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# Relation between the *SDD* invariant and other graph invariants

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**Abstract**

The *SDD* invariant is one of the 148 discrete Adriatic indices contributed many years ago. In this paper, we present the relations between the *SDD* invariant and other graph invariants.

**Keywords**

Degree, Zagreb invariant, symmetric division deg invariant.

**AMS Subject Classification**

05C12, 05C50.

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## Contents



## 1. Introduction

Molecular descriptors have found applications in modeling several physicochemical properties in *QSAR* and *QSPR* studies [2, 8]. A particularly many type of molecular descriptors are defined as functions of the structure of the underlying molecular graph, such as the Wiener invariant [21], the Zagreb invariants [4] and Balaban invariant [1]. Damir Vukicević et al. [19] proved that many of these descriptors are defined the sum of individual bond contributions. Among the 148 discrete Adriatic invariants studied in [19], whose predictive properties were evaluated against the benchmark datasets of the International Academy of Mathematical Chemistry [15], 20 invariants were selected as significant predictors of physicochemical properties. One of these useful discrete adriatic indices is the symmetric division deg (*SDD*) invariant which is defined as $SDD(\Gamma)=\sum_{xy\in E(\Gamma)}\left(\frac{\lambda_\Gamma(x)}{\lambda_\Gamma(y)}+\frac{\lambda_\Gamma(y)}{\lambda_\Gamma(x))}\right)$, where $\lambda_\Gamma(x)$ and $\lambda_\Gamma(y)$ are the degrees of vertices $x$ and $y$, respectively. Among all the existing molecular descriptors, *SDD* invariant has the best correlating ability for predicting the total surface area of polychlorobiphenys [19].

Vasilyev [20] provided the different types of lower and upper bounds of symmetric division deg invariant in some classes of graphs and determined the corresponding extremal graphs. Palacios [7] found a new upper bound for the symmetric division deg invariant of a graph $\Gamma$ with $n$ vertices, in terms of the inverse degree invariant,that is attained by all regular,all complete multipartite graphs, $K_{b_1,b_2,\ldots,b_l}$, and all $(s-1,t)$-regular graphs of order $s$, where $1=t<s-1$. Several papers have been appeared in literature addressing the mathematical aspects of this descriptor; for example see [5, 6, 10, 11]. In this paper, we present the relations between the *SDD* invariant and other graph invariants.

## 2. Preliminaries

The minimum and minimum vertex degrees of $\Gamma$, respectively, denoted by $\delta$ and $\Delta$.

- First and second Zagreb invariants: $M_1(\Gamma)=\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)+\lambda_\Gamma(y))$ and $M_2(\Gamma)=\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)\lambda_\Gamma(y))$.
- The $\alpha$-Randić invariant is then defined as $R_\alpha(\Gamma)=\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)\lambda_\Gamma(y))^\alpha$.
- The $\alpha$-$F$-invariant: $F_\alpha(\Gamma)=\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^\alpha$.

## 3. Bounds for *SDD*

In this section, we find the results related to *SDD* invariant and other graph invariants.

**Theorem 3.1.** *Let $\Gamma$ be a connected graph with m edges. Then* $SDD(\Gamma) \geq 2m\Big(1 + ln\left(\frac{\delta^2}{\Delta^2}\right)\Big)$ *with equality if and only if $\Gamma$ is either cycle graph or a path of length one.*

**Proof.** Consider the function $f(X) = X - lnX - 1$. We can easily find $f(X) \geq 0$. Hence for any edge $xy$ in $E(\Gamma)$,

$$\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} - ln\left(\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)}\right) - 1 \geq 0.$$

So,

$$\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} \geq 1 + ln\left(\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)}\right).$$

By taking the summation over all edges of the graph, we get

$$\begin{aligned} &\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} \qquad (3.1)\\ \geq\ & \sum_{xy\in E(\Gamma)} (1) + \sum_{xy\in E(\Gamma)} ln\left(\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)}\right)\\ =\ & m + ln\left(\prod_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)}\right)\\ \geq\ & m + ln\left(\frac{\delta^2}{\Delta^2}\right)^m\\ =\ & m\Big(1 + ln\left(\frac{\delta^2}{\Delta^2}\right)\Big). \end{aligned}$$

By the definition of *SDD* index, we have

$$SDD(\Gamma) \geq 2m\Big(1 + ln\left(\frac{\delta^2}{\Delta^2}\right)\Big).$$

To show the equality, let $f(X) = 0$, then $X = 1$. Thus $\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} = 1$ for any edge $xy \in E(\Gamma)$. Hence $\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2 = 2\lambda_G(x)\lambda_\Gamma(y)$ for every edge $xy \in E(\Gamma)$. Therefore $\Gamma$ is either cycle graph (or) a path of length one.

**Theorem 3.2.** *Let $\Gamma$ be a connected graph. Then* $SDD(\Gamma) \geq 4m - 4S_{-1,-1}(\Gamma)$ *with equality if and only if $\Gamma$ is either cycle graph (or) a path of length one.*

**Proof.** Consider the function $f(X) = X + \frac{1}{X} - 2$. One can easily show that $f(X) \geq 0$. Hence for any edge $xy$ in $\Gamma$,

$$\frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} + \frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2} - 2 \geq 0.$$

By taking the summation over the edges of the graph, we get

$$\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} + \sum_{xy\in E(\Gamma)} \frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2} - \sum_{xy\in E(\Gamma)} (2) \geq 0.$$

By the definition of *SDD* index, we have

$$\begin{aligned} \frac{SDD(\Gamma)}{2} &\geq 2m - \sum_{xy\in E(\Gamma)} \frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}\\ &\geq 2m - 2S_{-1,-1}(\Gamma). \end{aligned}$$

Hence

$$SDD(\Gamma) \geq 4m - 4S_{-1,-1}(\Gamma).$$

To show the equality, let $f(X) = 0$, then $X = 1$. Thus $\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2\lambda_G(x)\lambda_\Gamma(y)} = 1$ for any edge $xy \in E(\Gamma)$. Hence $\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2 = 2\lambda_G(x)\lambda_\Gamma(y)$ for every edge $xy \in E(\Gamma)$. Therefore $\Gamma$ is either cycle graph (or) a path of length one.

**Theorem 3.3.** *Let $\Gamma$ be a connected graph with $s_1$ pendant vertices, maximal degree $\Delta$ and minimal non-pendant vertex degree $\delta_1$. Then*

$$SDD(\Gamma) \leq \sqrt{\Big(F_2(\Gamma) - s_1(1+\delta_1^2)\Big)\Big(R_{-2}(\Gamma) - \frac{s_1}{\Delta^2}\Big)}.$$

**Proof.** For any edge $xy$ in $\Gamma$ with $\lambda_\Gamma(x), \lambda_\Gamma(y) \geq 2$, let $a_{xy} = \lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2$ and $b_{xy} = \frac{1}{\lambda_\Gamma(x)\lambda_\Gamma(y)}$. Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\Big(\sum_{xy\in E(\Gamma),\lambda_\Gamma(x),\lambda_\Gamma(y)\neq 1} \frac{\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ \leq\ & \sum_{xy\in E(\Gamma),\lambda_\Gamma(x),\lambda_\Gamma(y)\neq 1} \Big(\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2\Big)^2\\ & \sum_{xy\in E(\Gamma),\lambda_\Gamma(x),\lambda_\Gamma(y)\neq 1} \Big(\frac{1}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ =\ & \Big[\sum_{xy\in E(\Gamma)} \Big(\lambda_\Gamma(x)^2 + \lambda_\Gamma(y)^2\Big)^2 - \sum_{xy\in E(\Gamma),\lambda_\Gamma(x)=1} \Big(1 + \lambda_\Gamma(y)^2\Big)^2\Big]\\ & + \Big[\sum_{xy\in E(\Gamma)} \Big(\frac{1}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2 - \sum_{xy\in E(\Gamma),\lambda_\Gamma(x)=1} \Big(\frac{1}{\lambda_\Gamma(y)}\Big)^2\Big]\\ \leq\ & \Big(F_2(\Gamma) - s_1(1+\delta_1^2)\Big)\Big(R_{-2}(\Gamma) - \frac{s_1}{\Delta^2}\Big). \end{aligned}$$

By the definition of *SDD* index, we have

$$(SDD(\Gamma))^2 \leq \Big(F_2(\Gamma) - s_1(1+\delta_1^2)\Big)\Big(R_{-2}(\Gamma) - \frac{s_1}{\Delta^2}\Big).$$

Hence

$$SDD(\Gamma) \leq \sqrt{\Big(F_2(\Gamma) - s_1(1+\delta_1^2)\Big)\Big(R_{-2}(\Gamma) - \frac{s_1}{\Delta^2}\Big)}.$$

By setting $s_1 = 0$ in above theorem, we have the following corollary.

**Corollary 3.4.** *Let $\Gamma$ be a connected graph. Then* $SDD(\Gamma) \leq \sqrt{\Big(F_2(\Gamma)R_{-2}(\Gamma)}.$

**Theorem 3.5.** *Let $\Gamma$ be a connected graph. Then* $\frac{2\delta^2}{\Delta}R_{-\frac{1}{2}}(\Gamma) \leq SDD(\Gamma) \leq \frac{2\Delta^2}{\delta}R_{-\frac{1}{2}}(\Gamma).$

**Proof.** By the definition of *SDD* index,

$$\begin{aligned}(SDD(\Gamma))^2 &= \Big(\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ &= \Big(\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}}\Big)^2\\ &\geq \Big(\frac{2\delta^2}{\Delta}\sum_{xy\in E(\Gamma)} \frac{1}{\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}}\Big)^2.\end{aligned}$$

Hence

$$SDD(\Gamma) \geq \frac{2\delta^2}{\Delta} R_{-\frac{1}{2}}(\Gamma)$$

Similarly,

$$\begin{aligned}(SDD(\Gamma))^2 &= \Big(\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ &= \Big(\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}}\Big)^2\\ &\leq \Big(\frac{2\Delta^2}{\delta}\sum_{xy\in E(\Gamma)} \frac{1}{\sqrt{\lambda_\Gamma(x)\lambda_\Gamma(y)}}\Big)^2\end{aligned}$$

and this leads to the desired upper bound.

The generalized *SDD* index of a connected graph $\Gamma$ is defined as $S_{\alpha,\beta}(\Gamma) = \sum_{xy\in E(\Gamma)} \frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^\alpha}{(\lambda_\Gamma(x)\lambda_\Gamma(y))^\beta}$.

**Theorem 3.6.** *Let $\Gamma$ be a connected graph. Then* $\Big(\frac{1}{2\Delta}\Big)S_{\frac{3}{2},1}(\Gamma) \leq SDD(\Gamma) \leq \Big(\frac{1}{2\delta}\Big)S_{\frac{3}{2},1}(\Gamma)$.

**Proof.** By the definition of *SDD* index,

$$\begin{aligned}&(SDD(\Gamma))^2 \qquad (3.2)\\ =&\ \Big(\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ =&\ \Big(\sum_{xy\in E(\Gamma)} \frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)\sqrt{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}}{\lambda_\Gamma(x)\lambda_\Gamma(y)\sqrt{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}}\Big)^2\\ \geq&\ \Big(\sum_{xy\in E(\Gamma)} \frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^{\frac{3}{2}}}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big(\frac{1}{\sqrt{2\Delta^2}}\Big)\Big)^2\\ =&\ \Big(\frac{1}{2\Delta}\Big)S_{\frac{3}{2},1}(\Gamma).\end{aligned}$$

Similarly, we can find upper bound.

$$\begin{aligned}&(SDD(\Gamma))^2 \qquad (3.3)\\ =&\ \Big(\sum_{xy\in E(\Gamma)} \frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)\sqrt{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}}{\lambda_\Gamma(x)\lambda_\Gamma(y)\sqrt{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}}\Big)^2\\ \leq&\ \Big(\sum_{xy\in E(\Gamma)} \frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^{\frac{3}{2}}}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big(\frac{1}{\sqrt{2\delta^2}}\Big)\Big)^2\\ =&\ \Big(\frac{1}{2\delta}\Big)S_{\frac{3}{2},1}(\Gamma).\end{aligned}$$

**Lemma 3.7.** *Let $k \geq 1$ be an integer and $\{x_i\}_{i=1}^k$ be some non-negative real numbers such that $x_1 \geq x_2 \geq \ldots \geq x_k$. Then $(x_1+x_1+\ldots+x_k)(x_1+x_k) \geq x_1^2+x_2^2+\ldots+x_n^2+kx_1x_k$.*

**Theorem 3.8.** *Let $\Gamma$ be a connected graph. Then* $SDD(\Gamma) \geq \frac{\frac{F_2(\Gamma)}{\delta^4}+\frac{m\Delta^2\delta^2}{4\Delta\delta}}{\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)}$.

**Proof.** For any edge $xy$ in $E(\Gamma)$, let $a_{xy} = \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}$, $a_1 = \frac{\Delta^2}{2\delta}$ and $a_n = \frac{\delta^2}{2\Delta}$. Apply Lemma 3.7 and by the definition of *SDD* index, we have

$$\begin{aligned}&\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)\\ &\geq \sum_{xy\in E(\Gamma)}\Big(\frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2+\frac{m\Delta^2\delta^2}{4\Delta\delta}.\\ &\Rightarrow SDD(\Gamma)\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)\\ &\geq \sum_{xy\in E(\Gamma)}\Big(\frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2+\frac{m\Delta^2\delta^2}{4\Delta\delta}.\\ &\Rightarrow SDD(\Gamma)\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)\\ &\geq \sum_{xy\in E(\Gamma)}\Big(\frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^2}{\delta^4}\Big)+\frac{m\Delta^2\delta^2}{4\Delta\delta}.\end{aligned}$$

$$\Rightarrow SDD(\Gamma) \geq \frac{\sum_{xy\in E(\Gamma)}\Big(\frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2)^2}{\delta^4}\Big)+\frac{m\Delta^2\delta^2}{4\Delta\delta}}{\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)}.$$

Hence

$$SDD(\Gamma) \geq \frac{\frac{F_2(\Gamma)}{\delta^4}+\frac{m\Delta^2\delta^2}{4\Delta\delta}}{\Big(\frac{\Delta^2}{2\delta}+\frac{\delta^2}{2\Delta}\Big)}.$$

**Theorem 3.9.** *Let $\Gamma$ be a connected graph. Then* $SDD(\Gamma) \geq \frac{2\delta}{\Delta^2}\Big(M_1(\Gamma)-\frac{1}{\delta}M_2(\Gamma)-4ISI(\Gamma)\Big)$.

**Proof.** One can observe that

$$\begin{aligned}&\sum_{xy\in E(\Gamma)} \frac{\lambda_\Gamma(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)} \qquad (3.4)\\ =&\ \sum_{xy\in E(\Gamma)} \frac{\frac{1}{4}\Big((\lambda_\Gamma(x)+\lambda_\Gamma(y))^2-(\lambda_\Gamma(x)-\lambda_\Gamma(y))^2\Big)}{(\lambda_\Gamma(x)+\lambda_\Gamma(y))}\\ =&\ \frac{1}{4}\sum_{xy\in E(\Gamma)}\Big(\frac{(\lambda_\Gamma(x)+\lambda_\Gamma(y))^2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}-\frac{(\lambda_\Gamma(x)-\lambda_\Gamma(y))^2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)\\ =&\ \frac{1}{4}\Big(\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)+\lambda_\Gamma(y))-\sum_{xy\in E(\Gamma)}\frac{(\lambda_\Gamma(x)-\lambda_\Gamma(y))^2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)\\ \geq&\ \frac{1}{4}\Big(\sum_{xy\in E(\Gamma)}(\lambda_\Gamma(x)+\lambda_\Gamma(y))-\sum_{xy\in E(\Gamma)}\frac{(\lambda_\Gamma(x)-\lambda_\Gamma(y))^2}{2\delta}\Big).\end{aligned}$$

By the definition of inverse sum indeg index and first Zagreb index, we have

$$\begin{aligned} &ISI(\Gamma)\\ \geq\ & \frac{1}{4}\Big(M_1(\Gamma)-\sum_{xy\in E(\Gamma)}\frac{(\lambda_\Gamma(x)-\lambda_\Gamma(y))^2}{2\delta}\Big)\\ =\ & \frac{1}{4}\Big(M_1(\Gamma)-\sum_{xy\in E(\Gamma)}\frac{(\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2-2\lambda_G(x)\lambda_\Gamma(y)}{2\delta}\Big)\\ \geq\ & \frac{1}{4}\Big(M_1(\Gamma)-\frac{\Delta^2}{2\delta}\sum_{xy\in E(\Gamma)}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\\ &-\frac{1}{\delta}\sum_{xy\in E(\Gamma)}\lambda_\Gamma(x)\lambda_\Gamma(y)\Big). \end{aligned}$$

By the definition of *SDD* index and second Zagreb index, we have

$$ISI(\Gamma)\geq\frac{1}{4}\Big(M_1(\Gamma)-\frac{\Delta^2}{2\delta}SDD(\Gamma)-\frac{1}{\delta}M_2(\Gamma)\Big).$$

$$\Rightarrow\frac{\Delta^2}{2\delta}SDD(\Gamma)\geq M_1(\Gamma)-\frac{1}{\delta}M_2(\Gamma)-4ISI(\Gamma).$$

This implies

$$SDD(\Gamma)\geq\frac{2\delta}{\Delta^2}\Big(M_1(\Gamma)-\frac{1}{\delta}M_2(\Gamma)-4ISI(\Gamma)\Big).$$

**Theorem 3.10.** *Let $\Gamma$ be a connected graph. Then $SDD(\Gamma)\geq\frac{4\delta}{\Delta^2}ISI(\Gamma)$.*

**Proof.** One can see that for any two real numbers $a$ and $b$, $ab\leq\frac{a^2}{2}+\frac{b^2}{2}$. Hence

$$\begin{aligned} &\sum_{xy\in E(\Gamma)}\frac{\lambda_\Gamma(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)} \qquad (3.5)\\ \leq\ &\sum_{xy\in E(\Gamma)}\frac{\frac{\lambda_\Gamma(x)^2}{2}+\frac{\lambda_\Gamma(y)^2}{2}}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\\ =\ &\frac{1}{2}\sum_{xy\in E(\Gamma)}\Big(\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)\Big(\frac{\lambda_\Gamma(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)\\ \leq\ &\frac{\Delta^2}{4\delta}\sum_{xy\in E(\Gamma)}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}. \end{aligned}$$

By the definition of inverse sum indeg index and *SDD* index, we have

$$ISI(\Gamma)\leq\frac{\Delta^2}{4\delta}SDD(\Gamma).$$

Therefore

$$SDD(\Gamma)\leq\frac{4\delta}{\Delta^2}ISI(\Gamma).$$

**Lemma 3.11.** *For any positive numbers x and y, define $A(x,y)=\frac{x+y}{2}$, $\Gamma(x,y)=\sqrt{xy}$, $H(x,y)=\frac{2xy}{x+y}$, $N_1(x,y)=\Big(\frac{\sqrt{x}+\sqrt{y}}{2}\Big)^2$, $N_2(x,y)=\Big(\frac{\sqrt{x}+\sqrt{y}}{2}\Big)\Big(\sqrt{\frac{x+y}{2}}\Big)$, $N_3(x,y)=\frac{x+y+\sqrt{xy}}{3}\Big)$, $S(x,y)=\sqrt{\frac{x^2+y^2}{2}}\Big)$ and $M_{PQ}=P(x,y)-q(x,y)$.*

*Then $\frac{1}{2}M_{AH}(x,y)\leq\frac{1}{2}M_{SG}(x,y)$, $\frac{1}{8}M_{AH}(x,y)\leq\frac{1}{2}M_{N_2N_1}(x,y)$ and $M_{SH}(x,y)\leq 2M_{SN_1}(x,y)$.*

**Theorem 3.12.** *Let $\Gamma$ be a connected graph. Then $SDD(\Gamma)\geq m\Big(\frac{1}{2}+\frac{6\delta^3}{\Delta^2}\sqrt{\frac{1}{2}}-\frac{m\delta^2}{2}\Big)$.*

**Proof.** By Lemma 3.11, we have

$$\begin{aligned} &\frac{1}{3}\Big(\sqrt{\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2}}-\frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)\\ &\leq\frac{1}{2}\Big(\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{2}-\frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big). \end{aligned}$$

Thus

$$\frac{2\lambda_G(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\leq 3\Big(\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{2}\Big)-2\sqrt{\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2}}.$$

Squaring on both sides, we get

$$\begin{aligned} &4\Big(\frac{\lambda_\Gamma(x)\lambda_\Gamma(y)}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)^2\leq 9\Big(\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{2}\Big)^2\\ &+4\Big(\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2}\Big)\\ &-12\Big(\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{2}\Big)\Big(\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2}\Big). \end{aligned}$$

$$\begin{aligned} &\Rightarrow\Big(\frac{2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)^2\leq\frac{9}{4}\Big(\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\Big)^2\\ &+2\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{(\lambda_\Gamma(x)\lambda_\Gamma(y))^2}-6\frac{\lambda_\Gamma(x)+\lambda_\Gamma(y)}{(\lambda_\Gamma(x)\lambda_\Gamma(y))^2}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{2}. \end{aligned}$$

$$\begin{aligned} &\Rightarrow\Big(\frac{2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)^2\leq\frac{9}{4}\Big(\frac{2\delta}{\Delta^2}\Big)^2+\frac{2}{\Delta^2}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}\\ &-6\frac{2\delta}{\Delta^4}\sqrt{\frac{\delta^4}{2}}. \end{aligned}$$

Taking summation over all edges of $\Gamma$ on both sides, we have

$$\begin{aligned} &\sum_{xy\in E(\Gamma)}\Big(\frac{2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)^2\leq\frac{9m}{4}\Big(\frac{2\delta}{\Delta^2}\Big)^2\\ &+\frac{2}{\Delta^2}\sum_{xy\in E(\Gamma)}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}-6m\frac{2\delta}{\Delta^4}\sqrt{\frac{\delta^4}{2}}. \end{aligned}$$

But

$$\sum_{xy\in E(\Gamma)}\Big(\frac{2}{\lambda_\Gamma(x)+\lambda_\Gamma(y)}\Big)^2\geq\frac{m}{\Delta^2}.$$

Hence

$$\frac{m}{\Delta^2}-\frac{m\delta^2}{\Delta^2}+\frac{12m\delta^3}{\Delta^4}\sqrt{\frac{1}{2}}\leq\frac{2}{\Delta^2}\sum_{xy\in E(\Gamma)}\frac{\lambda_\Gamma(x)^2+\lambda_\Gamma(y)^2}{\lambda_\Gamma(x)\lambda_\Gamma(y)}.$$

By the definition of *SDD* index, we have

$$SDD(\Gamma) \geq m\Big(\frac{1}{2}+\frac{6\delta^3}{\Delta^2}\sqrt{\frac{1}{2}-\frac{m\delta^2}{2}}\Big).$$

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