{1+y'^2+z'^2}}+\frac{(g_x-z')}{\sqrt{1+y'^2+z'^2}}z'\right]_{x=x_1}=0$

$\left[\frac{y'}{\sqrt{1+y'^2+z'^2}}+\frac{z'}{\sqrt{1+y'^2+z'^2}}g_y\right]_{x=x_1}=0.$

Simplifying the above equations we get $[1+g_xz']_{x=x_1}=0$ and $\left[y'+g_yz'\right]_{x=x_1}=0$.

From the above two equations we get

$$\frac{1}{g_x}=\frac{y'}{g_y}=\frac{z'}{-1}\text{ at }x=x_1.$$

□

4. Solutions of variational problems related to shortest distance using variational Iteration method and moving boundaries

Example 1: Consider the variational problem to find the shortest distance between $(1,0)$ and the ellipse

$$4x^2+9y^2=36. \tag{4.1}$$

Let A be $(1,0)$ and $B(x_1,y_1)$ be a point moving on the ellipse.

The variational problem for the shortest distance of the considered problem is

$$V(y(x))=\int_1^{x_1}\sqrt{1+y'^2}dx. \tag{4.2}$$

Since (x_1, y_1) is a point moving on the ellipse we have

$$4x_1^2 + 9y_1^2 = 36. \tag{4.3}$$

The Euler-Lagrange's equation of (4.2) gives

$$y''(x) = 0. \tag{4.4}$$

Therefore the extremals are $y = c_1 x + c_2$. As $(1,0)$ is a point on $y = c_1 x + c_2$ we have $c_2 = -c_1$.

Suppose $c_1 = a$, then

$$y = ax - a. \tag{4.5}$$

Choose $y^{(0)}(x) = a - ax$. The variation iteration formula is

$$y^{(n+1)}(x) = y^{(n)}(x) + \int_0^t (t-x)\frac{d^2}{dt^2}\left(y^{(n)}(t)\right)dt. \tag{4.6}$$

Substituting $n = 0$ in (4.6) we obtain $y^{(1)}(x) = y^{(0)}(x) + \int_0^t (t-x)\left(\frac{d^2}{dt^2}y^{(0)}(t)\right)dt$.

Since $y^{(0)}(x) = ax - a$ we have $\frac{d^2}{dt^2}\left\{y^{(0)}(t)\right\} = 0$. Therefore, $y^{(1)}(x) = y^{(0)}(x)$.

It can be shown that

$$y^{(k)}(x) = y^{(0)}(x) = ax - a(k \geq 1 \quad \text{is a positive integer}). \tag{4.7}$$

Now we have to find the value of a and x_1 to know the shortest distance.

Since the point (x_1, y_1) lies on the extremals we have

$$y_1 = ax_1 - a. \tag{4.8}$$

From (4.3) and (4.8) we have

$$4x_1^2 + 9a^2(x_1-1)^2 = 36 \Rightarrow 9a^2(x_1-1)^2 = 36 - 4x_1^2$$
$$\Rightarrow \frac{3a(x_1-1)}{2} = \sqrt{9 - x_1^2} \tag{4.9}$$

The transversality condition (3.2) for the problem considered is

$$y' = -\frac{1}{g_x} \quad \text{at} \quad x = x_1. \tag{4.10}$$

Since $y = ax - a$, we have $y' = a$, substituting it and g_x in the equation (4.10) we get

$$a = \frac{3\sqrt{9 - x_1^2}}{2x_1}. \tag{4.11}$$

From (4.9) and (4.11) we get

$$x_1 = \frac{9}{5}. \tag{4.12}$$

From (4.9) and (4.12) we get $a = 2$. Hence the extremal is $y = 2x - 2$ and shortest distance is $\int_1^{9/5}\sqrt{1+2^2}dx = \frac{4}{\sqrt{5}}$.

We can observe that the exact solution is obtained at the first iteration itself.

Example 2: Consider another variational problem to find the shortest distance between $(2,2,2)$ and the sphere $x^2 + y^2 + z^2 = 1$. The point $(2,2,2)$ is exterior to the sphere.

The variational problem for the shortest distance of the considered problem is

$$V(y(x), z(x)) = \int_{x_1}^{2}\sqrt{1 + y' + z'^2}dx \tag{4.13}$$

Let $F(x, y, z, y, y', z') = \sqrt{1 + y'^2 + z'^2}$. The Euler-Lagrange's equations of (4.13) are

$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2+z'^2}} = 0.$

$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0 \Rightarrow \frac{d}{dx}\frac{z'}{\sqrt{1+y'^2+z'^2}} = 0.$ From the above two equations we have $\frac{y'}{\sqrt{1+y'^2+z'^2}} = k$ and $\frac{z'}{\sqrt{1+y'^2+z'^2}} = l$, where k and l are arbitrary constants. These equations can be expressed as

$$y' - k\sqrt{1 + y'^2 + z'^2} = 0 \tag{4.14}$$

$$z' - l\sqrt{1 + y'^2 + z'^2} = 0. \tag{4.15}$$

The order of the above differential equations is one. Therefore the general Lagrange multiplier $\lambda = -1$. Hence, the variational iteration formulae for the above two equations are

$$y^{(n+1)}(x) = y^{(n)}(x) - \int_0^x\left[\left(y^{(n)}(t)\right)' - k\sqrt{1 + \left\{\left(y^{(n)}(t)\right)'\right\}^2 + \left\{\left(z^{(n)}(t)\right)'\right\}^2}\right]dt \tag{4.16}$$

$$z^{(n+1)}(x) = z^{(n)}(x) - \int_0^x\left[\left(z^{(n)}(t)\right)' - l\sqrt{1 + \left\{\left(y^{(n)}(t)\right)'\right\}^2 + \left\{\left(z^{(n)}(t)\right)'\right\}^2}\right]dt \tag{4.17}$$

where $y^{(n+1)}(x)$ and $z^{(n+1)}(x)$ denote the $(n+1)^{\text{th}}$ approximate solution of $y(x)$ and $z(x)$ respectively. Choose $y^{(0)}(x) = ax + b$ and $z^{(0)}(x) = cx + d$ so that the point $(2,2,2)$ lies on these approximations. Therefore, we have $2 = 2a + b$ and $2 = 2c + d$. Hence $b = 2(1-a)$ and $d = 2(1-c)$. Thus $y^{(0)}(x) = ax + 2(1-a)$ and $z^{(0)}(x) = cx + 2(1-c)$. Substituting $n = 0$ in (4.16) we get

$$\begin{aligned} y^{(1)}(x) &= y^{(0)}(x) - \int_0^x\left[\left(y^{(0)}(t)\right)' - k\sqrt{1 + \left\{\left(y^{(0)}(t)\right)'\right\}^2 + \left\{\left(z^{(0)}(t)\right)'\right\}^2}\right]dt \\ &= ax + (1-a) - \int_0^x\left[a - k\sqrt{1 + \{a\}^2 + \{c\}^2}\right]dt \\ &= ax + (1-a) - \left[ax - k\sqrt{1 + a^2 + c^2}x\right] \end{aligned}$$

$$y^{(1)}(x) = 2(1-a) + k\sqrt{1 + a^2 + c^2}x. \tag{4.18}$$

Similarly, we have

$$z^{(1)}(x) = 2(1-c) + l\sqrt{1 + a^2 + c^2}x. \tag{4.19}$$

The point $(2,2,2)$ lies on the extremals. Therefore, $k\sqrt{1+a^2+c^2} = 2a$ and $l\sqrt{1+a^2+c^2} = 2c$. From these two equations we get

$$k = \frac{2a}{\sqrt{1+a^2+c^2}}, l = \frac{2c}{\sqrt{1+a^2+c^2}} \tag{4.20}$$

In view of the above equation it is observed that

$$y^{(1)}(x) = 2(1-a) + 2ax \tag{4.21}$$

$$z^{(1)}(x) = 2(1-c) + 2cx. \tag{4.22}$$

The transversality condition (3.7) gives

$$\frac{\sqrt{1-x^2-y^2}}{-x} = \frac{y'\sqrt{1-x^2-y^2}}{-y} = \frac{z'}{-1} at x = x_1 \tag{4.23}$$

$\frac{z_1}{-x_1} = \frac{2az_1}{-y_1} = \frac{2c}{-1}$.
From these equations we have

$$y_1 = 2ax_1. \tag{4.24}$$

and

$$z_1 = 2cx_1. \tag{4.25}$$

The point (x_1, y_1, Z_1) is also a point on the extremals $y^{(1)}$ and $z^{(1)}$, therefore,

$$y_1 = 2(1-a) + 2ax_1 \tag{4.26}$$

From (4.24) and the above equation we get $a = 1$.Similarly we have $c = 1$. Thus the extremals are $y = x$ and $z = x$.

The point (x_1, y_1, z_1) lies on the sphere and also on the extremals. Hence, $x_1^2 + y_1^2 + z_1^2 = 1, y_1 = x_1$ and $z_1 = x_1$. From these equations, we obtain $x_1 = \frac{1}{\sqrt{3}}$ $y_1 = \frac{1}{\sqrt{3}}$ and $z_1 = \frac{1}{\sqrt{3}}$. Shortest distance

$$= \int_{x_1}^{2} \sqrt{1+y'^2+z'^2}dx = \int_{\left(\frac{1}{\sqrt{3}}\right)}^{2} \sqrt{3}dx = 2\sqrt{3} - 1$$

and it is equal to exact solution.

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