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On transitive CI-algebras

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Abstract

The idea of transitive CI-algebras was first introduced by B. L. Meng. In this paper we continue to study transitive CI-algebras and try to establish some properties in some specific CI-algebras. We also prove that every absorptive CI-algebra is transitive but the converse is not true.

Keywords

CI-algebra, BE-algebra, transitive, absorptive.

AMS Subject Classification

06F35, 03G25, 08A30.

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Contents

1. Introduction

Y. Imai and K. Iseki ([2, 3]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. BCI-algebras as a class of logical algebras are the algebraic formulations of the set difference together with its properties in set theory and the implication functor in logical systems. They are closely related to partially ordered commutative monoids as well as various logical algebras. After that some more concepts of similar type like BCH ([1]), BH ([4]), d ([9]), etc have been developed and studied by a number of authors in the last two decades. In 2006, H. S. Kim nad Y. H. Kim introduced the notion of BE-algebras ([6]) as a generalization of dual BCK-algebras ([5]). In 2010 ([7]) B. L. Meng defined the notion of a new algebraic structure called CI-algebras as a generalization of BE-algebras. In his paper Meng defined the transitivity property of CI-algebras. BE-algebras and CI-algebras are already studied in detail by some researchers and some fundamental properties of CI-algebras are discussed. The structure of Cartesian product of CI-algebras has been developed by us in 2013 ([10]). The notion of absorptive CI-algebra ([11]) has been introduced by us. In 2018, we have formulated some special types of CI-algebras ([12]) obtained from a given CI-algebra. In this paper we continue to study transitive CI-algebras and investigate some new properties of it in these specific CI–algebras. We also prove that every absorptive CI-algebra is transitive but the converse is not true.

2. Preliminaries

Definition 2.1 ([8])**.** *An algebraic system $(S;^*,1)$ is called a BE -algebra if it satisfies the following axioms:*

(A1) $p*p=1$

(A2) $p*1=1$

(A3) $1*p=P$

(A4) $p*(t*r)=t*(p*r)$ *for all* $p,t,r\in S$

Definition 2.2 ([7])**.** *An algebraic system $(S;*,1)$ is called a CI-algebra if it satisfies the following axioms:*

(A1) $p*p=1$

(A2) $1*p=p$

(A3) $p*(t*r)=t*(p*r)$ *for all* $p,t,r\in S.$

Example 2.3 ([8])**.** *Let $S=R^+=\{p\in R:p>o\}$ For $p,t\in S$, we define*

$$p*t=t\div p$$

Then $(S;*,1)$ is a CI-algebra which is not a BE -algebra. In S, we can define a binary relation $\leq$ by $p\leq t$ iff $p*t=1$.

Lemma 2.4 ([7]). *In a CI-algebra $(S;*,1)$ following results are true:*

1. $p*((p*t)*t)=1$
2. $(p*t)*1=(p*1)*(t*1)$ *for all* $p,t\in S$.

Theorem 2.5 ([7]). *Let $(S;*,1)$ be a BE -algebra and let $r\notin S$. A binary operation $\circ$ is defined on $S\cup\{r\}$ as follows: For any $p,t\in S\cup\{r\}$*

$$p\circ t=\begin{cases} p*t & \text{if } p,t\in S\\ r & \text{if } p=r,t\neq r\\ r & \text{if } p\neq r,t=r\\ 1 & \text{if } p=t=r\end{cases}$$

Then $(S\cup\{r\};\circ,1)$ is a CI-algebra.

Note 2.6. *The above result provides a method to extend a BE-algebra into a CI-algebra, by adjoining an element not in the given BE-algebra.*

Theorem 2.7 ([10]). *Let $(S;*,1)$ be a system consisting of a non-empty set S, a binary operation $*$ and a distinct element 1. Let $Q=S\times S=\{(p_1,p_2):p_1,p_2\in S\}$. For $p,t\in Q$ with $p=(p_1,p_2),t=(t_1,t_2)$, we define an operation $\otimes$ in Q as*

$$p\otimes t=(p_1*t_1,p_2*t_2)$$

Then $(Q;\otimes,(1,1))$ is a CI-algebra iff $(S;*,1)$ is a CI-algebra.

Corollary 2.8 ([10]). *If $(S;*,1)$ and $(Q;o,e)$ are two CI-algebras, then $R=S\times Q$ is also a CI-algebra under the binary operation defined as follows: For $p=(p_1,t_1)$ and $t=(p_2,t_2)$ in R*

$$p\otimes t=(p_1*p_2,t_1\circ t_2)$$

Here the distinct element of R is $(1,e)$.

Theorem 2.9 ([12]). *Let $(S;*,1)$ be a CI-algebra and let $F(S)$ be the class of all functions $f:S\to S$. Let a binary operation o be defined in $F(S)$ as follows:*

$$\text{For } f,f_1\in F(S) \text{ and } p\in S$$
$$(fo\ f_1)(p)=f(p)*f_1(p)$$

Then $(F(S);o,1^{\sim})$ is a CI-algebra where $1^{\sim}$ is defined as $1^{\sim}(p)=1$ for all $p\in S$. Here two functions $f,f_1\in F(S)$ are equal iff $f(p)=f_1(p)$ for all $p\in S$.

Notation 2.10 ([12]).

(a) *For any set $S_1\subseteq S$, let $F(S_1)$ denote the set of all functions $f\in F(S)$ such that $f(p)\in S_1$ for all $p\in S$.*

(b) *For any $t\in S$, we consider $f_t\in F(S)$ defined as $f_{t(p)}=t$ for all $p\in S$.*

Definition 2.11 ([11]). *A CI-algebra $(S;*,1)$ is said to be absorptive if for any $p,t,r\in S$*

$$(p*t)*(p*r)=(t*r)$$

3. Transitive CI-Algebra

Definition 3.1 ([7]). *A CI-algebra $(S_i*,1)$ is said to be transitive if for all $p,t,r\in A$*

$$(t*r)*((p*t)*(p*r))=1$$

Lemma 3.2 ([8]). *Let $(S_i*,1)$ be a transitive CI-algebra. Then for all $p,t,r\in S$,*

(i) $t\leq r$ implies $p*t\leq p*r$

(ii) $t\leq r$ implies $r*p\leq t*p$.

Theorem 3.3. *Let S and Q be CI-algebras as considered in theorem (2.7). Then Q is transitive iff S is transitive.*

Proof. First suppose that Q is transitive. Let p,t,r be three arbitrary elements of S. We consider $u_1=(p,1),u_2=(t,1)$ and $u_3=(r,1)$ of Y. Since Q is transitive, we have

$$u_2\otimes u_3\leq(u_1\otimes u_2)\otimes(u_1\otimes u_3)$$
$$\Rightarrow(u_2\otimes u_3)\otimes((u_1\otimes u_2)\otimes(u_1\otimes u_3))=(1,1)$$

This gives $((t*r)*((p*t)*(p*r)),1)=(1,1)$

$$\Rightarrow(t*r)*((p*t)*(p*r))=1$$

$$\Rightarrow(t*r)\leq(p*t)*(p*r)$$

Hence S is transitive.

Conversely, suppose that $(S;*,1)$ is a transitive CI-algebra and $u_1=(p_1,t_1),u_2=(p_2,t_2)$ and $u_3=(p_3,t_3)$ be any three arbitrary elements of Q. Then

$$\begin{aligned}&(u_2\otimes u_3)\otimes((u_1\otimes u_2)\otimes(u_1\otimes u_3))\\&=(u_2\otimes u_3)\otimes((p_1*p_2,t_1*t_2)\otimes(p_1*p_3,t_1*t_3))\\&=(u_2\otimes u_3)\otimes(((p_1*p_2)*(p_1*p_3)),((t_1*t_2)*(t_1*t_3)))\\&=(p_2*p_3,t_2*t_3)\otimes(((p_1*p_2)*(p_1*p_3)),\\&\qquad((t_1*t_2)*(t_1*t_3)))\\&=((p_2*p_3)*((p_1*p_2)*(p_1*p_3)),(t_2*t_3)*\\&\qquad((t_1*t_2)*(t_1*t_3)))\\&=(1,1).\end{aligned}$$

This implies $u_2\otimes u_3\leq(u_1\otimes u_2)\otimes(u_1\otimes u_3)$. So Q is transitive. □

Corollary 3.4. *If $R=S\times Q$ be a CI-algebra considered in corollary (2.8). Then R is transitive iff both S and Q are transitive.*

Theorem 3.5. *Let $(S;_4*,1)$ be a CI-algebra and let $\left(F(S);o,1^2\right)$ be function CI-algebra discussed in theorem (2.9). Then $F(S)$ is transitive if and only if S is transitive.*

Proof. Let S be transitive. Then for any $p,t,r\in S$, we have

$$(t*r)*((p*t)*(p*r))=1$$

Let $f, f_1, f_2 \in F(S)$. Then for any $p \in S$, we have

$$\begin{aligned}((f_1 \circ f_2) \circ ((f \circ f_1) \circ (f o f_3)))(p) \\ = (f_1 \circ f_2)(p) * ((f o f_1) \circ (f \circ f_2))(p) \\ = (f_1(p) * f_2(p)) * ((f(p) * f_1(p)) * (f(p) * f_2(p))) \\ = 1 = 1^{\sim}(p)\end{aligned}$$

So, $((f_1 \circ f_2) \circ ((f \circ f_1) \circ (f o f_3))) = 1^{\sim}$.
This proves that $F(S)$ is transitive.

Conversely, let $F(S)$ be transitive. Then for any $f, f_1, f_2 \in F(S)$, we have

$$(f_1 \circ f_2) \circ ((f \circ f_1) o (f \circ f_2)) = 1^2$$

Let $p, t, r \in S$. We consider $f_p, f_t, f_x \in F(S)$.

Then from above we have

$$((f_t \circ f_x) \circ ((f_p \circ f_t) o (f_p \circ f_x))(u) = 1 \text{ for all } u \in S.$$

This gives $(t * r) * ((p * t) * (p * r)) = 1$. (notation $(2.10)(b)$)
Hence S is transitive. □

Theorem 3.6. *Every absorptive CI-algebra is transitive.*

Proof. Let $(S; *, 1)$ be an absorptive CI-algebra. Then for all $p, t, r \in S$, we have

$$(p * t) * (p * r) = t * r$$

This gives

$$(t * r) * ((p * t) * (p * r)) = (t * r) * (t * r) = 1$$

Hence $(S; *, 1)$ is transitive. □

Note 3.7. *Converse of the above result is not true. For this we have the following example:*

Example 3.8. *We consider a system $(S; *, 1)$ where $S = \{1, t_1, t_2, t_3\}$ and the binary operation * is given by*

$*$	1	t_1	t_2	t_3
1	1	t_1	t_2	t_3
t_1	1	1	t_2	t_3
t_2	1	t_1	1	t_3
t_3	1	t_1	t_2	1

This $(S; *, 1)$ is a self distributive and transitive BE -algebra. Using theorem (2.5) we see that $(Q; o, 1)$ is a $CI-$ algebra where $Q = S \cup \{t\}$ and the binary operation $*$ is given by

$\circ$	1	t_1	t_2	t_3	t
1	1	t_1	t_2	t_3	t
t_1	1	1	t_2	t_3	t
t_2	1	t_1	1	t_3	t
t_3	1	t_1	t_2	1	t
t	t	t	t	t	1

Now we cheek the transitive condition

$$(t_2 \circ t_3) \circ ((t_1 \circ t_2) \circ (t_1 \circ t_3)) = 1$$

for elements of Q. Since S is transitive, in order to prove that Q is transitive, we verify the above condition for $t_1 = t, t_2 = t$ and $t_3 = t$ one by one. We have

$$\begin{aligned}(t_2 \circ t_3) \circ ((t \circ t_2) \circ (t \circ t_3)) = t_3 \circ (t \circ t) = t_3 \circ 1 = 1 \\ (t \circ t_3) \circ ((t_1 \circ t) \circ (t_1 \circ t_3)) = t \circ (t \circ t_3) = t \circ t = 1 \\ (t_2 \circ t) \circ ((t_1 \circ t_2) \circ (t_1 \circ t)) = t \circ (t_2 \circ t) = t \circ t = 1\end{aligned}$$

Hence $(Q; o, 1)$ is transitive. But $(Q; o, 1)$ is not absorptive. For, $(t \circ t_1) \circ (t \circ t_2) = t \circ t = 1$ and $t_1 \circ t_2 = t_2$.

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