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Prime labeling of torch graph

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Abstract

Let $G=(V(G),E(G))$ be a graph with n vertices. A bijection $f:V(G)\rightarrow\{1,2,...,n\}$ is called a prime labeling if for each edge $e=uv, gcd(f(u),f(v))=1$. A graph which admits prime labeling is called a prime graph. In this paper we prove that Torch graph O_n is a prime graph.

Keywords

Prime Labeling, Duplication, Torch Graph.

AMS Subject Classification

05C10.

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Contents

1. Introduction

We consider only simple, finite, undirected and non-trivial graph $G=(V(G),E(G))$ with the vertex set $V(G)$ and the edge set $E(G)$. For notations and terminology we refer to Bondy and Murthy [2].The notion of a prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout.A [7]. Many researchers have studied prime graph. For a simple graph G with n vertices in the vertex set $V(G)$, a prime labeling is an assignment of the integers 1 to n as labels of the vertices such that each pair of labels from adjacent vertices is relatively prime. A graph that has such a labeling is called prime. Gallian's dynamic graph labeling survey [4] contains a detailed list of graphs that have been proven to be prime.

2. Preliminaries

Definition 2.1. *Duplication of a vertex v_k of a graph G produces a new graph G_1 by adding a vertex $v_k^{'}$ with $N(v_k^{'})=N(v_k)$. In other words, a vertex $v_k^{'}$ is said to be a duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to $v_k^{'}$.*

Definition 2.2. *Torch Graph O_n has $n+4$ vertices and $2n+3$ edges. The set of vertices and edges respectively are*

$$\begin{aligned}
&V(O_n)=\{v_i|1\leq i\leq n+4\} \text{ and}\\
&E(O_n)=\{v_iv_{n+1}|2\leq i\leq n-2\}\cup\{v_iv_{n+3}|2\leq i\leq n-2\}\\
&\quad\cup\{v_1v_i|n\leq i\leq n+4\}\\
&\quad\quad\cup\{v_{n-1}v_n, v_nv_{n+2}, v_nv_{n+4}, v_{n+1}v_{n+3}\}
\end{aligned}$$

The following Figure 1 shows a Torch Graph O_3.

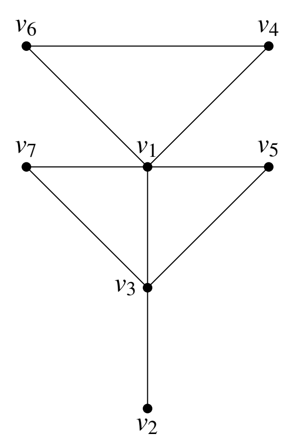

Figure 1. Torch Graph O_3

The following Figure 2 shows a Torch Graph O_n for $n \geq 4$.

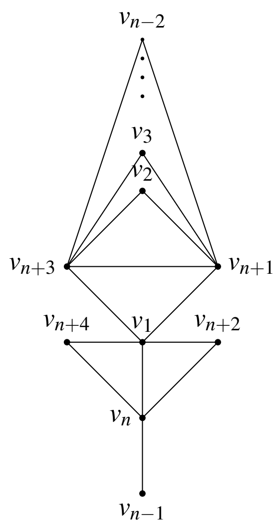

Figure 2. Torch Graph O_n for $n \geq 4$.

Definition 2.3. *If the vertices of the graph are assigned values subject to certain conditions then it is known as (vertex) graph labeling.*

Definition 2.4. *Let $G = (V(G), E(G))$ be a graph with n vertices. A bijection $f : V(G) \to \{1, 2, ..., n\}$ is called a prime labeling if for each edge $e = uv, gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.*

3. Main Results

3.1 Theorem

The torch graph O_n $(n \geq 3)$ is a prime graph.
Proof:
Let $V(O_n) = \{v_i | 1 \leq i \leq n+4\}$ and
$E(O_n) = \{v_i v_{n+1} | 2 \leq i \leq n-2\} \cup \{v_i v_{n+3} | 2 \leq i \leq n-2\} \cup \{v_1 v_i | n \leq i \leq n+4\} \cup \{v_{n-1}v_n, v_n v_{n+2}, v_n v_{n+4}, v_{n+1} v_{n+3}\}$
There are $n+4$ vertices and $2n+3$ edges.
Define $f : V(G) \to \{1, 2, ..., n+4\}$ as follows,
Case(i): when $3 \leq n \leq 8$,

$$\begin{aligned}
f(v_1) &= 3 \\
f(v_n) &= 5 \\
f(v_{n-1}) &= 6 \\
f(v_{n+1}) &= 1 \\
f(v_{n+2}) &= 2 \\
f(v_{n+3}) &= 7 \\
f(v_{n+4}) &= 4 \\
f(v_i) &= i+6; \qquad 2 \leq i \leq n-2
\end{aligned}$$

Clearly, vertex labels are distinct.

$$\begin{aligned}
gcd(f(v_i), f(v_{n+1})) &= gcd(i+6, 1); \; 2 \leq i \leq n-2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
gcd(f(v_i), f(v_{n+3})) &= gcd(i+6, 7); \; 2 \leq i \leq n-2 \\
&= 1 \\
gcd(f(v_1), f(v_n)) &= gcd(3, 5) \\
&= 1 \\
gcd(f(v_1), f(v_{n+1})) &= gcd(3, 1) \\
&= 1 \\
gcd(f(v_1), f(v_{n+2})) &= gcd(3, 2) \\
&= 1 \\
gcd(f(v_1), f(v_{n+3})) &= gcd(3, 7) \\
&= 1 \\
gcd(f(v_1), f(v_{n+4})) &= gcd(3, 4) \\
&= 1 \\
gcd(f(v_{n-1}), f(v_n)) &= gcd(6, 5) \\
&= 1 \\
gcd(f(v_n), f(v_{n+2})) &= gcd(5, 2) \\
&= 1 \\
gcd(f(v_n), f(v_{n+4})) &= gcd(5, 4) \\
&= 1 \\
gcd(f(v_{n+1}), f(v_{n+3})) &= gcd(1, 7) \\
&= 1
\end{aligned}$$

Case(ii): when $n \geq 9$.

$$\begin{aligned}
f(v_1) &= 1 \\
f(v_n) &= 7 \\
f(v_{n-1}) &= 5 \\
f(v_{n+2}) &= 9 \\
f(v_{n+4}) &= 3 \\
f(v_{i+1}) &= 2i; \qquad 1 \leq i \leq \left\lfloor \frac{n+4}{2} \right\rfloor
\end{aligned}$$

Now, the labeling of remaining vertices as follows,

Suppose $n+4$ is prime number then $f(v_{n+1}) = n+4$ and $f(v_{n+3}) = j$ where j is the predecessor prime number of $n+4$ and we label the vertices (if exist) v_{i+1} $(\lfloor \frac{n+4}{2} \rfloor + 1 \leq i \leq n-3)$ by the integers $2t-1$ $(6 \leq t \leq \lceil \frac{n+4}{2} \rceil)$ except $n+4$ and j.

Suppose $n+4$ is not prime number then $f(v_{n+1}) = l$ and $f(v_{n+3}) = m$ where $l > m$, l and m be the greatest two consecutive prime number less than $n+4$ and we label the vertices (if exist) v_{i+1} $(\lfloor \frac{n+4}{2} \rfloor + 1 \leq i \leq n-3)$ by the integers $2t-1$ $(6 \leq t \leq \lceil \frac{n+4}{2} \rceil)$ except l and m.

Clearly, vertex labels are distinct.

$$\begin{aligned} gcd(f(v_1),f(v_n)) &= gcd(1,7)\\ &=1\\ gcd(f(v_1),f(v_{n+2})) &= gcd(1,9)\\ &=1\\ gcd(f(v_1),f(v_{n+4})) &= gcd(1,3)\\ &=1\\ gcd(f(v_{n-1}),f(v_n)) &= gcd(5,7)\\ &=1\\ gcd(f(v_n),f(v_{n+2})) &= gcd(7,9)\\ &=1\\ gcd(f(v_n),f(v_{n+4})) &= gcd(7,3)\\ &=1 \end{aligned}$$

If $n+4$ is prime,

$$\begin{aligned} gcd(f(v_{n+1}),f(v_{n+3})) &= gcd(n+4,j)(\because j\text{is the predecessor}\\ &\quad \text{prime number of} n+4)\\ &=1\\ gcd(f(v_1),f(v_{n+1})) &= gcd(1,n+4)\\ &=1\\ gcd(f(v_1),f(v_{n+3})) &= gcd(1,j)\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+1})) &= gcd(2i,n+4); \quad 1\le i\le \left\lfloor \frac{n+4}{2}\right\rfloor\\ &\quad (\because n+4 \text{ is a prime number})\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+1})) &= gcd\ (2t-1,n+4);\ 6\le t\le \left\lceil \frac{n+4}{2}\right\rceil \text{and}\\ &\quad \left\lfloor \frac{n+4}{2}\right\rfloor+1\le i\le n-3\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+3})) &= gcd(2i,j); \quad 1\le i\le \left\lfloor \frac{n+4}{2}\right\rfloor\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+3})) &= gcd\ (2t-1,j);\ 6\le t\le \left\lceil \frac{n+4}{2}\right\rceil \text{and}\\ &\quad \left\lfloor \frac{n+4}{2}\right\rfloor+1\le i\le n-3\\ &=1 \end{aligned}$$

If $n+4$ is not a prime,

$$\begin{aligned} gcd(f(v_{n+1}),f(v_{n+3})) &= gcd(l,m).\\ &\quad (\because l \text{ and } m \text{ be the greatest}\\ &\quad \text{two consecutive prime number}\\ &\quad \text{less than} n+4)\\ &=1\\ gcd(f(v_1),f(v_{n+1})) &= gcd(1,l)\\ &=1\\ gcd(f(v_1),f(v_{n+3})) &= gcd(1,m)\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+1})) &= gcd(2i,l);\ 1\le i\le \left\lfloor \frac{n+4}{2}\right\rfloor\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+1})) &= gcd(2t-1,l);\\ &\quad 6\le t\le \left\lceil \frac{n+4}{2}\right\rceil \text{and}\\ &\quad \left\lfloor \frac{n+4}{2}\right\rfloor+1\le i\le n-3\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+3})) &= gcd(2i,m);\ 1\le i\le \left\lfloor \frac{n+4}{2}\right\rfloor\\ &=1\\ gcd(f(v_{i+1}),f(v_{n+3})) &= gcd(2t-1,m);\\ &\quad 6\le t\le \left\lceil \frac{n+4}{2}\right\rceil \text{and}\\ &\quad \left\lfloor \frac{n+4}{2}\right\rfloor+1\le i\le n-3\\ &=1 \end{aligned}$$

Thus f admits prime labeling.Hence O_n is a prime graph.

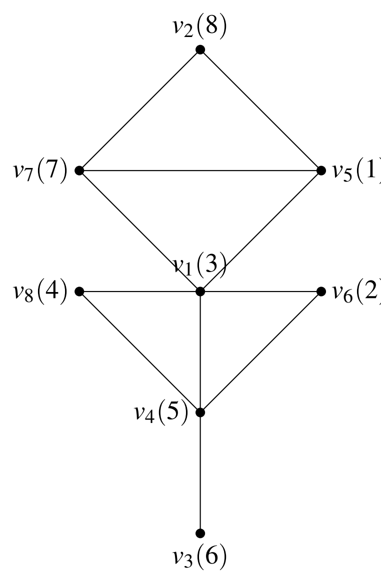

Figure 3. prime labeling of O_4

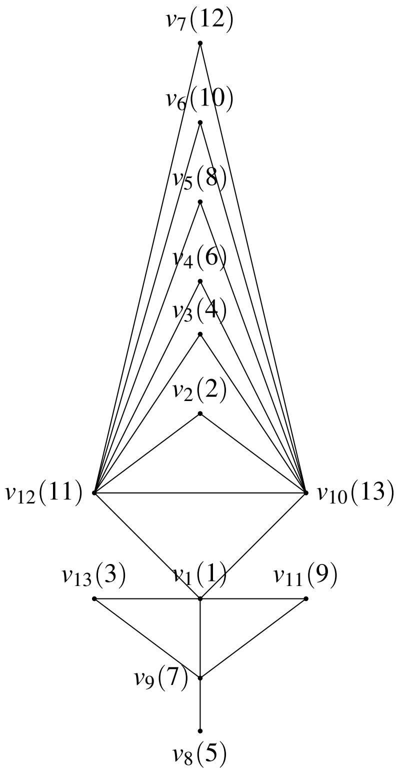

Figure 4. prime labeling of O_9

3.2 Theorem

The graph obtained by duplication a vertex v_{n-2} to v'_{n-2} of the torch graph $O_n, (n \geq 4)$ is a prime graph.

Proof:

Let G be the torch graph $O_n, (n \geq 4)$

Let G_{n-2} be the graph obtained by duplicating the vertex v_{n-2} in O_n.

Let v'_{n-2} be the duplication of v_{n-2} in G_{n-2}.

Then $|V(G_{n-2})| = n+5$ and $|E(G_{n-2})| = 2n+5$.

$$\text{Let } V(G_{n-2}) = \{v_i | 1 \leq i \leq n+4 \text{ and } v'_{n-2}\}$$

$$\begin{aligned}\text{Let } E(G_{n-2}) = &\{v_i v_{n+1} | 2 \leq i \leq n-2\}\\ &\cup \{v_i v_{n+3} | 2 \leq i \leq n-2\}\\ &\cup \{v_1 v_i | n \leq i \leq n+4\}\\ &\cup \{v_{n-1}v_n, v_n v_{n+2}, v_n v_{n+4}, v_{n+1}v_{n+3}\}\\ &\cup \{v'_{n-2}v_{n+1}, v'_{n-2}v_{n+3}\}\end{aligned}$$

Define $f : V(G_{n-2}) \to \{1, 2, ..., n+5\}$ as follows,

Case(i) when $4 \leq n \leq 7$,

$$\begin{aligned}f(v_1) &= 3\\ f(v_n) &= 5\\ f(v_{n-1}) &= 6\\ f(v_{n+1}) &= 1\\ f(v_{n+2}) &= 2\\ f(v_{n+3}) &= 7\\ f(v_{n+4}) &= 4\\ f(v_i) &= i+6; \qquad 2 \leq i \leq n-2.\\ f(v'_{n-2}) &= n+5\end{aligned}$$

Clearly, vertex labels are distinct.

$$\begin{aligned}gcd(f(v_i), f(v_{n+1})) &= gcd(i+6, 1);\ 2 \leq i \leq n-2\\ &= 1\\ gcd(f(v_i), f(v_{n+3})) &= gcd(i+6, 7);\ 2 \leq i \leq n-2\\ &= 1\\ gcd(f(v_1), f(v_n)) &= gcd(3, 5)\\ &= 1\\ gcd(f(v_1), f(v_{n+1})) &= gcd(3, 1)\\ &= 1\\ gcd(f(v_1), f(v_{n+2})) &= gcd(3, 2)\\ &= 1\\ gcd(f(v_1), f(v_{n+3})) &= gcd(3, 7)\\ &= 1\\ gcd(f(v_1), f(v_{n+4})) &= gcd(3, 4)\\ &= 1\\ gcd(f(v_{n-1}), f(v_n)) &= gcd(6, 5)\\ &= 1\\ gcd(f(v_n), f(v_{n+2})) &= gcd(5, 2)\\ &= 1\\ gcd(f(v_n), f(v_{n+4})) &= gcd(5, 4)\\ &= 1\\ gcd(f(v_{n+1}), f(v_{n+3})) &= gcd(1, 7)\\ &= 1\\ gcd(f(v'_{n-2}), f(v_{n+1})) &= gcd(n+5, 1)\\ &= 1\\ gcd(f(v'_{n-2}), f(v_{n+3})) &= gcd(n+5, 7)\\ &= 1\end{aligned}$$

Case(ii) when $n \geq 8$.

$$\begin{aligned}f(v_1) &= 1\\ f(v_n) &= 7\\ f(v_{n-1}) &= 5\\ f(v_{n+2}) &= 9\\ f(v_{n+4}) &= 3\\ f(v'_{n-2}) &= 2\left\lfloor \frac{n+5}{2} \right\rfloor\\ f(v_{i+1}) &= 2i; \quad 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor\end{aligned}$$

Now, the labeling of remaining vertices as follows,

Suppose $n+5$ is prime number then $f(v_{n+1}) = n+5$ and

$f(v_{n+3}) = j$, where j is the predecessor prime number of $n+5$ and we label the vertices (if exist) v_{i+1} ($\lfloor \frac{n+3}{2} \rfloor + 1 \leq i \leq n-3$) by the integers $2t-1$ ($6 \leq t \leq \lceil \frac{n+5}{2} \rceil$) except $n+5$ and j.

Suppose $n+5$ is not prime number then $f(v_{n+1}) = l$ and $f(v_{n+3}) = m$ where $l > m$, l and m be the greatest two consecutive prime number less than $n+5$ and we label the vertices (if exist) v_{i+1} ($\lfloor \frac{n+3}{2} \rfloor + 1 \leq i \leq n-3$) by the integers $2t-1$ ($6 \leq t \leq \lceil \frac{n+5}{2} \rceil$) except l and m.
Clearly, vertex labels are distinct.

$$\begin{aligned}
gcd(f(v_1), f(v_n)) &= gcd(1,7) \\
&= 1 \\
gcd(f(v_1), f(v_{n+2})) &= gcd(1,9) \\
&= 1 \\
gcd(f(v_1), f(v_{n+4})) &= gcd(1,3) \\
&= 1 \\
gcd(f(v_{n-1}), f(v_n)) &= gcd(5,7) \\
&= 1 \\
gcd(f(v_n), f(v_{n+2})) &= gcd(7,9) \\
&= 1 \\
gcd(f(v_n), f(v_{n+4})) &= gcd(7,3) \\
&= 1
\end{aligned}$$

If $n+5$ is prime,

$$\begin{aligned}
gcd(f(v_{n+1}), f(v_{n+3})) &= gcd(n+5, j) \ (\because j \text{ be the} \\
&\quad \text{predecessor prime number of } n+5) \\
&= 1 \\
gcd(f(v_1), f(v_{n+1})) &= gcd(1, n+5) \\
&= 1 \\
gcd(f(v_1), f(v_{n+3})) &= gcd(1, j) \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+1})) &= gcd(2i, n+5);\ 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+1})) &= gcd(2t-1, n+5); 6 \leq t \left\lceil \frac{n+5}{2} \right\rceil \text{ and} \\
&\quad \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq i \leq n-3. \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+3})) &= gcd(2i, j); \quad 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+3})) &= gcd(2t-1, j); 6 \leq t \leq \left\lceil \frac{n+5}{2} \right\rceil \text{ and} \\
&\quad \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq i \leq n-3. \\
&= 1 \\
gcd(f(v'_{n-2}), f(v_{n+1})) &= gcd\ (2\left\lfloor \frac{n+5}{2} \right\rfloor, n+5) \\
&= 1 \\
gcd(f(v'_{n-2}), f(v_{n+3})) &= gcd\ (2\left\lfloor \frac{n+5}{2} \right\rfloor, j) \\
&= 1
\end{aligned}$$

If $n+5$ is not prime,

$$\begin{aligned}
gcd(f(v_{n+1}), f(v_{n+3})) &= gcd(l, m) \ (\because l \text{ and } m \text{ be the two} \\
&\quad \text{consecutive prime number} \\
&\quad \text{less than } n+5) \\
&= 1 \\
gcd(f(v_1), f(v_{n+1})) &= gcd(1, l) \\
&= 1 \\
gcd(f(v_1), f(v_{n+3})) &= gcd(1, m) \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+1})) &= gcd(2i, l); 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+1})) &= gcd(2t-1, l); \\
&\quad 6 \leq t \leq \left\lceil \frac{n+5}{2} \right\rceil \text{ and} \\
&\quad \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq i \leq n-3. \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+3})) &= gcd(2i, m); 1 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor \\
&= 1 \\
gcd(f(v_{i+1}), f(v_{n+3})) &= gcd(2t-1, m); \\
&\quad 6 \leq t \leq \left\lceil \frac{n+5}{2} \right\rceil \text{ and} \\
&\quad \left\lfloor \frac{n+3}{2} \right\rfloor + 1 \leq i \leq n-3. \\
&= 1 \\
gcd(f(v'_{n-2}), f(v_{n+1})) &= gcd\ (2\left\lfloor \frac{n+5}{2} \right\rfloor, l) \\
&= 1 \\
gcd(f(v'_{n-2}), f(v_{n+3})) &= gcd\ (2\left\lfloor \frac{n+5}{2} \right\rfloor, m) \\
&= 1
\end{aligned}$$

Thus f admits prime labeling. Hence O_n is a prime graph.

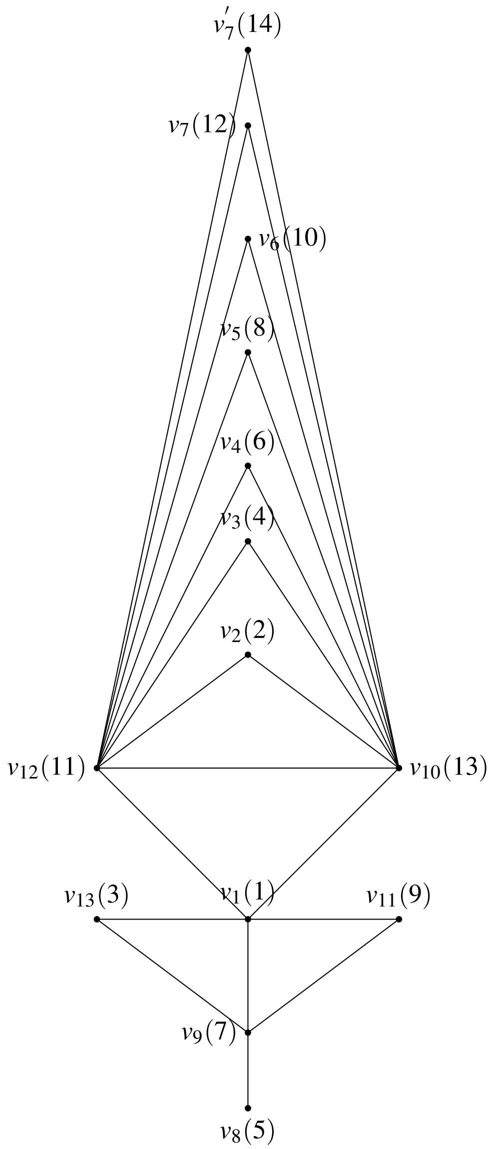

Figure 5. prime labeling of O_9

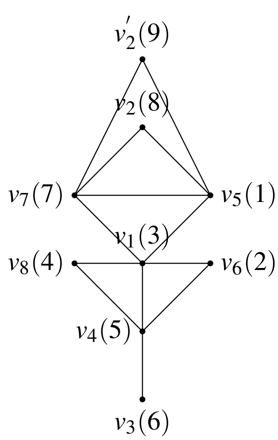

Figure 6. prime labeling of O_4

References

[1] R. Anantha Lakshmi, K.Jayalakshmi and T. Madhavi, prime Labeling of Jahangir Graphs, *International Journal of Engineering & Technology*, 7(4.10)(2018), 389–392.

[2] J.A. Bondy and U.S.R. Murthy, *Graph Theory and Applications*, North Holland, New York 1976.

[3] A. Delman, S. Koilraj, P.Lawrence Rozario Raj, Neighbourhood prime labeling of some special graphs,*International Journal of Pure and Applied Mathematics,* 120(4)(2018), 601–607.

[4] A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinations*,(2019),#6.

[5] Jona Martinus Manulang and Kiki Ariyanti Sugeng, Graceful labeling on torch graph, *Indonesian Journal of Combinatorics*, 2(1)(2018), 14–19.

[6] L. Meenakshi Sundaram and A. Nagarajan, Prime Labeling of Some Special Class of graphs, *Ultra Scientist,* 28(4)(A)(2016), 208–212.

[7] A.Tout, A.N. Dabboucy and K. Howalla, Prime labeling of graphs, *Nat. Acad. Sci letters*, 11(1982), 365–368.

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