Malaya Journal of Matematik, Vol. 9, No. 1, 896-901, 2021

https://doi.org/10.26637/MJM0901/0158

Transient analysis of an inventory model with instantaneous replenishment and Catastrophes

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Abstract

We consider an (s,S) inventory model with instantaneous replenishment and catastrophes. Inventory is replenished according to (s,S) policy, replenishment being instantaneous. Further no shortage is permitted. We have derived explicit expressions for the transient probabilities using Laplace Transform method and have computed such probabilities numerically. The various performance like number of customers in the system, variance of number of customers in the system, expected inventory level, effective flush out rate, etc. are measured. Steady state probabilities and stability conditions are also derived. Numerical illustrations are used to discuss the system performance measures.

Keywords

Inventory model, Transient and steady state analysis, Laplace Transform method, Catastrophes.

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Article History: Received **14** January **2021**; Accepted **19** March **2021**

Contents

1. Introduction

Queueing inventory models were first studied by (Melikov and Molchano 1992) and (Sigman and Simchi- Levi 1992). Later Berman and et.al [1] considered an inventory system where a processing time is required for serving the inventory. This was a deterministic model. Berman and Kim [2] and Berman and Sapna [3] were the first to discuss inventory with exponential distribution and with arbitrary distribution respectively. Krishnamoorthy and his co-authors used Matrix Analytic Methods to study inventory models [4–12] where service time for providing the inventoried item is assumed.

Below we present a review of the work on the studies of transient analysis of queueing systems. In paper [13] a transient solution of an M/M/1 queue with catastrophes is considered. In paper [14] a transient analysis of a single server queue with catastrophes, failures and repairs is considered by the authors. In paper [15] a closed form solution for a queueing network model with catastrophes is obtained. We refer to paper [16] for a complete survey on various inventory queueing models.

By a catastrophe we mean a kind of negative arrivals to the system.The effect of a negative arrival can be to remove some or all the regular customers in the system.Here we assume that all the customers are flushed out of the system.As an example we may consider a computer system or a network affected by a virus. If a job transmits a virus we need to perform a clearing operation .This will remove all the queued operations in the system.

2. Mathematical Model

The system is described as under. Customers arrive to a counter according to a Poisson process of rate λ where inventory is served. Service is exponentially distributed with rate μ,

but when it ends the whole customers are emptied. Inventory is replenished according to (s,S) policy, replenishment being instantaneous. Further no shortage is permitted. At time t let N(t) be the number of customers in the system including the one being served and L(t) be the inventory level.
Then $\Omega = \{X(t) : t \geq 0\} = \{(N(t), L(t)), t \geq 0\}$ will be a Markov chain with state space $E = \{(0,k) : s \leq k \leq S-1\} \cup \{(i,k) : i \geq 1, s+1 \leq k \leq S\}$

The state space of the Markov chain can be partitioned into levels $\tilde{i}$ defined as $\tilde{0}$ = {(0, s), (0, s + 1), ...,(0, S - 1)} and $\tilde{i}$ = {(i, s + 1), (i, s + 2),...,(i, S)} for $i \geq 1$. In the following sequel Q stands for S-s,In denotes an identity matrix of order n and e denotes a column matrix of 1's of appropriate order. Now the infinitesimal matrix of the process is $T = \begin{bmatrix} B_0 & B_1 & 0 & 0 & 0 & 0 \\ A_2 & A_1 & A_0 & 0 & 0 & 0 \\ A_2 & 0 & A_1 & A_0 & 0 & 0 \\ A_2 & 0 & 0 & A_1 & A_0 & 0 \\ . & & & & & \\ .... & & & & & \end{bmatrix}$ where, $B_0 = -\lambda I_Q$;
$A_2 = \mu I_Q$; $A_1 = -(\lambda+\mu)I_Q$; $A_0 = \lambda I_Q$

$$B_1 = \begin{bmatrix} 0 & 0 & .... & \lambda \\ \lambda & 0 & .... & 0 \\ 0 & \lambda ... & & \\ .... & & & \\ 0 & 0....... & \lambda & 0 \end{bmatrix}$$

3. Transient Solution

The Chapman-Kolmogorov equations are given by

$$\pi'(0,i)(t) = -\lambda\pi(0,i)(t) + \mu[\pi(1,i+1)(t) + \pi(2,i+1)(t) + ...]; \quad s \leq i \leq S-1 \quad (3.1)$$

$$\pi'(j,i)(t) = -(\lambda+\mu)\pi(j,i)(t) + \lambda\pi(j-1,i)(t); \quad j \geq 2, s+1 \leq i \leq S \quad (3.2)$$

$$\pi'(1,S)(t) = -(\lambda+\mu)\pi(1,S)(t) + \lambda\pi(0,s)(t) \quad (3.3)$$

$$\pi'(1,i)(t) = -(\lambda+\mu)\pi(1,i)(t) + \lambda\pi(0,i)(t); \quad s+1 \leq i \leq S-1 \quad (3.4)$$

Applying Laplace transform to (3.1), (3.2), (3.3) and (3.4) we obtain the following

$$-\lambda\tilde{\pi}(0,i)(a) + \mu[\tilde{\pi}(1,i+1)(a) + \tilde{\pi}(2,i+1)(a) + ...] = a\tilde{\pi}(0,i)(a); s+1 \leq i \leq S-1 \quad (3.5)$$

$$\lambda\tilde{\pi}(0,s)(a) + \mu[\tilde{\pi}(1,s+1)(a) + \tilde{\pi}(2,s+1)(a) + ...] = a\tilde{\pi}(0,s)(a) - 1 \quad (3.6)$$

$$\lambda\tilde{\pi}(j,i)(a) = (\lambda+\mu+a)\tilde{\pi}(j+1,i)(a); j \geq 1, s+1 \leq i \leq S \quad (3.7)$$

$$\lambda\tilde{\pi}(0,i)(a) = (\lambda+\mu+a)\tilde{\pi}(1,i)(a); j \geq 1, s+1 \leq i \leq S-1 \quad (3.8)$$

$$\lambda\tilde{\pi}(0,s)(a) = (\lambda+\mu+a)\tilde{\pi}(1,S)(a) \quad (3.9)$$

where $\tilde{\pi}(j,i)(a)$ stands for the Laplace transform of $\pi(j,i)(t)$. By taking inverse Laplace transform we obtain the following

$$\pi(i+1,j)(t) = \pi(i,j)(t) * \lambda\exp -(\lambda+\mu)t; \quad i \geq 1; s+1 \leq j \leq S \quad (3.10)$$

$$\pi(1,j)(t) = \pi(0,j)(t) * \lambda\exp -(\lambda+\mu)t; \quad s+1 \leq j \leq S-1 \quad (3.11)$$

$$\pi(1,S)(t) = \pi(0,s)(t) * \lambda\exp -(\lambda+\mu)t \quad (3.12)$$

$$\frac{\lambda\mu}{\mu-\lambda}[\exp(-\lambda t) - \exp(-\mu t)] * \pi(0,s+1)(t) = \pi(0,s)(t) - \exp(-\lambda t) \quad (3.13)$$

$$\frac{\lambda\mu}{\mu-\lambda}[\exp(-\lambda t) - \exp(-\mu t)] * \pi(0,s)(t) = \pi(0,S-1)(t) \quad (3.14)$$

$$\frac{\lambda\mu}{\mu-\lambda}[\exp(-\lambda t) - \exp(-\mu t)] * \pi(0,i+1)(t) = \pi(0,i)(t); \; s+1 \leq i \leq S-2 \quad (3.15)$$

Finally $\pi(0,s)(t)$ is the inverse Laplace transform of

$$\frac{\mu+a}{\lambda\mu}\frac{\chi^{Q-1}}{\chi^{Q}-1} \quad (3.16)$$

and $\chi = \frac{(\lambda+a)(\mu+a)}{\lambda\mu}$. Here $*$ stands for the convolution of two functions and we asssume the initial distribution is (1, 0, 0,...,0).
The inverse Laplace transform of the above expression is given by sum of residues of

$$\frac{\exp(st)(\mu+a)^{Q}(\lambda+a)^{(}Q-1)}{(\mu+a)^{Q}(\lambda+a)^{Q} - (\lambda\mu)^{Q}} \quad (3.17)$$

The poles of this expression are 0 and $-(\lambda+\mu)$ each of order Q.

4. Numerical Analysis

Even though we have obtained explicit expressions for the transient probabilities we compute the above probabilities numerically for various times. Also the following performance measures are evaluated numerically.

1. Expected number of customers in the system $= E[N(t)] = \sum_{j=1}^{\infty}\sum_{j=s+1}^{\infty} j\pi(j,i)(t)$
2. Variance of number of customers in the system $= E[N(t)^2] - E[N(t)]^2$
3. Expected inventory level in the system $= E[L(t)] = \sum_{j=1}^{\infty}\sum_{i=s+1}^{S} i\pi(j,i)(t) + \sum_{i=s}^{S-1} i\pi(0,i)(t)$
4. Effective Flush out rate EFR $= \sum_{j=1}^{\infty}\sum_{i=s+1}^{S} \mu\pi(j,i)(t)$
5. Probability that the system is empty $= P[N(t)=0] = 1 - \sum_{j=1}^{\infty}\sum_{i=s+1}^{S} \pi(j,i)(t)$
6. Probability that the system is not empty $= P[N(t)>0] = \sum_{j=1}^{\infty}\sum_{i=s+1}^{S} \pi(j,i)(t)$

5. Steady State Analysis

Stability Condition:Define $A = A_0 + A_1 + A_2$ and let $\pi = (\pi_1, \pi_2, ..., \pi_Q)$ be the steady state vector of the generation matrix A. The QBD process with generation matrix T is stable if and only if the rate of drift to the left is larger than rate of drift to the right; ie, $\pi A_0 e < \pi A_2 e$(see Neuts) that is if and only if $\frac{\lambda}{\mu} < 1$.
Computation of steady state vector: We find the steady state vector of Ω explicitly. Let $\pi = (\pi_0, \pi_1, ...)$ be the steady state vector where $\pi_0 = (\pi_0(0,s), \pi_0(0,s+1), ..., \pi_0(0,S-1))$ and $\pi_i = (\pi_i(i,s+1), \pi_i(i,s+2), ..., \pi_i(i,S)); i \geq 1$.
$\pi T = 0$ implies, $\pi_0 B_0 + \pi_1 A_2 + \pi_2 A_2 + ... = 0; \pi_0 B_1 + \pi_1 A_1 = 0$; $\pi_i A_0 + \pi_{i+1} A_1 = 0; i \geq 1$.
From the above equations we easily obtain
$\pi_0(0,s) = \pi_0(0,s+1) = ... = \pi_0(0,S-1)$ and $\pi_i(i,s+1) = \pi_i(i,s+2) = ...\pi_i(i,S)$ for all $i \geq 1$. For solving the above system of equations we first consider a system where units arrive according to a Poisson Process with rate λ and where service is exponentially distributed with rate μ, but when it ends the whole buffer is emptied. Let $x = (x(0), x(1), ...)$ be the steady state of this system. Then $x(j) = p(1-p)^j$ where $p = \frac{\lambda}{\lambda+\mu}$.
Now $\pi_0(0,k) = \frac{1}{Q}x(0); s \leq k \leq S-1)$ and $\pi_i(i,k) = \frac{1}{Q}x(i); s+1 \leq k \leq S; i \geq 1$.

6. Numerical Illustration

In this section we provide numerical illustration of the system performance as underlying parameters vary at different instance of time.

6.1 Variation of various performance measures with arrival rate

In table 6.1.1 we see that as λ increases the expected number of customers in the system and expected inventory level both increase. But in 6.1.2 and 6.1.3 the expected number of customers in the system initially increases and then decreases whereas the expected inventory level decreases. In table 6.1.4 the expected number of customers and the expected inventory level both decreases with increase in λ.The expected number of customers in the system in steady state can be calculated as μ/λ. Hence as time t is large enough is the expected number of customers in the system should decrease with increase in λ as is seen in table 6.1.4. For smaller values of time t the system is in transient phase and so the performance measures depends on t and the initial distribution.

Table 6.1.1. $\mu = 2, s = 2, S = 8, t = 1$

λ	E[N(t)]	V[N(t)]	EIL	P[N(t)] = 0	P[N(t)] > 0
3	1.295	2.2723	5.1353	0.404	0.596
4	1.7139	3.3825	5.3816	0.3347	0.6653
5	2.0981	4.5106	5.4983	0.2851	0.7149
6	2.4164	5.5319	5.5053	0.2468	0.7532
7	2.6434	6.3825	5.4051	0.2151	0.7849
8	2.7692	7.058	5.2043	0.1875	0.8125

Table 6.1.2. $\mu = 2, s = 2, S = 8, t = 5$

λ	E[N(t)]	V[N(t)]	EIL	P[N(t)] = 0	P[N(t)] > 0
3	1.386	3.1113	3.8756	0.3841	0.6159
4	1.6093	4.1062	3.8713	0.2918	0.7875
5	1.6169	4.7523	3.4414	0.2125	0.7875
6	1.4434	4.9308	2.7573	0.1468	0.8534
7	1.1662	4.587	2.0209	0.096	0.904
8	0.8646	3.8304	1.3758	0.0595	0.9405

Table 6.1.3. $\mu = 2, s = 2, S = 8, t = 10$

λ	E[N(t)]	V[N(t)]	EIL	P[N(t)] = 0	P[N(t)] > 0
3	1.3027	3.0328	3.6708	0.361	0.639
4	1.3382	3.7773	3.1544	0.2427	0.7573
5	1.0988	3.7988	2.2496	0.1444	0.8556
6	0.7415	3.0534	1.3617	0.0754	0.9246
7	0.4211	1.9702	0.7145	0.0342	0.9653
8	0.2054	1.0453	0.3283	0.0141	0.9859

Table 6.1.4. $\mu = 2, s = 2, S = 8, t = 15$

λ	E[N(t)]	V[N(t)]	EIL	P[N(t)] = 0	P[N(t)] > 0
3	1.2244	2.9464	3.4617	0.3395	0.6607
4	1.3382	3.7773	3.1544	0.2427	0.7573
5	1.0988	3.7988	2.2496	0.1444	0.8556
6	0.7415	3.0534	1.3617	0.0754	0.9246
7	0.4211	1.9702	0.7145	0.0342	0.9653
8	0.2054	1.0453	0.3283	0.0141	0.9859

6.2 Variation of various performance measures with service rate

In all the four tables 6.2.1, 6.2.2, 6.2.3 and 6.2.4, we observe that the number of customers and expected inventory level decreases with increase in the service rate μ. This is as expected since when a service completion happens all the customers are flushed out of the system.

Table 6.2.1. $\lambda = 2, s = 2, S = 8, t = 1$

μ	E[N(t)]	V[N(t)]	EIL	P[N(t)] $= 0$	P[N(t)] > 0
3	0.6334	0.9437	6.0913	0.6027	0.3973
4	0.4908	0.704	3.6973	0.6675	0.3325
5	0.3973	0.5465	3.4208	0.7145	0.2855
6	0.3325	0.4403	3.2182	0.7501	0.2499
7	0.2855	0.366	3.0645	0.7778	0.2222
8	0.2499	0.3121	2.9442	0.8	0.2

Table 6.2.2. $\lambda = 2, s = 2, S = 8, t = 5$

μ	E[N(t)]	V[N(t)]	EIL	P[N(t)] $= 0$	P[N(t)] > 0
3	0.6665	1.1001	3.3558	0.5994	0.4006
4	0.4997	0.7482	3.2042	0.6665	0.33345
5	0.3999	0.5596	3.0482	0.7142	0.2858
6	0.3333	0.4443	2.9151	0.75	0.25
7	0.2857	0.3673	2.808	0.7778	0.2222
8	0.25	0.3125	2.7222	0.8	0.2

Table 6.2.3. $\lambda = 2, s = 2, S = 8, t = 10$

μ	E[N(t)]	V[N(t)]	EIL	P[N(t)] $= 0$	P[N(t)] > 0
3	0.66395	1.0991	3.3761	0.5985	0.4015
4	0.4995	0.748	3.174	0.6663	0.3337
5	0.3995	0.5596	3.0057	0.7142	0.2858
6	0.3333	0.4443	2.8771	0.75	0.25
7	0.2857	0.3673	2.7781	0.7778	0.2222
8	0.25	0.3125	2.6998	0.8	0.2

Table 6.2.4. $\lambda = 2, s = 2, S = 8, t = 15$

μ	E[N(t)]	V[N(t)]	EIL	P[N(t)] $= 0$	P[N(t)] > 0
3	0.6629	1.098	3.3817	0.5975	0.4025
4	0.4994	0.7479	3.1653	0.6661	0.3339
5	0.3999	0.5595	2.9996	0.7141	0.2859
6	0.3333	0.4443	2.8747	0.7499	0.2501
7	0.2857	0.3673	2.7776	0.7778	0.2222
8	0.25	0.3125	2.6999	0.8	0.2

Figure 6.2.1. Arrival rate λ verses expected number of customers at time, $t = 1$

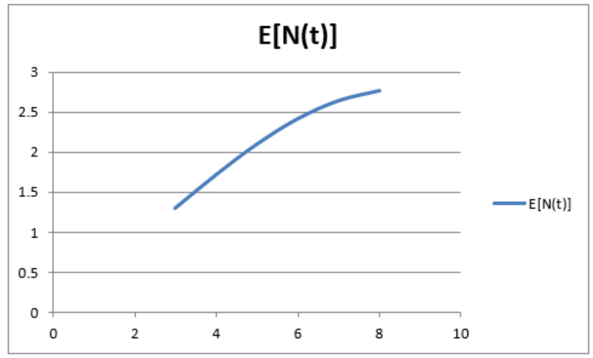


Figure 6.2.2. Arrival rate λ verses expected number of customers at time, $t = 5$

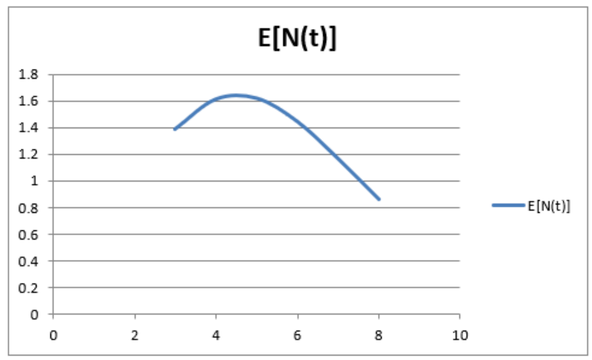


Figure 6.2.3. Arrival rate λ verses expected number of customers at time, $t = 10$

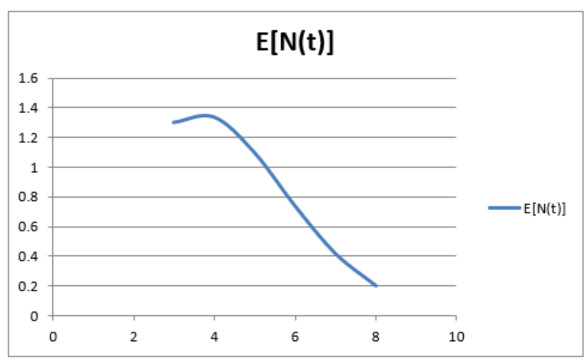


Figure 6.2.4. Arrival rate λ verses expected number of customers at time, $t = 15$

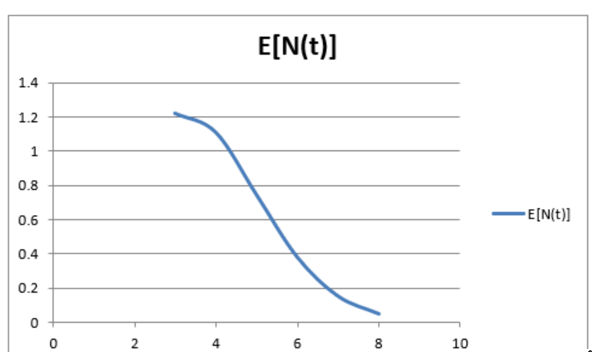


Figure 6.2.5. Service rate μ verses expected number of customers at time, $t = 1$

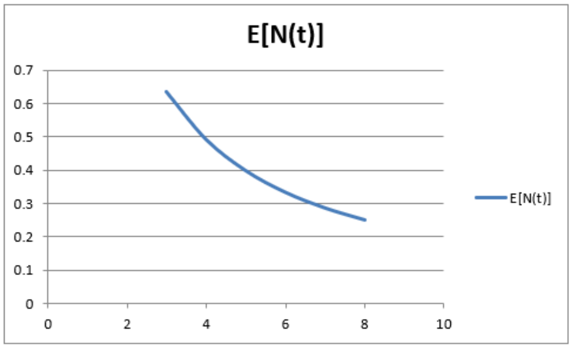


Figure 6.2.6. Service rate μ verses expected number of customers at time, $t = 5$

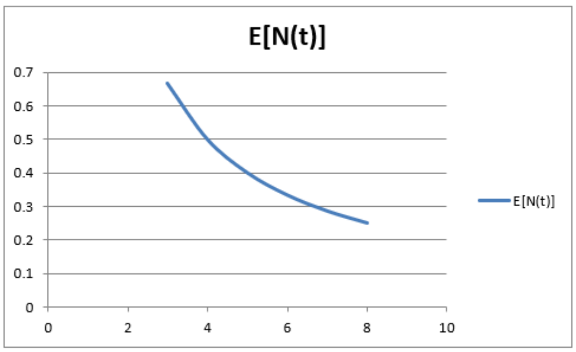


Figure 6.2.7. Service rate μ verses expected number of customers at time, $t = 10$

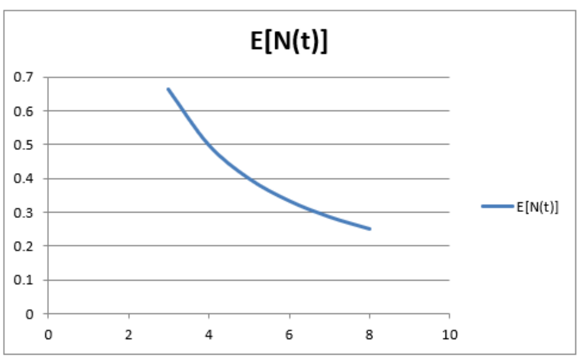
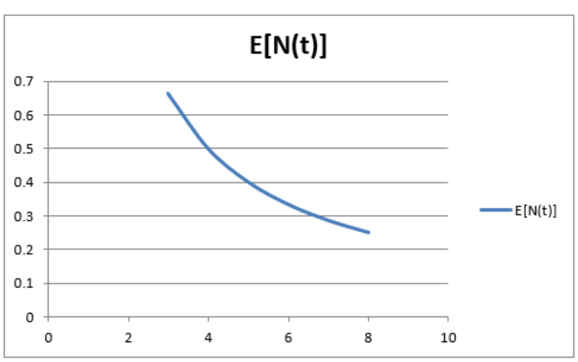


Figure 6.2.8. Service rate μ verses expected number of customers at time, $t = 15$



7. Conclusion

In this paper we could derive an explicit expression for the transient probability vector of an inventory queueing model. Here we assumed zero lead time. We wish to extend this paper by considering positive lead time as well which may have several applications in real life situations.

References

[1] O.Berman,E.Kim and D.G Shimshack, Deterministic approximations for inventory management at service facilities, IIE Transactions 5(1993), no:5, 98-104.

[2] O.Berman and E.Kim ,Stochastic models for inventory management at service facilities, Communication in Statistics. Stochastic Models 15(1999), no.4, 695-718.

[3] O.Berman and K.P Sapna ,Inventory management at service facilities for systems with arbitrarily distributed service times, Communications in Statistics. Stochastic Models 16(2000), no.3-4, 343-360.

[4] TG Deepak, A.Krishnamoorthy ,C.N Viswanath and K.Vineetha ;Inventory with service time and transfer of customers and/inventory; Annals of Operations Research ,PP.191- 213,Volume 160,Number 1/April 2008.

[5] A.Krishnamoorthy,TG Deepak,C.N Viswanath and K.Vineetha; Effective utilization of idle time in an (s,S)inventory with positive service time,Journal of Applied Mathematics and Stochastic Analysis(JAMSA), vol.2006, Article ID 69068, 13 pages,2006.doi10.1155/JAMSA/2006/69068

[6] A.Krishnamoorthy,ME Islam and C.N Viswanath; Retrial inventory with Batch Markovian Arrival and Positive Service Time, Stochastic Modelling and Applications Vol 9,Number 2,pp.38-53,2006.

[7] A.Krishnamoorthy, K.P Jose and C.N Viswanath; Numerical investigation of a PH/PH/1Inventory system with Positive service time and Shortage; Neural, Parallel & Scientific Computations; Volume 16,Number 4,December 2008,Dynamic Publishers, USA.

[8] A.Krishnamoorthy, S.N Sajeev and C.N Viswanath; An Inventory Model with Retrial and Orbital Search; Bulletin of Kerala Mathematical Association ,Special Issue,47-65,October 2009;Guest Editor: S.R.S Varadhan FRS.

[9] A.Krishnamoorthy, C.N Viswanath,T.G Deepak and K.Vineetha; Control Policies for Inventory with Service Time; Stochastic analysis and Applications; Volume 24, Number 4/2006, PP.889-899.

[10] A.Krishnamoorthy, C.N Viswanath and M.E Islam; On a Production Inventory with Service Time and retrial of Customers; Proceedings of the 11th International Conference on Analytical and Stochastic Modelling Techniques and Analysis; ASMTA 2004; pp.238-247-Publishing Company- Magdeburg

[11] A.Krishnamoorthy, C.N Viswanath and M.E Islam;Retrial Production Inventory with MAP and Service Times ; Queues, Flows, Systems, Networks. Proceed-

ings of the International Conference "Modern Mathematical Methods of Analysis and Optimization of Telecommunication Networks", Gomel, 2003, pp.148-156.

[12] C.N Viswanath,T.G Deepak ,A.Krishnamoorthy and B.Krishnakumar; On an (s,S) Inventory system with service time vacation on the server an correlated lead time; Quality Technology and Quantitative Management (QTQM), PP.129-143, Volume 5, Number 2 June 2008.

[13] Krishna Kumar B, Arivudainambi D.Transient solution of an M/M/1 queue with catastrophes. Comput Math Appl 2000; 40:1233–1240.

[14] B. Krishna Kumar, A. Krishnamoorthy, S. Pavai Madheswari•S. Sadiq Basha :Transient analysis of a single server queue with catastrophes, failures and repairs, Queueing Systems 2007.

[15] ChaoX.Aqueueing network model with catastrophes and product form solution. OR Lett 1995;18:75–79.

[16] A.Krishnamoorthy, Dhanya Shajin, ViswanathC. Narayanan: Inventory with Potive Service Time: A SurveyQueueing Theory 2, coordinated by Vladimir ANISIMOV, Nikolaos LIMNIOS. © ISTE Editions 2020

[17] Atencia I, Moreno P. The discrete time Geo/Geo/1 queue with negative customers and disasters. Comput Oper Res 2004; 31(9):1537–1548.

[18] Di Crescenzo A, Giorno V, Nobile AG, Ricciardi LM. On the M/M/1 queue with catastrophes and its continuous approximation.Queueing Syst2003; 43:329–347.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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